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TRANSVERSALITY AND THE INVERSE IMAGE OF A SUBMANIFOLD WITH CORNERS

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1. Introduction.

It is a well-known theorem about usual differential manifolds that the inverse image of a submanifold by a map which intersects the submanifold transversally is a submanifold of the same codimension. This paper generalizes the theorem to the case where the manifolds are allowed to have corners (Section 6). The key step in the proof is a generalization of the theorem about local linearization of submersions (Section 5). It is assumed that the map preserves local facets relatively to the submanifold. This property is defined in Section 4. It is convenient to make extensive use of germs, and therefore some notation related to germs is introduced in Section 2. Elementary facts about manifolds with corners are reviewed in Section 3. Finally, the special case of manifolds and submanifolds with boundary of codimension one is discussed in Section 7.

2. Germs.

In this section X and Y are topological spaces, $x \in X, y \in Y$. The set of germs at x of subsets $A \subset X$ will be denoted $\mathrm{SG}(X,x)$. The germ of A at x will be denoted $[X,A,x]$. If $a,b \in \mathrm{SG}(X,x)$, define the inclusion $a \subset b$ by $a \subset b$ iff there exist $A \in a$ and $B \in b$ such that $A \subset B$. Define the intersection $a \cap b$ by $a \cap b = [X, A \cap B, x]$, where $A \in a$ and $B \in b$. If $f\colon (Y,y) \to (X,x)$ is a continuous local map, let $[f,y]$ denote the germ of f at y. If $F\colon (Y,y) \to (X,x)$ is the germ at y of a continuous local map, and $a \in \mathrm{SG}(X,x)$, define $F^{-1}(a) \in \mathrm{SG}(Y,y)$ by $F^{-1}(a) = [f^{-1}(A), y]$, where $f \in F$ and $A \in a$.

3. Manifolds, submanifolds, and regular germs.

By a "manifold" we mean a manifold with corners in the sense of Cerf [1, ch. 1, pp. 241–246] (variétè à bord géneralisée) or Mather [3 § 1, pp. 255–259]. Consult those papers for further details. If $0 \leqq p \leqq m$, put $\mathbf{R}^m_{(p)} = \mathbf{R}^p \times [0,\infty[^{m-p}$. If X is a *manifold of class* $s \geqq 1$ *and of dimension* m, then for every $x \in X$ there is

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a p and a local C^s chart $\varphi: (X,x) \to (\mathsf{R}^m_{(p)}, 0)$ such that $\operatorname{im}(\varphi)$ is open in $\mathsf{R}^m_{(p)}$ and φ maps $\operatorname{dom}(\varphi)$ C^s-diffeomorphically onto $\operatorname{im}(\varphi)$. The number p does not depend on the specific choice of φ. It is called the index of x in X and denoted $\operatorname{ind}(X,x)$. From now on X, Y, and Z are manifolds of class $s \geqq 1$ and of dimension m, n, and k.

If $0 \leqq q \leqq r \leqq k \leqq m+r-p$, $r \leqq p$, put

$$\mathsf{R}^m_{(p;q,r,k)} = \mathsf{R}^q \times [0,\infty[^{r-q} \times \{0\}^{p-r} \times [0,\infty[^{k-r} \times \{0\}^{m+r-k-p} .$$

Let $A \subset X$. A C^s *submanifold chart* on $(X, A;\, \mathsf{R}^m_{(p)}, \mathsf{R}^m_{(p;q,r,k)})$ is a local C^s chart $\varphi: X \to \mathsf{R}^m_{(p)}$ such that $\varphi(A \cap \operatorname{dom}(\varphi)) = \operatorname{im}(\varphi) \cap \mathsf{R}^m_{(p;q,r,k)}$. We call A a *k-dimensional C^s submanifold of X* if $A \neq \emptyset$ and every $x \in A$ is contained in the domain of a C^s submanifold chart φ on $(X, A;\, \mathsf{R}^m_{(p)}, \mathsf{R}^m_{(p;q,r,k)})$ for some p, q, r (dependent on x). This φ may be chosen such that $\varphi(x) = 0$. A germ $a \in \operatorname{SG}(X,x)$ is *C^s-regular of dimension k and index q* if it has a representative A, which is a k-dimensional C^s submanifold of X and such that $\operatorname{ind}(A,x) = q$. We write $\dim a = k$, $\operatorname{ind} a = q$. For $p = 0, \ldots, m$ put

$$\partial_p X = \{x \in X : \operatorname{ind}(X,x) = p\} .$$

A connected component of $\partial_p X$ is called a *p-stratum of X*. A subset of X, which is a p-stratum of X for some p, is called a *stratum of X*. The set of strata of X clearly form a partition of X. If $x \in X$, let $\partial^x X$ denote the stratum of X containing x. Then $\partial^x X$ is a C^s submanifold of X of dimension $\operatorname{ind}(X,x)$. The set $X \setminus \partial_m X$ is called *the boundary of X* and it is denoted $\operatorname{bd} X$. If $\operatorname{bd} X = \emptyset$, then X is a usual manifold or a *manifold without boundary*. If $\operatorname{bd} X = \partial_{m-1} X$, then X is a *manifold with boundary*. This is not equivalent to $\operatorname{bd} X \neq \emptyset$ as the term might suggest.

