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ON ROTATION-AUTOMORPHIC FUNCTIONS

RAUNO AULASKARI and TUOMAS SORVALI

1.

In this paper we shall consider classes of analytic or meromorphic functions of a complex variable which are generalizations of classical automorphic functions, the main interest being in the normality of the functions.

Let (X,d) and (Y,ϱ) be metric spaces. Consider a family $\mathscr{F}$ of continuous functions $f: X \to Y$. The family $\mathscr{F}$ is *normal* if every sequence $(f_n) \subset \mathscr{F}$ contains a subsequence (f_k) which converges uniformly on every compact subset of X.

Let Ω be a group of isometric automorphisms of X.

DEFINITION 1. A function $f: X \to Y$ is *normal* (with respect to Ω) if $\{f \circ T \mid T \in \Omega\}$ is a normal family.

If f is a meromorphic function, then the normality of f is closely related to the following property of f.

DEFINITION 2. A function $f: X \to Y$ is of *bounded stretching* if there exists an $M > 0$ such that

$$\limsup_{x' \to x} \frac{\varrho(f(x), f(x'))}{d(x,x')} \leqq M \tag{1.1}$$

for all $x \in X$.

Let Ω' be the group of the isometric automorphisms of Y.

DEFINITION 3. A function $f: X \to Y$ is *homomorphism-automorphic* with respect to a subgroup $\Gamma \subset \Omega$ if for every $T \in \Gamma$ there exists a $j(T) \in \Omega'$ such that

$$f \circ T = j(T) \circ f\,. \tag{1.2}$$

REMARK 1. Let f be a homomorphism-automorphic function and let F be a fundamental set of Γ. Then f is of bounded stretching if and only if there exists an $M > 0$ such that (1.1) holds for all $x \in F$.

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2.

Let X be the unit disk D and let d be the non-euclidean metric of D. For the space (Y, ϱ) we choose the Rieman sphere $\hat{\mathbf{C}}$ with the spherical metric ϱ. Then, by (1.1), a meromorphic function $f\colon D \to \hat{\mathbf{C}}$ is of bounded stretching if and only if

$$\sup_{z \in D} (1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} \leqq M \tag{2.1}$$

for some $M > 0$.

Let Ω be the group of the Möbius transformations $T\colon D \to D$. Then we have ([3]):

THEOREM 1. *A meromorphic function $f\colon D \to \hat{\mathbf{C}}$ is normal in D if and only if* (2.1) *holds*, i.e., *f is of bounded stretching.*

3.

Suppose that Γ is a Fuchsian group acting on D, and let f be a non-constant meromorphic function in D. Suppose that f is homomorphism-automorphic with respect to Γ. Since f is meromorphic and T is conformal, it follows from (1.2) that $j(T)$ is sense-preserving, i.e., $j(T)$ is a rotation. On the other hand, if T is given, then (1.2) defines $j(T)$ uniquely. It follows that $\Gamma' = \{j(T) \mid T \in \Gamma\}$ is a group of rotations of the sphere and $j\colon \Gamma \to \Gamma'$ is a homomorphism. In this case we call f a *rotation-automorphic function.* The following special cases are well-known:

(1) If Γ' contains only rotations $w \mapsto e^{i\varphi}w$, then f is a character-automorphic function (cf. [4]).

(2) If Γ' is trivial, then f is an automorphic function.

Combining Remark 1 and Theorem 1 we obtain

THEOREM 2. *Let $f\colon D \to \hat{\mathbf{C}}$ be a rotation-automorphic function with respect to a Fuchsian group Γ, and let F be a fundamental set of Γ. Then f is normal if and only if*

$$\sup_{z \in F} (1-|z|^2)\frac{|f'(z)|}{1+|f(z)|^2} \leqq M \tag{3.1}$$

holds for some $M > 0$.

4.

Let (Y, ϱ) be the finite plane $\mathbf{C}$ with the euclidean metric, and let $f\colon D \to \mathbf{C}$ be a non-constant analytic function. Then f is of bounded stretching if and only if

$$\sup_{z \in D} (1-|z|^2)|f'(z)| \leqq M$$

for some $M>0$, i.e., if and only if f is a Bloch function.

