MATH. SCAND. 49 (1981), 229–249

CHARACTERIZATIONS OF POISSON INTEGRALS ON SYMMETRIC SPACES

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Introduction.

It is well known that for the hyperbolic unit disk $U=\{|z|<1\}$ the eigenfunctions u of the Laplacian are given by generalized Poisson integrals of hyperfunctions T on the boundary ([4, Ch. IV]). More generally, such an integral representation holds for the joint eigenfunctions of the invariant differential operators on a symmetric space X, see [6]. If X has rank 1, the kernels used here are powers of the ordinary Poisson kernel. In that case, T is a distribution if and only if u grows at most exponentially with the distance, see Lewis [J. Funct. Anal. 29 (1978), 287–307]. For U, the functional T is given by an L^p function on the boundary, $1<p<\infty$, if and only if the L^p norm of u/φ on $|z|=r$ is bounded as $r \to 1$, as follows by standard arguments, cf. Remark 2 in Section 4. Here φ is the circular mean value of u (assuming $u(0) \neq 0$), i.e., the generalized Poisson integral of a constant function on the boundary. The present paper deals with another way of characterizing those u for which T is an L^p function. These characterizations use weak L^p spaces and work for symmetric spaces of arbitrary rank. They extend the author's work in [12] on ordinary Poisson integrals in $\mathbf{R}^n$.

In U, the results read as follows. Let dm_s be the measure $(1-|z|^2)^{-1-s}dxdy$, so that $s=1$ gives the invariant measure and $s=-1$ Lebesgue measure. Then T is an L^p function if and only if $(1-|z|^2)^{s/p}u/\varphi$ is in weak $L^p(m_s)$. When $p>1$, this holds for all $s \neq 0$, when $p=1$ only for $s \notin [0,1]$.

For $\operatorname{rank} X = r > 1$, it turns out that $r-1$ logarithmic factors must be introduced in the weak L^p condition, which can be done in several ways. The case (called $\lambda=0$) corresponding to the square root of the ordinary Poisson kernel must be treated differently, although the results are essentially the same.

Lohoué and Rychener [9] have proved a special case of our results and applied it to convolution operators on L^p in a Lie group. Most of our techniques are suitable generalizations of those of [12]. See also [13], where more general kernels are considered.

The preparatory Section 2 contains, among other things, some known facts

Received June 9, 1980.

about the behaviour of joint eigenfunctions, and there the measures and weak L^p spaces we use are defined. In Section 3, we prove two auxiliary technical results in the setting of a "half-space" over a nilpotent Lie group. The idea of the proof of Theorem 3.1 is taken from that of Theorem 1 in [12].

The main results are given in Section 4 ($\lambda \neq 0$) and Section 5 ($\lambda = 0$). For $\lambda = 0$, we also give in Section 5 a method of recovering f from its Poisson integral, which replaces the ordinary convergence result at the boundary. Finally, those results which hold only for $p > 1$ are studied in the last section, which also contains two counterexamples.

2. Preliminaries.

Let $X = G/K$ be a Riemannian symmetric space of noncompact type. Here G is a connected semisimple Lie group with finite center, and K a maximal compact subgroup of G. In fact, all our results are valid even when X is reducible, but for simplicity we treat only the irreducible case. Denoting by $\mathfrak{g}$ and $\mathfrak{k}$ the Lie algebras of G and K, we choose a Cartan decomposition $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ and let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$. Then $r = \dim \mathfrak{a}$ is the rank of X. This gives a root space decomposition of $\mathfrak{g}$. Choosing as usual a positive Weyl chamber $\mathfrak{a}_+$ and the associated ordering of the roots, we call $\mathfrak{n}$ ($\bar{\mathfrak{n}}$) the sum of the root spaces corresponding to positive (negative) roots. Letting A, N, and $\bar{N}$ be the subgroups of G having Lie algebras $\mathfrak{a}$, $\mathfrak{n}$, and $\bar{\mathfrak{n}}$, respectively, we have Iwasawa decompositions $\mathfrak{g} = \mathfrak{k} + \mathfrak{a} + \mathfrak{n}$ and $G = KAN$. Thus, any $g \in G$ can be written uniquely as $g = k(g) \exp H(g) n(g)$ with $k(g) \in K$, $H(g) \in \mathfrak{a}$, and $n(g) \in N$. We let e be the unit element of G, K, or N, and set $o = eK \in X$.