If $a \in \operatorname{SG}(X,x)$ is C^s-regular and $A \in a$, then the germ $[X, \partial^x X, x] \cap a = [X, A \cap \partial^x X, x]$ is C^s-regular. If $\varphi: (X,x) \to (\mathsf{R}^m_{(p)}, 0)$ is a C^s submanifold chart on $(X, A;\, \mathsf{R}^m_{(p)}, \mathsf{R}^m_{(p;q,r,k)})$, then the parameters have the following significance: $p = \operatorname{ind}(X,x)$, $q = \operatorname{ind} a$, $r = \dim [X, \partial^x X, x] \cap a$, $k = \dim a$, $m = \dim X$. Hence they are all uniquely determined. The parameters determine the pair (X, a) up to the germ of a local C^s-diffeomorphism.

If $A \subset X$ is a C^s submanifold and $x \in A$, let $T_x A$ denote the tangent space to A at x, and put $\hat{T}_x A = T_x \partial^x A$. If $a \in \operatorname{SG}(X,x)$ is C^s-regular and $A \in a$ is a C^s submanifold of X, put $Ta = T_x A$ and $\hat{T}a = \hat{T}_x A$. If $f: (Y,y) \to (X,x)$ is a local C^s map or the germ at y of such a map, let $T_y f: T_y Y \to T_x X$ denote the differential of f at y. Note that $(T_y f)(\hat{T}_y Y) \subset \hat{T}_x X$.

4. Local faces.

A *p-face of X* is the closure of a p-stratum of X. A subset of X which is a p-face of X for some p is called a *face of X*. A face of X is not necessarily a submanifold of X. This leads to the following approach. If U is a convex open neighborhood of 0 in $\mathbf{R}^m_{(p)}$, then we can describe the faces of U as follows: If $\alpha \subset \{p+1,\ldots,m\}$, put

$$F^m_{(p,\alpha)} = \{x \in \mathbf{R}^m_{(p)} : x_i = 0 \quad \text{for } i \in \{p+1,\ldots,m\} \setminus \alpha\} .$$

Then the q-faces of U are the sets $F^m_{(p,\alpha)} \cap U$, where $p + \operatorname{card} \alpha = q$. A *local q-face* of X at x is a germ $[X, F, x]$, where F is a q-face of the domain of a local C^s chart φ: $(X,x) \to (\mathbf{R}^m_{(p)}, 0)$ such that im (φ) is convex (here $p = \operatorname{ind}(X,x)$). Two such local C^s charts give rise to the same local q-faces of X at x. The local q-faces of X at x are C^s-regular of dimension q and index p. A *local face* of X at x is a local q-face of X at x for some q. The set of local faces of X at x, partially ordered by the inclusion defined in Section 2, is a lattice. The intersection of two local faces of X at x is a local face of X at x. If φ: $(X,x) \to (\mathbf{R}^m_{(p)}, 0)$ is a C^s chart such that im (φ) is convex, then the map

$$F^m_{(p,\alpha)} \cap \operatorname{im}(\varphi) \mapsto [X, \varphi^{-1}(F^m_{(p,\alpha)} \cap \operatorname{im}(\varphi)), x]$$

is an isomorphism from the lattice of faces of im (φ) to the lattice of local faces of X at x. This isomorphism maps the intersection of two faces of im (φ) to the intersection of the corresponding local faces of X at x. If $a, b \in \operatorname{SG}(X,x)$ are local faces of X at x, $a \subset b$, and $\dim a = (\dim b) - 1$, then a is a *local facet in b*. A local facet in $[X, X, x]$ is called a *local facet of X at x*.

If G: $(Z,z) \to (X,x)$ is the germ of a continuous local map, define F_G as the smallest local face of X at x such that $[Z,Z,z] \subset G^{-1}(F_G)$. Put $p = \operatorname{ind}(X,x)$ and recall that $k = \dim Z$. Note that there are exactly $(\dim F_G - p)$ local facets in F_G. We say that G *preserves local facets* if there are $(\dim F_G - p)$ distinct local facets F'_i, $i = 1,\ldots,(\dim F_G - p)$, of Z at z, such that $F'_i \subset G^{-1}(F_i)$ for all i, where F_i, $i = 1,\ldots,(\dim F_G - p)$, are the local facets in F_G. We want to give a description of this property in local coordinates. Assume that $q \leqq r \leqq k$, $p + k - r \leqq m$. A local map f: $(\mathbf{R}^k_{(q)}, 0) \to (\mathbf{R}^m_{(p)}, 0)$ is said to *satisfy condition* (r) if $f_i(x) = 0$ for all $x \in \operatorname{dom}(f)$, all $i = p+k-r+1,\ldots,m$, and if $x_j = 0$ implies $f_{j+p-r}(x) = 0$ for all $x \in \operatorname{dom}(f)$, all $j = r+1,\ldots,k$. It is not too difficult to see that if G: $(\mathbf{R}^k_{(q)}, 0) \to (\mathbf{R}^m_{(p)}, 0)$ is the germ of a local map, then G preserves local facets iff there are permutations S: $\mathbf{R}^k_{(q)} \to \mathbf{R}^k_{(q)}$ of the coordinates $q+1,\ldots,k$ and T: $\mathbf{R}^m_{(p)} \to \mathbf{R}^m_{(p)}$ of the coordinates $p+1,\ldots,m$, and a representative $g \in G$ such that the local map TgS^{-1}: $(\mathbf{R}^k_{(q)}, 0) \to (\mathbf{R}^m_{(p)}, 0)$ satisfies condition (r) with $r = p + k - \dim F_G$.