A Bloch function f need not be normal in the sense of Definition 1. However, every sequence $f \circ T_n$, $T_n \in \Omega$, contains a subsequence which converges either to an analytic function or to ∞, uniformly on compact subsets of D. Hence a Bloch function is normal if the euclidean metric of $\mathbf{C}$ is replaced by the spherical metric.

Let f: $D \to \mathbf{C}$ be a homomorphism-automorphic function with respect to a Fuchsian group Γ. Then Γ' consists of translations $w \mapsto w+\omega$, and f is called an *additive automorphic function* (cf. [2]). Especially, if Γ' is trivial then f is automorphic.

Let F be a fundamental set of Γ. By Remark 1, an additive automorphic function f is a Bloch function if and only if

$$\sup_{z \in F} (1-|z|^2)|f'(z)| \leqq M$$

holds for some $M>0$.

REMARK 2. Let (X, d) and (Y, ϱ) be unit disks with the non-euclidean metric d and let f: $D \to D$ be an analytic homomorphism-automorphic function with respect to a Fuchsian group Γ. If Γ' is a Fuchsian group, then f induces an analytic function from D/Γ into D/Γ'. In this sense, analytic functions between Riemann surfaces are special cases of homomorphism-automorphic functions.

5.

In this section we consider the existence of homomorphism-, rotation-, and additive automorphic functions.

Let X and Y be Riemann surfaces and f: $X \to Y$ a non-constant analytic function. Suppose that the covering surface (X, f) of Y is universal, let Γ_0 be the cover transformation group and denote by H' the group of all conformal self-mappings of Y. A conformal mapping T: $X \to X$ is a lifting of $S \in H'$ if $f \circ T = S \circ f$. It follows that every $S \in H'$ has liftings. Especially, all cover transformations are obtained as liftings of the identity mapping of Y.

Let H be the set of all liftings of the mappings of H'. Then H is a group and $j: H \to H'$, $j(T)=S$, is a homomorphism whose kernel is the cover transformation group Γ_0. This can be stated differently also as follows:

THEOREM 3. *The mapping f: $X \to Y$ is automorphic with respect to Γ_0 and homomorphism-automorphic with respect to H. The group H contains as a subgroup every group Γ with respect to which f is homomorphism-automorphic.*

If Γ' is a subgroup of H', then we can similarly as above define Γ as the group of all liftings of the mappings of Γ'. Then $\Gamma \subset H$ and f is homomorphism-automorphic with respect to Γ. Suppose that Γ' is properly discontinuous in Y. Then the same holds true of Γ in X. In this case Γ can also be characterized as a cover transformation group (cf. [1, II.4C]):

THEOREM 4. *Γ is the cover transformation group of $(X, \varphi \circ f)$ over $Z = Y/\Gamma'$ where $\varphi: Y \to Z$ is the canonical projection.*

PROOF. Evidently Γ is a subgroup of the cover transformation group in question. From the construction of Z it follows that whenever x_1 and x_2 are two points of X lying over the same point of Z, there is an $S \in \Gamma$ for which $S(x_1) = x_2$. From this it follows that Γ contains all cover transformations of $(X, \varphi \circ f)$.

Note that $(X, \varphi \circ f)$ is a universal covering surface of Z only if Γ' is in addition also fixed point free in Y.

Suppose now that Y is a domain in $\hat{\mathbf{C}}$ having the unit disk D as a universal covering surface. If Γ' is a properly discontinuous group of conformal self-mappings of Y, then Γ is a Fuchsian group and the projection $f: D \to Y$ is homomorphism-automorphic with respect to Γ.

In order to construct a rotation-automorphic function $f: D \to \hat{\mathbf{C}}$, choose Y such that there exists a properly discontinuous group Γ' of rotations of the sphere mapping Y onto itself. This can be done e.g. by choosing for the group Γ' a group of a regular solid and letting Y be the sphere $\hat{\mathbf{C}}$ punctured at the fixed points of Γ'.

To obtain an additive automorphic function $f: D \to \mathbf{C}$, let Γ' be the group generated by the translations $z \to z + \omega_1$ and $z \to z + \omega_2$, $\operatorname{Im}(\omega_1/\omega_2) > 0$, and let Y be the plane $\mathbf{C}$ punctured at the points $m\omega_1 + n\omega_2$, m and n integers.