Denoting $\bar{A}_+ = \exp \bar{\mathfrak{a}}_+$, where $\bar{\mathfrak{a}}_+$ is the closure of $\mathfrak{a}_+$ in $\mathfrak{a}$, we have the Cartan decomposition $G = K\bar{A}_+K$. This means that for any $g \in G$ or $x \in X$ there is a unique element H in $\bar{\mathfrak{a}}_+$ such that g or x belongs to $K(\exp H)K$. We then set $H = H'(g)$ or $H = H'(x)$, respectively. A function $F = F(H)$ defined on $\bar{\mathfrak{a}}_+$ will be considered also as a function on G and on X, by means of $F(g) = F(H'(g))$ and $F(x) = F(H'(x))$.

Let $\mathfrak{a}^*$ ($\mathfrak{a}^*_{\mathbf{C}}$) be the real (complex) dual of $\mathfrak{a}$. Then

$$_+\mathfrak{a}^* = \{\lambda \in \mathfrak{a}^* : \lambda(H) > 0 \quad \text{for } H \in \mathfrak{a}_+\}$$

is the open cone generated by the positive roots, by the bipolar theorem. The Killing form makes $\mathfrak{a}$ into an inner product space, so that there is a canonical map $\mathfrak{a} \to \mathfrak{a}^*$, and we denote by $\mathfrak{a}^*_+$ the image of $\mathfrak{a}_+$ under this map. Then $\mathfrak{a}^*_+ \subset {}_+\mathfrak{a}^*$, as proved in Harish–Chandra [1, Lemma 35, p. 279]. Let 2ϱ be the sum of the positive roots, so that $\varrho \in \mathfrak{a}^*_+$ (see [1, p. 281]). We denote by S the "slice" $\{H_0 \in \mathfrak{a}_+ : 2\varrho(H_0) = 1\}$, and often write any $H \in \mathfrak{a}_+$ as $H = tH_0$, with $t > 0$ and $H_0 \in S$, and put $t = |H|$. Thus, $|\cdot| = |H'(x)|$ is also a function on X. A restricted domain in X is defined to be one of type $\{x \in X : H'(x) \in \mathbf{R}_+S'\}$, where S' is a nonempty, open, and relatively compact subset of S and $\mathbf{R}_+S'$ the open cone it generates in $\mathfrak{a}$. In such a domain, we see that $H'(x)$ stays far from the boundary of the positive Weyl chamber, except for x near o. We call $\mathbf{R}_+S'$ a restricted cone.

The (maximal) boundary of X is by definition the quotient K/M, where M is the centralizer of A in K. This boundary has a unique normalized K-invariant measure dkM. If $\lambda \in \mathfrak{a}_{\mathbf{C}}^*$, the associated Poisson kernel is

$$P_\lambda(g,k) = e^{-(i\lambda+\varrho)(H(g^{-1}k))}, \quad \text{for } g \in G,\ k \in K\,.$$

Since this expression is right K-invariant in g and right M-invariant in k, we may also consider P_λ as a function on $X \times K/M$ and write $P_\lambda(x, kM)$ for $x \in X$. The λ-Poisson integral of a Borel measure μ in K/M is defined by

$$P_\lambda\mu(x) = \int P_\lambda(x,kM)\,d\mu(kM)\,.$$

For integrable functions f on the boundary, $P_\lambda f$ means $P_\lambda(f\,dkM)$. Then the (spherical) function $\varphi_\lambda = P_\lambda 1$ is a left K-invariant function on X. As is well known, any $P_\lambda\mu$ is an eigenfunction for all K-invariant differential operators on X, and the eigenvalues depend only on λ. Let $\mathscr{E}_\lambda$ be the space of all eigenfunctions for these operators with the same eigenvalues as φ_λ. Whenever convenient, we consider the functions in $\mathscr{E}_\lambda$ as defined on G rather than on X. These functions are smooth since some invariant operators are elliptic.