If G: $(Y,y) \to (X,x)$ is the germ of continuous local map, and $a \in \operatorname{SG}(X,x)$ is C^s-regular, then let $F_{(G,a)}$ denote the smallest local face of X at x, such that $a \subset F_{(G,a)}$ and $[Y,Y,y] \subset G^{-1}(F_{(G,a)})$. We say that G *preserves local facets relatively to a* if the following condition is fulfilled: Let $F_1,\ldots,F_h$ be those local facets in $F_{(G,a)}$ that contain a. Then there are distinct local facets

$F'_1,\ldots,F'_h$ of Y at y, so that $F'_i \subset G^{-1}(F_i)$, $i=1,\ldots,h$. If G: $(Y,y) \to (\mathbf{R}^m_{(p)},0)$ is the germ of a continuous local map, then G preserves local facets relatively to $[\mathbf{R}^m_{(p)}, \mathbf{R}^m_{(p;\,q,r,k)}, 0]$ iff

$$[P,0]G\ :\ (Y,y) \to (\mathbf{R}^{m+r-q-k}_{(p-q)},0)$$

preserves local facets, where P: $\mathbf{R}^m_{(p)} \to \mathbf{R}^{m+r-q-k}_{(p-q)}$ is the projection on the coordinates $q+1,\ldots,p;\ p+k-r+1,\ldots,m$. A local map f: $(Y,y) \to (X,x)$ *preserves local facets at* y if $[f,y]$ preserves local facets. If f preserves local facets at y for all $y \in \operatorname{dom}(f)$ then f *preserves local facets.* Let $A \subset X$ be a C^s submanifold, f: $Y \to X$ a local map, and $y \in \operatorname{dom}(f)$. Then f *preserves local facets relatively to* A *at* y if either $f(y) \notin A$ of $f(y) \in A$ and $[f,y]$ preserves local facets relatively to $[X, A, f(y)]$. If f preserves local facets relatively to A at y for all $y \in \operatorname{dom}(f)$, then f *preserves local facets relatively to* A.

5. Local linearization.

The following theorem is a strong version of the usual theorem about local linearization of submersions.

THEOREM 1. *Let* f: $(Z,z) \to (X,x)$ *be a local* C^s *map. Assume that*
(a) f *preserves local facets at* z,
(b) $(T_zf)(T_zZ) = T_xX$,
(c) $(T_zf)(\hat{T}_zZ) = \hat{T}_xX$.
Then for every local C^s *chart* ψ: $(X,x) \to (\mathbf{R}^m_{(p)},0)$ *there is a permutation* T: $\mathbf{R}^m_{(p)} \to \mathbf{R}^m_{(p)}$ *of the coordinates* $p+1,\ldots,m$ *and a local* C^s *chart* φ: $(Z,z) \to (\mathbf{R}^k_{(q)},0)$ *such that* $\operatorname{dom}(\varphi) \subset \operatorname{dom}(f)$, $f(\operatorname{dom}(\varphi)) \subset \operatorname{dom}(\psi)$, and

$$T\psi f\varphi^{-1}(y) = (y_1,\ldots,y_p;\ y_{k-m+p+1},\ldots,y_k) \quad \textit{for } y \in \operatorname{im}(\varphi)\,.$$

Here $p = \operatorname{ind}(X,x)$ *and* $q = \operatorname{ind}(Z,z)$.

PROOF. Let $\tilde{\varphi}$: $(Z,z) \to (\mathbf{R}^k_{(q)},0)$ be a local C^s chart. Since the local map $\psi f\tilde{\varphi}^{-1}$ preserves local facets at 0, there are permutations S: $\mathbf{R}^k_{(q)} \to \mathbf{R}^k_{(q)}$ of the coordinates $q+1,\ldots,k$ and T: $\mathbf{R}^m_{(p)} \to \mathbf{R}^m_{(p)}$ of the coordinates $p+1,\ldots,m$, such that $g = T\psi f\tilde{\varphi}^{-1}S^{-1}$ satisfies condition (r) near 0, where $r = k+p-\dim F_{[g,0]}$. The matrix $Dg(0)$ looks like this:

$$\begin{array}{r} \\ p \\ k-r \\ m-p+r-k \end{array} \begin{array}{c} \begin{array}{ccc} q & r-q & k-r \end{array} \\ \begin{bmatrix} A & B & C \\ 0 & 0 & D \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$