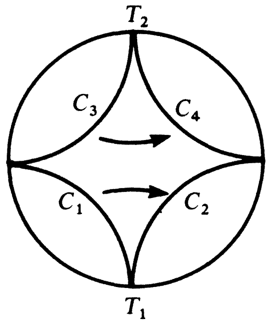
6.

Next we give an explicit example of a rotation-automorphic function such that Γ' is not fixed point free in Y.

Let Γ be the group generated by the parabolic transformations

$$T_1(z) = \frac{(1-i)z+1}{z+1+i}, \qquad T_2(z) = \frac{(1+i)z+1}{z+1-i}.$$

Then Γ has a fundamental polygon F whose sides lie on the circles



$$C_1 = \{z \mid |z-(-1-i)|=1\}$$
$$C_2 = \{z \mid |z-(1-i)|=1\}$$
$$C_3 = \{z \mid |z-(-1+i)|=1\}$$
$$C_4 = \{z \mid |z-(1+i)|=1\} .$$

Figure 1.

The transformation T_1 sends C_1 onto C_2 and T_2 sends C_3 onto C_4 (see Figure 1.).

Let S_2 be the rotation

$$S_2(w) = e^{2\pi i/3}w$$

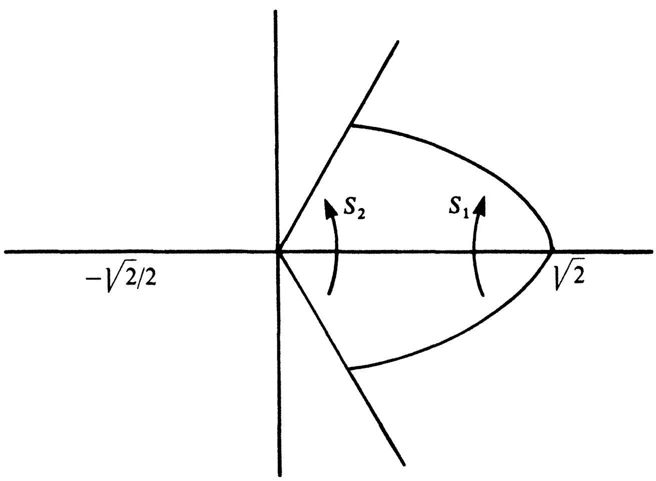


Figure 2.

and S_1 the rotation whose fixed points are $-\sqrt{2}/2$ and $\sqrt{2}$ and multiplier equals $e^{2\pi i/3}$. Let Γ' be the group generated by S_1 and S_2. Define a homomorphism j: $\Gamma \to \Gamma'$ by $j(T_i)=S_i$, $i=1,2$. Let F' be the domain bounded by the rays $w=te^{\pm\pi i/3}$, $t\geqq 0$, and by the circular arcs through the fixed points $\sqrt{2}$ and $-\sqrt{2}/2$ (Figure 2). Consider F and F' as quadrilaterals. By symmetry, both have the modulus one. Hence there is a conformal mapping f from F onto

F' which sends vertices onto vertices and equivalent sides onto equivalent sides, respectively. The function f can be continued to D by the formula (1.2). Then f is analytic in D. Moreover, f is rotation-automorphic with respect to Γ. Since Γ' is a subgroup of the cubic group, it is properly discontinuous in $\hat{\mathbf{C}}$. Moreover, the transformations of Γ' keep

$$Y = \hat{\mathbf{C}} \setminus \{0, \sqrt{2}, -\sqrt{2}/2, \sqrt{2}e^{2\pi i/3}, \sqrt{2}e^{-2\pi i/3}, (-\sqrt{2}/2)e^{2\pi i/3}, (-\sqrt{2}/2)e^{-2\pi i/3}, \infty\}$$

invariant. Since $S_1(S_2^{-1}(S_1(F'))) = F'$, the rotation $S_1 \circ S_2^{-1} \circ S_1 \neq id$ has a fixed point in F'. Hence Γ' is not fixed point free in Y.