In this paper, C will denote many different constants, and we will generally not indicate precisely which parameters C depends on at each occurrence. The relation $f \sim g$ means $C^{-1} \leqq f/g \leqq C$. The following lemma describes the asymptotic behavior of φ_λ.

LEMMA 2.1. *One has* $\varphi_\lambda(\exp H) \sim e^{(i\lambda-\varrho)(H)}$ *if* $i\lambda \in \mathfrak{a}_+^*$, *uniformly for* $H \in \mathfrak{a}_+$. *This can be written simply* $\varphi_\lambda \sim e^{i\lambda-\varrho}$, *with our conventions.*

PROOF. With $h = \exp H$, we set $\bar n^h = h\bar n h^{-1}$ and ${}^h\bar n = h^{-1}\bar n h$. Denoting by $d\bar n$ a suitable Haar measure on $\bar N$, we may transform the Poisson integral to an integral over $\bar N$ by means of $\bar n \to k(\bar n)M$, getting

$$\begin{aligned}
\varphi_\lambda(h) &= \int_{\bar N} e^{-(i\lambda+\varrho)(H(h^{-1}k(\bar n)))-2\varrho(H(\bar n))}\,d\bar n \\
&= e^{(i\lambda-\varrho)(H)} \int_{\bar N} e^{-(i\lambda+\varrho)(H({}^h\bar n))+2\varrho(H)+(i\lambda-\varrho)(H(\bar n))}\,d\bar n \\
&= e^{(i\lambda-\varrho)(H)} \int_{\bar N} e^{-(i\lambda+\varrho)(H(\bar n))+(i\lambda-\varrho)(H(\bar n^h))}\,d\bar n
\end{aligned}$$

(see e.g. Helgason [4, pp. 129–130]; distinguish between $H(.)$ and H). The last step here was the transformation $\bar{n} \to \bar{n}^h$ which has Jacobian $e^{-2\varrho(H)}$. If a canonical coordinate system is used in $\bar{N}$, the conjugation $\bar{n} \to \bar{n}^h$ has the effect of decreasing all coordinates when $h \in \bar{A}_+$, so $\bar{n}^h$ stays in a compact set as $\bar{n}$ varies in a compact set $L \subset \bar{N}$ and $h \in \bar{A}_+$. Therefore, the integrand in the last integral is ~ 1 in L. From this we get $\varphi_\lambda(\exp H) \geqq e^{(i\lambda-\varrho)(H)}/C$.

For the converse inequality, we estimate the same integrand from above by $e^{-(\varrho+\delta i\lambda)H(\bar{n})}$, where $\delta > 0$ is small, as in Helgason [4, p. 130] or Michelson [10, p. 262], and this expression is integrable and independent of H. The lemma is proved.

A measure μ in $\bar{N}$ may also be considered as a measure in K/M by means of the transformation $\bar{n} \to k(\bar{n})M$. From the proof just given, we then see that

$$(2.1) \qquad P_\lambda\mu(\bar{m}h) = e^{(i\lambda-\varrho)(H)} \int e^{-(i\lambda+\varrho)(H({}^h(\bar{m}^{-1}\bar{n}))) + 2\varrho(H) + (i\lambda+\varrho)(H(\bar{n}))}\, d\mu(\bar{n}) ,$$

for $\bar{m} \in N$, $h \in A$, which will be used later.

For $\lambda = 0$, we put

$$\varphi_0(\exp H) = e^{-\varrho(H)}\psi(H), \qquad H \in \bar{\mathfrak{a}}_+ .$$

Harish–Chandra [1, p. 279] has proved that there is a natural number q such that $\psi(H)/(1+|H|)^q$ is bounded on $\bar{\mathfrak{a}}_+$ and $\psi(tH_0) \sim t^q$ as $t \to \infty$ for each $H_0 \in S$, but this last relation is not uniform in H_0 when $r > 1$. If $r = 1$, then $q = 1$.