D is a diagonal matrix with diagonal elements $\geqq 0$. It follows from assumption

(b) that the rank of $Dg(0)$ is m, so that D is regular, its diagonal elements are >0, and $m-p+r-k=0$. It follows from assumption (c) that $(T_0 g)(\hat{T}_0 \mathbf{R}^k_{(q)}) = \hat{T}_0 \mathbf{R}^m_{(p)}$, so that the rank of A is p. Hence there is a permutation P: $\mathbf{R}^k_{(q)} \to \mathbf{R}^k_{(q)}$ of the coordinates $1, \ldots, q$, such that $D(gP)(0)$ looks like this:

$$\begin{array}{cc} & \begin{array}{cccc} p & q-p & r-q & k-r \end{array} \\ \begin{array}{c} p \\ m-p \end{array} & \left[\begin{array}{cccc} A_1 & A_2 & B & C \\ 0 & 0 & 0 & D \end{array}\right] \end{array}$$

where A_1 is regular. Since gP satisfies condition (r) near 0, and

$$\frac{\partial (gP)_j}{\partial x_{j-m+k}}(0) > 0 \qquad \text{for } j=p+1, \ldots, m\,,$$

there is a local C^s extension $\tilde{g}$: $(\mathbf{R}^k, 0) \to (\mathbf{R}^m, 0)$ of gP at 0, such that $\tilde{g}_j(x)$ and x_{j-m+k} have the same sign for $x \in \operatorname{dom}(\tilde{g})$, $j=p+1, \ldots, m$. Define a local C^s map h: $(\mathbf{R}^k_{(q)}, 0) \to (\mathbf{R}^k, 0)$ by choosing $\operatorname{dom}(h)$ small enough and putting

$$h_i(x) = \begin{cases} \tilde{g}_i(x) & \text{for } i=1, \ldots, p \\ x_i & \text{for } i=p+1, \ldots, r \\ \tilde{g}_{i-k+m}(x) & \text{for } i=r+1, \ldots, k\,. \end{cases}$$

Now $Dh(0)$ looks as follows:

$$\begin{array}{cc} & \begin{array}{cccc} p & q-p & r-q & k-r \end{array} \\ \begin{array}{c} p \\ q-p \\ r-q \\ k-r \end{array} & \left[\begin{array}{cccc} A_1 & A_2 & B & C \\ 0 & E & 0 & 0 \\ 0 & 0 & E & 0 \\ 0 & 0 & 0 & D \end{array}\right] \end{array}$$

This shows that the rank of $Dh(0)$ is k, and that

$$\frac{\partial h_i}{\partial x_i}(0) > 0 \qquad \text{for } i=q+1, \ldots, k\,.$$

Furthermore, $x_i=0$ implies $h_i(x)=0$ for x near 0 in $\mathbf{R}^k$, $i=q+1, \ldots, k$. Hence assuming that $\operatorname{dom}(h)$ is chosen small enough, h is a local C^s map $(\mathbf{R}^k_{(q)}, 0) \to (\mathbf{R}^k_{(q)}, 0)$, and

$$h(\operatorname{bd} \mathbf{R}^k_{(q)} \cap \operatorname{dom}(h)) \subset \operatorname{bd} \mathbf{R}^k_{(q)}\,.$$

It follows from the version of the inverse function theorem mentioned in Mather [3, § 1, pp. 255–259] that we may assume that h is a local C^s-diffeomorphism. Choose an open neighborhood U of z in $\operatorname{dom}(\tilde{\varphi}) \cap \operatorname{dom}(f)$, such that $f(U) \subset \operatorname{dom}(\psi)$, $P^{-1}S\tilde{\varphi}(U) \subset \operatorname{dom}(h)$, and put $\varphi = hP^{-1}S\tilde{\varphi}_{\operatorname{res} U}$. If $y \in \operatorname{im}(\varphi)$, then there is $x \in P^{-1}S\tilde{\varphi}(U)$, such that $h(x)=y$, which implies

$$g_iP(x) = y_i \quad \text{for } i=1,\ldots,p\,,$$

and

$$g_iP(x) = y_{i+k-m} \quad \text{for } i=p+1,\ldots,m\,.$$

Thus

$$\begin{aligned} T\psi f\varphi^{-1}(y) &= T\psi f\varphi^{-1}S^{-1}Ph^{-1}(y) \\ &= gP(x) = (y_1,\ldots,y_p;\, y_{p+k-m+1},\ldots,y_k)\,. \end{aligned}$$

6. Transversality and the inverse image.

Let $G\colon (Y,y) \to (X,x)$ be the germ of a local C^1 map and let $a \in \mathrm{SG}\,(X,x)$ be C^1-regular. Then G is *transversal to* a if $(TG)(T_yY)+Ta=T_xX$, and G is *stratum transversal to* a if $(TG)(\hat{T}_yY)+\hat{T}a=\hat{T}_xX$.