Since f omits eight values, f is normal in D.

7.

Let f be a rotation-automorphic function with respect to Γ. We consider conditions under which f is normal in D.

Let F be a normal fundamental polygon of Γ.

DEFINITION 4. (cf. [2]) The fundamental domain F is *thick* if for each sequence $z_n \in F$, $n = 1, 2, \ldots$, there is a sequence z_n' and constants $r > 0$, $r' > 0$ such that $d(z_n, z_n') < r$ and

$$U(z_n', r') = \{z \mid d(z_n', z) < r'\} \subset F .$$

LEMMA. *Let $z_n \in F$ such that $|z_n| \to 1$ as $n \to \infty$. If $r > 0$ and $0 < R < 1$, denote $U_n = U(z_n, r)$ and $D_R = \{z \mid |z| < R\}$. Then $T(U_n) \cap D_R \neq \emptyset$ for finitely many $T \in \Gamma$ and $n \in \mathbf{N}$ only.*

PROOF. Choose $R' \in]R, 1[$ such that $U(z, r) \cap D_R = \emptyset$ if $|z| \geqq R'$. Then $T(\bar{F}) \cap D_{R'} \neq \emptyset$ for finitely many transformations $T \in \Gamma$ only. On the other hand, for every $T \in \Gamma$ the set $\{T(z_n) \mid n = 1, 2, \ldots\} \cap D_{R'}$ is finite. Combining these trivial observations we infer that $|T(z_n)| < R'$ for at most finitely many $T \in \Gamma$ and $n \in \mathbf{N}$.

THEOREM 5. *If F is thick and*

$$\iint_F \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z < \infty , \tag{7.1}$$

then f is normal in D.

PROOF. Suppose, on the contrary, that f is not normal in D. Then (3.1) does not hold. Thus there is a sequence $z_n \in F$ such that $|z_n| \to 1$ and

$$(1-|z_n|^2)\frac{|f'(z_n)|}{1+|f(z_n)|^2} \to \infty \tag{7.2}$$

as $n \to \infty$.

Choose $r>0$. By the above lemma, the thickness of F and (7.1), we have

$$\lim_{n\to\infty} \iint_{U(z_n,r)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z = 0\,. \tag{7.3}$$

We can select a subsequence, also denoted by z_n, such that

$$\sum_{n=1}^{\infty} \iint_{U(z_n,r)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z < \pi\,. \tag{7.4}$$

Let

$$g_n(\zeta) = f\left(\frac{\zeta+z_n}{1+\bar{z}_n\zeta}\right).$$

Then

$$\iint_{U(0,r)} \left(\frac{|g_n'(\zeta)|}{1+|g_n(\zeta)|^2}\right)^2 d\sigma_\zeta = \iint_{U(z_n,r)} \left(\frac{|f'(z)|}{1+|f(z)|^2}\right)^2 d\sigma_z \tag{7.5}$$

is equal to the spherical area of the surface onto which g_n maps the disk $U(0,r)$. It follows from (7.4) and (7.5) that the family $\{g_n\}$ omits in $U(0,r)$ a set of positive spherical area. Hence the family $\{g_n\}$ is normal in $U(0,r)$. Therefore

$$\frac{|g_n'(0)|}{1+|g_n(0)|^2} = (1-|z_n|^2)\frac{|f'(z_n)|}{1+|f(z_n)|^2} \leqq M < \infty$$

for $n=1,2,\ldots$. This contradicts (7.2).

REFERENCES

1. L. V. Ahlfors and L. Sario, *Riemann surfaces*, (Princeton Mathematical Series 26) Princeton University Press, Princeton, N.J., 1960.
2. R. Aulaskari, *On normal additive automorphic functions*, Ann. Acad. Sci. Fenn. Ser.A I Math. Dissertationes 23 (1978), 1–53.
3. O. Lehto and K. I. Virtanen, *Boundary behaviour and normal meromorphic functions*, Acta Math. 97 (1957), 47–65.
4. Ch. Pommerenke, *On normal and automorphic functions*, Michigan Math. J. 21 (1974), 193–202.

JOENSUUN KORKEAKOULU
PL 111
80101 JOENSUU 10
FINLAND