LEMMA 2.2. *Given $i\lambda \in \overline{\mathfrak{a}_+^*}$ and a compact set $L \subset X$, there is a constant $C = C(L, \lambda)$ such that any nonnegative $u \in \mathscr{E}_\lambda$ satisfies*

$$u(gx) \leqq Cu(gy)$$

for all $x, y \in L$ and any $g \in G$.

This lemma is a form of Harnack's inequality. Except for some cases, it is a consequence of Lemma 2.1 in Michelson [10], but we indicate another proof: For $g = e$, the lemma follows by well-known elliptic operator techniques (cf. Serrin [11], or the fact that $\mathscr{E}_\lambda$ defines a sheaf satisfying Brelot's axiomatic potential theory). The general case is then immediate from the translation invariance of $\mathscr{E}_\lambda$.

We shall work with several positive measures on X. The invariant measure m is given by

$$\int_X \varphi \, dm = \int_{K/M \times \mathfrak{a}_+} \varphi(k \exp H) \prod_\alpha \sinh(\alpha(H))\, dkM\, dH ,$$

where the product is taken over the positive roots, counted according to multiplicity. We write $\varphi(k\exp H)$ rather than $\varphi(k(\exp H)K)$, and dH is a Euclidean measure in $\mathfrak{a}$.

Writing $H = tH_0$, we have $dH = t^{r-1}dtdH_0$ in $\mathfrak{a}_+$ for a fixed measure dH_0 in S which is an $(r-1)$-dimensional Euclidean measure on S. As $H \to \infty$, which means that $\alpha(H) \to \infty$ for all positive roots, $dm \sim e^{2\varrho}dkMdH$. Setting $dm_\sigma = e^{\sigma - 2\varrho}dm$ for $\sigma \in \mathfrak{a}^*$, we obtain a family of measures, and $dm_\sigma \sim e^{\sigma}dkMdH$ as $H \to \infty$. Notice that Lebesgue measure in the unit disk is included here, up to the $\sim$ relation, and that m_σ is finite if and only if $\sigma \in -{}_+\mathfrak{a}^*$.

Next, we define weak L^p spaces. If μ is any positive measure on X and f a μ-measurable real- or complex-valued function, the distribution function of f is

$$\lambda_f(\alpha) = \mu\{x \in X : |f(x)| > \alpha\}, \qquad \alpha > 0 .$$

The decreasing rearrangement of f is

$$f^*(t) = \inf\{\alpha : \lambda_f(\alpha) \leqq t\}, \qquad 0 < t < \mu(X) .$$

Notice the simple inequality

$$\int_E |f|\, d\mu \leqq \int_0^{\mu(E)} f^*(t)\, dt \tag{2.2}$$

valid for any μ-measurable set $E \subset X$. Weak L^p, denoted Λ_p, consists of those f for which

$$f^*(t) \leqq Ct^{-1/p}, \qquad 0 < t < \mu(X) ,$$

and the smallest possible C here is the quasi-norm of f in weak L^p. Setting

$$\log_*^b t = (1 + |\log t|)^b, \qquad \text{all } t > 0 \text{ and } b \geqq 0 ,$$

we define Λ_p^s by the inequality $f^*(t) \leqq Ct^{-1/p}\log_*^{s/p} t$, and call $\inf C$ the quasinorm as before. Notice that $\Lambda_p^0 = \Lambda_p$, that Λ_p^s may also be defined by

$$\lambda_f(\alpha) \leqq C\alpha^{-p}\log_*^s \alpha \tag{2.3}$$

with another C, and that

$$f \in \Lambda_p^s \Leftrightarrow |f|^p \in \Lambda_1^s . \tag{2.4}$$

When $\mu = m_\sigma$, we denote by $\Lambda_{p,\sigma}$ and $\Lambda_{p,\sigma}^s$ the spaces obtained. Finally, $\Lambda_{p,\sigma}^*$ is weak L^p with respect to the measure $(1 + |H|)^{1-r}dm_\sigma$; observe that this measure is slightly smaller than m_σ and behaves like $e^{\sigma}dkMdtdH_0$ as $H = tH_0 \to \infty$.