THEOREM 2. *Let* $G\colon (Y,y) \to (X,x)$ *be the germ of a local* C^s *map and let* $a \in \mathrm{SG}\,(X,x)$ *be* C^s*-regular. Put* $p=\mathrm{ind}\,(X,x)$, $q=\mathrm{ind}\,a$, $r=\dim\,[X,\partial^xX,x]\cap a$, $k=\dim a$, $b=\mathrm{ind}\,(Y,y)$. *Let* $P\colon \mathbf{R}^m_{(p)} \to \mathbf{R}^{m+r-q-k}_{(p-q)}$ *be the projection on the coordinates* $q+1,\ldots,p;\ p+k-r+1,\ldots,m$. *Then* i) *and* ii) *below are equivalent.*

i) a) *G preserves local facets relatively to a, and*

b) *G is transversal to a, and*

c) *G is stratum transversal to a.*

ii) *Let* $g \in G$ *be a* C^s *representative,* $A \in a$, *and* $\psi\colon (X,x) \to (\mathbf{R}^m_{(p)},0)$ *a* C^s *submanifold chart on* $(X,A;\, \mathbf{R}^m_{(p)}, \mathbf{R}^m_{(p;\,q,r,k)})$. *Then there is a permutation* $T\colon \mathbf{R}^m_{(p)} \to \mathbf{R}^m_{(p)}$ *of the coordinates* $p+k-r+1,\ldots,m$ *and a local* C^s *chart* $\varphi\colon (Y,y) \to (\mathbf{R}^n_{(b)},0)$, *such that* $\mathrm{dom}\,(\varphi)\subset \mathrm{dom}\,(g)$, $g(\mathrm{dom}\,(\varphi))\subset\mathrm{dom}\,(\psi)$, *and the local map*

$$PT\psi g\varphi^{-1} : (\mathbf{R}^n_{(b)},0) \to (\mathbf{R}^{m+r-q-k}_{(p-q)},0)$$

is given by $y \mapsto (y_1,\ldots,y_{p-q};\, y_{n-m+p+k-r+1},\ldots,y_n)$ *for* $y \in \mathrm{im}\,(\varphi)$. *It is assumed that* $p-q \leqq b \leqq n-m+p+k-r$.

Furthermore, i) *or* ii) *imply*

iii) a) *$G^{-1}(a)$ is C^s-regular,*

b) $\dim Y - \dim G^{-1}(a) = \dim X - \dim a$,

c) $\mathrm{ind}\,(Y,y) - \mathrm{ind}\,G^{-1}(a) = \mathrm{ind}\,(X,x) - \mathrm{ind}\,a$,

d) $\dim\,([Y,\partial^yY,y]\cap G^{-1}(a)) - \mathrm{ind}\,(Y,y)$
$= \dim\,([X,\partial^xX,x]\cap a) - \mathrm{ind}\,(X,x)$.

PROOF. i) implies ii): Put $f=\psi g$. Since G preserves local facets relatively to a, $[\psi,x]G$ preserves local facets relatively to $[\mathbf{R}^m_{(p)}, \mathbf{R}^m_{(p;\,q,r,k)}, 0]$, and therefore Pf preserves local facets at 0. i) c) implies that

$$(T_y f)(\hat{T}_y Y) + \{\mathbf{0}\} \times \mathbf{R}^q \times \{0\}^{m-q} = \{\mathbf{0}\} \times \mathbf{R}^p \times \{0\}^{m-p} .$$

Using $T_0 P$ we get

$$\{\mathbf{0}\} \times \mathbf{R}^{p-q} \times \{0\}^{m+r-p-k} = (T_0 P)(\{\mathbf{0}\} \times \mathbf{R}^p \times \{0\}^{m-p})$$

$$= (T_y(Pf))(\hat{T}_y Y) + (T_0 P)(\{\mathbf{0}\} \times \mathbf{R}^q \times \{0\}^{m-q}) = (T_y(Pf))(\hat{T}_y Y) .$$

Hence $(T_y(Pf))(\hat{T}_y Y) = \hat{T}_0 \mathbf{R}^{m+r-q-k}_{(p-q)}$ and in particular

$$\text{(I)} \qquad \{\mathbf{0}\} \times \mathbf{R}^{r-q} \times \{0\}^{m-k} \subset (T_y(Pf))(T_y Y) .$$

i) b) implies that

$$(T_y f)(T_y Y) + \{\mathbf{0}\} \times \mathbf{R}^r \times \{0\}^{p-r} \times \mathbf{R}^{k-r} \times \{0\}^{m-p-k+r} = \{\mathbf{0}\} \times \mathbf{R}^m .$$