3. Auxiliary theorems.

In this section, H_0 will be in S and we set $h_t = \exp tH_0$, $t \in \mathbf{R}$. For $\bar{n} \in \bar{N}$, we write $\bar{n}_t = h_t\bar{n}h_{-t}$. In the space $\bar{N} \times \mathbf{R}$, let dm'_σ denote the measure $e^{t\sigma(H_0)}d\bar{n}dt$

($\bar{n} \in \bar{N}$, $t \in \mathbf{R}$). Here $\sigma \in \mathfrak{a}^*$ as before. Because of (2.1), the integral formed in the following theorem is closely related to $P_\lambda \mu(mh_t)$.

THEOREM 3.1. *Let* $i\lambda \in \mathfrak{a}_+^*$, *and* $\sigma \in (2\varrho + {}_+\mathfrak{a}^*) \cup (-{}_+\mathfrak{a}^*)$. *With* $H_0 \in S$ *and* μ *a probability measure in* $\bar{N}$, *define a function* v *in* $\bar{N} \times \mathbf{R}$ *by*

$$v(\bar{m}, t) = e^{-t\sigma(H_0)} \int e^{-(i\lambda+\varrho)H((\bar{m}^{-1}\bar{n})_{-t})+t} \, d\mu(\bar{n}) .$$

Then v *is in weak* L^1 *with respect to* m'_σ *in* $\bar{N} \times \mathbf{R}$, *and the corresponding quasinorm is bounded uniformly for* $H_0 \in S$.

PROOF. The product $\bar{N} \times \mathbf{R}$ should be seen as a "half-space" $\{(\bar{m}, t') : \bar{m} \in \bar{N}, t' > 0\}$ over $\bar{N}$, but we use $t = -\log t'$ instead of t' as a coordinate. Call $\bar{N} \times [j, j+1] \subset \bar{N} \times \mathbf{R}$ the j-layer, for any integer j.

Let B be a compact neighborhood of $e \in \bar{N}$ which is symmetric ($B^{-1} = B$), of Haar measure 1, and such that $hBh^{-1} \subset B$ for all $h \in \bar{A}_+$. Of course, B_t means $h_t B h_{-t}$, and the Haar measure of B_t equals the Jacobian of the map $\bar{n} \to \bar{n}_t$, which is $e^{-2\varrho(tH_0)} = e^{-t}$. To obtain the claimed uniformity in H_0, we fix an element $H_1 \in S$, putting $h'_y = \exp yH_1$ and $B_{t,y} = h'_{-y} B_t h'_y$. Notice that $B_{t,y}$ is decreasing in t and increasing in y. Take $\beta > 0$ so that $BB \subset B_{0,\beta}$.

The sets B_t will serve as building-blocks to discretize the problem. For each integer j, we choose a maximal set $\{\bar{n}_l B_j\}_l$ of pairwise disjoint translates of B_j in $\bar{N}$, and each $\bar{n}_l B_j$ is called a j-base. Thus, for any $\bar{m} \in \bar{N}$, the translate $\bar{m}B_j$ must intersect some j-base. The sets $\bar{n}_l B_j \times [j, j+1]$ are called j-pieces, and they are disjoint and contained in the j-layer.

In the "half-space", the j-layer should be thought of as situated at height $\sim e^{-j}$ and having width $\sim e^{-j}$. And the j-pieces essentially correspond to a subdivision of the j-layer into cubes of side e^{-j}. (The j-pieces do not cover the j-layer, but this is unimportant.)