Using $T_0 P$ we get

$$\{\mathbf{0}\} \times \mathbf{R}^{m+r-q-k} = (T_0 P)(\{\mathbf{0}\} \times \mathbf{R}^m)$$

$$= (T_y(Pf))(T_y Y) + (T_0 P)(\{\mathbf{0}\} \times \mathbf{R}^r \times \{0\}^{p-r} \times \mathbf{R}^{k-r} \times \{0\}^{m-p-k+r})$$

$$= (T_y(Pf))(T_y Y) + \{\mathbf{0}\} \times \mathbf{R}^{r-q} \times \{0\}^{m-k} = (T_y(Pf))(T_y Y),$$

where the last equality follows from (I). Hence

$$(T_y(Pf))(T_y Y) = T_0 \mathbf{R}^{m+r-q-k}_{(p-q)} .$$

Thus all the assumptions of Theorem 1 are fulfilled by Pf, and therefore there is a permutation $\tilde{T}$: $\mathbf{R}^{m+r-q-k}_{(p-q)} \to \mathbf{R}^{m+r-q-k}_{(p-q)}$ of the coordinates $p-q+1,\ldots,m+r-q-k$, and a local C^s chart φ: $(Y,y) \to (\mathbf{R}^n_{(b)}, 0)$, such that $\operatorname{dom}(\varphi) \subset \operatorname{dom}(Pf)$ (which implies that $\operatorname{dom}(\varphi) \subset \operatorname{dom}(g)$ and $g(\operatorname{dom}(\varphi)) \subset \operatorname{dom}(\psi)$), and

$$\tilde{T} Pf \varphi^{-1}(y) = (y_1,\ldots,y_{p-q}; y_{n-m+p+k-r+1},\ldots,y_n) \quad \text{for } y \in \operatorname{im}(\varphi) .$$

Let T: $\mathbf{R}^m_{(p)} \to \mathbf{R}^m_{(p)}$ be the permutation of the coordinates $p+k-r+1,\ldots,m$ corresponding to $\tilde{T}$, so that $PT = \tilde{T}P$. Then T and φ are as claimed in ii).

ii) implies i): Choose g, A, ψ, T, and φ as in ii). Clearly $PT\psi g \varphi^{-1}$ preserves local facets at 0. Hence $T\psi g \varphi^{-1}$ preserves local facets relatively to $\mathbf{R}^m_{(p;\,q,r,k)}$ at 0, and therefore G preserves local facets relatively to a. Moreover,

$$(T_0 P)((T_0(T\psi g \varphi^{-1}))(T_0 \mathbf{R}^n_{(b)}) = T_0 \mathbf{R}^{m+r-q-k}_{(p-q)}$$

and

$$(T_0 P)(T_0(T\psi g \varphi^{-1}))(\hat{T}_0 \mathbf{R}^n_{(b)}) = \hat{T}_0 \mathbf{R}^{m+r-q-k}_{(p-q)} ,$$

so that

$$(T_0(T\psi g \varphi^{-1}))(T_0 \mathbf{R}^n_{(b)}) + T_0 \mathbf{R}^m_{(p;\,q,r,k)} = T_0 \mathbf{R}^m_{(p)}$$

and

$$(T_0(T\psi g \varphi^{-1}))(\hat{T}_0 \mathbf{R}^n_{(b)}) + \hat{T}_0 \mathbf{R}^m_{(p;\,q,r,k)} = \hat{T}_0 \mathbf{R}^m_{(p)} .$$

This implies i) b) and i) c).

ii) implies iii): Choose g, A, ψ, T, and φ as in ii). For $y \in \operatorname{im}(\varphi)$ we have

$$\begin{aligned}
& y \in \varphi(g^{-1}(A) \cap \operatorname{dom}(\varphi)) \\
& \text{iff } \psi g \varphi^{-1}(y) \in \mathbf{R}^m_{(p;\, q, r, k)} \\
& \text{iff } T\psi g \varphi^{-1}(y) \in \mathbf{R}^m_{(p;\, q, r, k)} \\
& \text{iff } PT\psi g \varphi^{-1}(y) \in [0, \infty[^{r-q} \times \{0\}^{p-r} \times \{0\}^{m-p-k+r} \\
& \text{iff } y \in [0, \infty[^{r-q} \times \{0\}^{p-r} \times \mathbf{R}^{b-p+q} \times [0, \infty[^{n-b-m+p+k-r} \times \{0\}^{m-p-k+r} .
\end{aligned}$$

Let $S\colon \mathbf{R}^n_{(b)} \to \mathbf{R}^n_{(b)}$ be a permutation of the coordinates $1, \ldots, b$, which places the coordinates $p-q+1, \ldots, b$ in the spaces $1, \ldots, b-p+q$, the coordinates $1, \ldots, r-q$ in the spaces $b-p+q+1, \ldots, b-p+r$, and the coordinates $r-q+1, \ldots, p-q$ in the spaces $b-p+r+1, \ldots, b$. Then for $y \in S(\operatorname{im}(\varphi))$ we have

$$\begin{aligned}
& y \in S\varphi(g^{-1}(A) \cap \operatorname{dom}(\varphi)) \\
& \text{iff } y \in \mathbf{R}^{b-p+q} \times [0, \infty[^{r-q} \times \{0\}^{p-r} \times [0, \infty[^{n-b-m+p+k-r} \times \{0\}^{m-p-k+r} \\
& \qquad = \mathbf{R}^n_{(b;\, b-p+q,\, b-p+r,\, n-m+k)} .
\end{aligned}$$

Hence $S\varphi$ is a C^s submanifold chart on

$$(Y, g^{-1}(A);\ \mathbf{R}^n_{(b)}, \mathbf{R}^n_{(b;\, b-p+q,\, b-p+r,\, n-m+k)}) ,$$

and this implies iii).

Let $f\colon Y \to X$ be a local C^1 map, and $A \subset X$ a C^1 submanifold. Then we say that *f intersects A transversally at y*, $y \in \operatorname{dom}(f)$, if either $f(y) \notin A$ or $f(y) \in A$ and $[f, y]$ is transversal to $[X, A, f(y)]$. If f intersects A transversally at y for all $y \in \operatorname{dom}(f)$, then *$f$ intersects A transversally*. We say that *f intersects A stratum transversally at y*, $y \in \operatorname{dom}(f)$, if either $f(y) \notin A$ or $f(y) \in A$ and $[f, y]$ is stratum transversal to $[X, A, f(y)]$. If f intersects A stratum transversally at y for all $y \in \operatorname{dom}(f)$, then *$f$ intersects A stratum transversally*.

THEOREM 3. *Let $A \subset X$ be a C^s submanifold, and $f\colon Y \to X$ a local C^s map, which preserves local facets relatively to A and intersets A transversally and stratum transversally. Then either $f^{-1}(A) = \varnothing$, or*

a) *$f^{-1}(A)$ is a C^s submanifold of Y, and*

b) $\dim Y - \dim f^{-1}(A) = \dim X - \dim A$, *and*

c) $\operatorname{ind}(Y, y) - \operatorname{ind}(f^{-1}(A), y) = \operatorname{ind}(X, f(y)) - \operatorname{ind}(A, f(y))$ *for all $y \in f^{-1}(A)$, and*

d) $\dim([Y, f^{-1}(A) \cap \partial^y Y, y]) - \operatorname{ind}(Y, y) = \dim([X, A \cap \partial^{f(y)} X, f(y)]) - \operatorname{ind}(X, f(y))$ *for all $y \in f^{-1}(A)$.*

PROOF. This theorem follows easily from Theorem 2.

Note that if Y, X, and A are manifolds without boundary, then Theorem 3 expresses the well-known theorem. The following examples show that it is not possible to cancel the assumptions in Theorem 3 that f intersects A transversally, that f intersects A stratum transversally. and that f preserves local facets relatively to A.

EXAMPLE 1. Put $X = \mathbf{R}^2$, $Y = \mathbf{R} \times]-\infty, 0]$, $A = \mathbf{R} \times [0, \infty[$, and define f: $Y \to X$ by

$$f(x,y) = (x, y + x^2) .$$

Here f preserves local facets relatively to A, and f intersects A transversally, but f does not intersect A stratum transversally at $(0,0)$, and $f^{-1}(A)$ is not a submanifold of Y.

EXAMPLE 2. Define a C^∞ map g: $\mathbf{R} \to \mathbf{R}$ by

$$g(x) = \exp(-1/x^2) \sin(1/x) \quad \text{if } x \neq 0, \quad \text{and} \quad g(0) = 0 .$$

Put $X = Y = \mathbf{R} \times [0, \infty[$, $A = \mathbf{R} \times \{0\}$, and define f: $Y \to X$ by

$$f(x,y) = (x, (g(y))^2) .$$

Clearly f preserves local facets relatively to A and intersects A stratum transversally, but f does not intersect A transversally at $(0,0)$. This explains the fact that $f^{-1}(A) = \mathbf{R} \times g^{-1}(\{0\})$ is not a submanifold of Y.

EXAMPLE 3. Let X, Y, A, and g be as in Example 2. Define f: $Y \to X$ by

$$f(x,y) = (x, (g(x))^2 + y) .$$

Clearly f intersects A transversally and stratum transversally, but f does not preserve local facets relatively to A, and $f^{-1}(A) = g^{-1}(\{0\}) \times \{0\}$ is not a submanifold of Y.

7. A special case: Boundary of codimension one.

M. Hirsch states an inverse image theorem for manifolds with boundary in [2, p. 31]. We are going to take a closer look at this theorem. First we give an example, which shows that Theorem 3 cannot be stated so nicely when attention is restricted to manifolds with boundary.

Example 4. Put $Y = \mathbf{R} \times [0, \infty[$, $X = \mathbf{R}^2$, $A = \{x \in \mathbf{R}^2 : \|x\| \leqq 1\}$, and let f: $Y \to X$ be the inclusion map. Here X, Y, and A are manifolds with boundary, f preserves local facets relatively to A, and f intersects A transversally and stratum transversally, but $f^{-1}(A)$ is not a manifold with boundary.