We shall need three observations. First, Lemma 2.2 gives a property of v. For if $\bar{p} \in B_t$ and $|\tau - t| \leqq 1$, then $\bar{m}\bar{p}h_\tau = \bar{m}h_t \bar{p}_{-t} h_{\tau-t}$, and the last two factors here belong to a compact set. Applying Lemma 2.2 to the Poisson kernel, we conclude that for any $k \in K$ and any $\bar{m} \in \bar{N}$,

$$e^{-(i\lambda+\varrho)(H((\bar{m}h_t)^{-1}k))}$$

does not change more than by a factor C if $\bar{m}$ is replaced by $\bar{m}\bar{p}$ and t by τ. Since $(\bar{m}^{-1}\bar{n})_{-t} = (\bar{n}^{-1}\bar{m}h_t)^{-1}h_t$, this implies

$$v(\bar{m}\bar{p}, \tau) \sim v(\bar{m}, t) \quad \text{for } \bar{p} \in B_t, \ |t - \tau| \leqq 1 . \tag{3.1}$$

Next, we see that

$$\int e^{-(i\lambda+\varrho)(H(\bar{m}_{-t}))+t}\,d\bar{m} \quad \text{is independent of } t\,, \tag{3.2}$$

by transforming $\bar{m} \to \bar{m}_t$. This integral is known to be finite ([1, Lemma 45, p. 289]). Finally

$$\int_{\bar{N}\setminus B_{0,y}} e^{-(i\lambda+\varrho)(H(\bar{m}))}\,d\bar{m} = O(e^{-ay}), \qquad y \to \infty\,, \tag{3.3}$$

for some $a>0$. This follows from the facts that

$$\int e^{-(1-\eta)(i\lambda+\varrho)(H(\bar{m}))}\,d\bar{m} < \infty$$

and $e^{-(i\lambda+\varrho)H(\bar{m})} = O(|\bar{m}|^{-a'})$ for some $\eta, a'>0$ and some norm on $\bar{N}$ (Knapp–Williamson [7, Proposition 5.5]).

Now let $s=\sigma(H_0)$ so that $s \notin [0,1]$ and s is bounded away from $[0,1]$, uniformly as $H_0 \in S$. Consider first the case $s>1$. We have $v \leqq e^{-(s-1)t}$ everywhere since $H(\bar{m}) \in {}_{+}\bar{\mathfrak{a}}$ for any $\bar{m} \in \bar{N}$ (see [1, Lemma 43, p. 287]). Take $\alpha > 0$. Since $v \to 0$ as $t \to \infty$, we may let j_0 be the largest integer for which the set $\{(\bar{m},t): v(\bar{m},t)>\alpha\}$ intersects the j_0-layer.

By induction in decreasing j, we shall now construct for $j=j_0, j_0-1, \ldots$ measures ν_j in $\bar{N}\times \mathsf{R}$, and $\operatorname{supp}\nu_j$ will be a set of j-pieces. Each time we decide to place ν_k-mass in a certain k-piece, we simultaneously forbid placing mass near this k-piece in the sequel, i.e. for $j<k$. This is done by introducing "above" the k-piece a forbidden region which becomes wider as we move upwards (j decreases). At each step in the construction, mass is placed in a piece if and only if this piece intersects $\{v>\alpha\}$ and is not already in a forbidden region.

When we now carry out this in detail, F_j, $j=j_0, j_0-1, \ldots$, will be sets of (forbidden) pieces. For some $k \leqq j_0$, assume ν_j and F_j defined for $j_0 \geqq j > k$. Then we let ν_k be the restriction of the measure m'_σ to the union of all those k-pieces which intersect $\{(\bar{m},t): v(\bar{m},t)>\alpha\}$ and do not belong to $\bigcup_{j_0\geqq j>k} F_j$, i.e. which are not already forbidden. (In case $k=j_0$, this union is empty.) Set $P_k = \pi(\operatorname{supp}\nu_k)$, where $\pi: \bar{N}\times\mathsf{R} \to \bar{N}$ is the projection. Then P_k is a union of k-bases. Now F_k is defined as the set of those j-pieces, all $j<k$, whose projections intersect the set $P_k B_{j,\varkappa(k-j)+\beta}$, where $\varkappa$ is a fixed number satisfying $0<\varkappa<s-1$. This defines ν_j, $j \leqq j_0$.

We claim that $\nu = \sum_{-\infty}^{j_0} \nu_j$ satisfies

(i) $m'_\sigma\{v>C\alpha\} \leqq C\|\nu\|$
(ii) $v>\alpha/C$ in $\operatorname{supp}\nu$
(iii) $U^\nu \leqq C$ in $\bar{N}$,

where

$$U^{\nu}(\bar{n}) = \int e^{-st-(i\lambda+\varrho)(H((\bar{m}^{-1}\bar{n})_{-t}))+t}\,d\nu(\bar{m},t)\,.$$

These three inequalities imply Theorem 3.1, since

$$(3.4)\quad m'_{\sigma}\{v>C\alpha\} \leqq C\|\nu\| \leqq C\alpha^{-1}\int v\,d\nu = C\alpha^{-1}\int U^{\nu}\,d\mu \leqq C\alpha^{-1}\,,$$

by Fubini's theorem; cf. [12, p. 183].

To prove (i), we first observe that the m'_{σ} measure of a k-piece is $Ce^{(s-1)k}$. Take a point $(\bar{m},t)$ with $v(\bar{m},t)>C\alpha$, and let $k=[t]$. Then $\bar{m}B_k\times[k,k+1]$ intersects some k-piece $\bar{n}_lB_k\times[k,k+1]$, and because of (3.1), one has $v>\alpha$ in the intersection, if C is suitably chosen. It also follows that $\bar{m}\in\bar{n}_lB_kB_k^{-1}$ $=\bar{n}_lB_kB_k\subset\bar{n}_lB_{k,\beta}$. But the Haar measure of $\bar{n}_lB_{k,\beta}$ is at most C times that of $\bar{n}_lB_k$, and it follows that $m'_{\sigma}\{v>C\alpha\}$ is bounded by C times the total m'_{σ} measure of those pieces which intersect $\{v>\alpha\}$. And such a piece is either in $\operatorname{supp}\nu$ or in $\bigcup F_k$ by construction, and $m'_{\sigma}(\operatorname{supp}\nu)=\|\nu\|$. Thus, it remains to estimate the total measure of the pieces in $\bigcup F_k$ by $C\|\nu\|$. To this end, notice that a j-piece in F_k, $j<k$, must intersect $\pi(Q)B_{j,\varkappa(k-j)+\beta}\times[j,j+1]$, for some k-piece Q in $\operatorname{supp}\nu_k$. Therefore, this j-piece is contained in $\pi(Q)B_{j,\varkappa(k-j)+C}\times[j,j+1]$ for some C. The Haar measure of $\pi(Q)B_{j,\varkappa(k-j)+C}$ is $O(e^{-j+\varkappa(k-j)})$. Hence, the total m'_{σ} measure of all pieces in F_k associated with Q in this way is at most

$$C\sum_{j<k}e^{sj-j+\varkappa(k-j)} \leqq Ce^{(s-1)k}\sum_{j<k}e^{-(s-1-\varkappa)(k-j)} \leqq Ce^{(s-1)k} = Cm'_{\sigma}(Q)\,.$$

Summing over all the pieces Q in $\operatorname{supp}\nu_k$ and then over k, we see that the total measure of the pieces in $\bigcup F_k$ is at most

$$Cm'_{\sigma}(\operatorname{supp}\nu) = C\|\nu\|\,.$$

Thus, (i) is proved.

Inequality (ii) is an immediate consequence of (3.1).

To prove (iii), we need two lemmas. The first one expresses that if the projection P_j of $\operatorname{supp}\nu_j$ is far from $\bar{n}$, then U^{ν_j} is small at $\bar{n}$.