In the remainder of this paper X and Y are manifolds with boundary. Hirsch defines a *neat C^s submanifold* in such a way that $A \subset X$ is a neat C^s submanifold of X iff

1) A is a C^s submanifold of X,
2) A is a manifold with boundary,
3) $\operatorname{bd} A = A \cap \operatorname{bd} X$,
4) $T_x A \not\subset T_x(\operatorname{bd} X)$ for all $x \in \operatorname{bd} A$ (cf. [2, pp. 30–31]).

Lemma 1. *Let $A \subset X$ be a neat C^s submanifold, $x \in A$. Then*

a) $\operatorname{ind}(A, x) = \dim([X, A \cap \partial^x X, x])$
b) $(\hat{T}_x X) \cap (T_x A) = \hat{T}_x A$
c) $\operatorname{ind}(X, x) + \dim A = \operatorname{ind}(A, x) + \dim X$
d) *Every continuous local map f: $Y \to X$ preserves local facets relatively to A.*

Proof. We may assume that $X = \mathbf{R}^m_{(p)}$, $A = \mathbf{R}^m_{(p;\, q, r, k)}$, and $x = 0$. Since $\operatorname{bd} A = A \cap \operatorname{bd} X$, we have $q = r$, and this implies a) and b). c) is equivalent to $p + k - r = m$. If this were not true, then $A \subset \operatorname{bd} X$, contradicting 4) in the definition. Since $p + k - r = m$, the only local face of X at x containing $[X, A, x]$ is $[X, X, x]$. This proves d).

The following theorem is a restatement of Hirsch [2, Theorem 4.2 p. 31]. Recall that X and Y are manifolds with boundary.

Theorem. *Let $A \subset X$ be a C^s submanifold with boundary. Suppose that* a) *A is neat, or* b) $A \subset X \setminus \operatorname{bd} X$, *or* c) $A \subset \operatorname{bd} X$. *If f: $Y \to X$ is a C^s map such that both f and $f_{\mathrm{res}(\operatorname{bd} Y)}$ intersect A transversally, then $f^{-1}(A)$ is a C^s submanifold (with boundary) of Y, and* $\operatorname{bd} f^{-1}(A) = f^{-1}(\operatorname{bd} A)$.

We shall comment successively on the cases a), b), and c) of this Theorem.
a) It is not generally true that $\operatorname{bd} f^{-1}(A) = f^{-1}(\operatorname{bd} A)$. The proper conclusion is that $f^{-1}(A)$ is a C^s submanifold (with boundary) of Y (if $f^{-1}(A) \neq \emptyset$), and that $\operatorname{bd} f^{-1}(A) = f^{-1}(A) \cap \operatorname{bd} Y$.

Proof. Since f and $f_{\mathrm{res}\,(\operatorname{bd} Y)}$ intersect A transversally,

$$(T_y f)(\hat{T}_y Y) + T_{f(y)} X \quad \text{for all } y \in f^{-1}(A)\,.$$

Since $(T_y f)(\hat{T}_y Y) \subset \hat{T}_x X$, it follows that

$$\hat{T}_{f(y)} X = (T_y f)(\hat{T}_y Y) + (\hat{T}_{f(y)} X) \cap (T_{f(y)} A) ,$$

and by b) of Lemma 1 this is equal to $(T_y f)(\hat{T}_y Y) + \hat{T}_{f(y)} A$. Hence f intersects A stratum transversally, and Theorem 3 applies. Let $y \in f^{-1}(A)$. Formulae c) and d) of Theorem 3 and c) of Lemma 1 yield

$$\dim Y - \operatorname{ind}(Y, y) = \dim f^{-1}(A) - \operatorname{ind}(f^{-1}(A), y) .$$

Hence $f^{-1}(A)$ is a manifold with boundary, and $y \in \operatorname{bd} f^{-1}(A)$ iff $y \in (\operatorname{bd} Y) \cap f^{-1}(A)$.

COUNTEREXAMPLE 5. Put $X = Y = \mathbf{R} \times [0, \infty[$, $A = \{0\} \times [0, \infty[$, and define f: $Y \to X$ by $f(x, y) = (x, y+1)$. The assumptions of the a) version of the theorem are satisfied, but

$$\operatorname{bd} f^{-1}(A) = \{(0,0)\} \neq \varnothing = f^{-1}(\{(0,0)\}) = f^{-1}(\operatorname{bd} A) .$$

b) This is not true.

COUNTEREXAMPLE 6. Let X, Y, A, f be as in Example 1. The assumptions of the b) version of the theorem are satisfied, but $f^{-1}(A)$ is not a submanifold of Y.

c) The proper conclusion is that $f^{-1}(A) = \varnothing$.

PROOF. Suppose $y \in f^{-1}(A)$. Then

$$(T_y f)(T_y \partial^y Y) + T_{f(y)} A \subset T_{f(y)}(\operatorname{bd} X) \subsetneqq T_{f(y)} X .$$

If $y \in \operatorname{bd} Y$, this contradicts the assumption that $f_{\operatorname{res}(\operatorname{bd} Y)}$ intersects A transversally, and if $y \in Y \setminus \operatorname{bd} Y$ it contradicts the assumption that f intersects A transversally.

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