LEMMA 3.2. *Let $b>0$. If $\bar{n}\in\bar{N}$ and $P_j\cap\bar{n}B_{j,\varkappa b}=\emptyset$, then $U^{\nu_j}(\bar{n})\leqq C_0e^{-\varepsilon b}$, where $\varepsilon>0$ and C_0 are constants.*

PROOF. In view of the reasoning leading to (3.1),

$$U^{\nu_j}(\bar{n}) \leqq C\int_{P_j} e^{-(i\lambda+\varrho)(H((\bar{m}^{-1}\bar{n})_{-j}))+j}\,d\bar{m}\,.$$

By assumption, $P_j^{-1}\bar{n} \subset \bar{N} \setminus B_{j,\varkappa b}$, so a transformation $\bar{m} \to \bar{n}\bar{m}_j^{-1}$ takes us to the integral in (3.3). The lemma follows.

By making C_0 larger if necessary, we may assume

$$U^{\nu_j}(\bar{n}) \leqq C_0 \tag{3.5}$$

for all j and all $\bar{n}$, because of (3.2). Inequality (iii) is a consequence of the following lemma, where ε and C_0 are as just described.

LEMMA 3.3. *For any $\bar{n} \in \bar{N}$ and any $j \leqq j_0$, it is possible to rearrange the sum $\sum_{k=j}^{j_0} U^{\nu_k}(\bar{n})$ so that it becomes dominated term by term by $\sum_{k=0}^{j_0-j} C_0 e^{-\varepsilon k}$.*

PROOF. The case $j = j_0$ is clear from (3.5), so assume the lemma holds for $j+1$. Let m be the nonnegative integer satisfying

$$C_0 e^{-\varepsilon(m+1)} < U^{\nu_j}(\bar{n}) \leqq C_0 e^{-\varepsilon m}. \tag{3.6}$$

Lemma 3.2 then implies that $P_j \cap \bar{n}B_{j,\varkappa(m+1)} \neq \varnothing$ so that $\bar{n} \in P_j B_{j,\varkappa(m+1)}$. If $k \geqq j + m + 1$ we have

$$\begin{aligned}\bar{n}B_{k,\varkappa(k-j)} &\subset P_j B_{j,\varkappa(m+1)} B_{k,\varkappa(k-j)} \subset P_j B_{j,\varkappa(k-j)} B_{j,\varkappa(k-j)} \\ &\subset P_j B_{j,\varkappa(k-j)+\beta} \subset \bar{N} \setminus P_k,\end{aligned}$$

the last inclusion by the construction of F_k. So for $j_0 \geqq k \geqq j+m+1$, Lemma 3.2 implies $U^{\nu_k}(\bar{n}) \leqq C_0 e^{-\varepsilon(k-j)}$. By our induction assumption, the terms $U^{\nu_k}(\bar{n})$, $j+m+1 > k > j$, are in some order dominated by $C_0 e^{-\varepsilon k}$, $0 \leqq k \leqq m-1$. These two estimates together with the right-hand inequality of (3.6) end the induction step. Lemma 3.3, (iii) and Theorem 3.1 (case $s > 1$) are proved. The claimed uniformity in H_0 follows since none of the constants used depend on H_0.

When $s < 0$ in Theorem 3.1, we need only modify a few details. Then j_0 is the smallest integer j for which the j-layer intersects $\{v > \alpha\}$, and the construction is carried out from smaller to greater j-values. The set F_k consists of those j-pieces, $j > k$, whose projections intersect $P_k B_{k,\varkappa(j-k)+\beta}$. Here $0 < \varkappa < -s$. We leave the rest to the reader.

This ends the proof of Theorem 3.1.

For $\lambda = 0$ we replace Theorem 3.1 by a weaker local result. Let $\mathbf{R}_+ = \{t \in \mathbf{R}: t > 0\}$.

THEOREM 3.4. *Fix a compact set $L \subset \bar{N}$, and let μ be a probability measure carried by L. Let σ, H_0, and v be as in Theorem 3.1 but set $\lambda = 0$. Then*

$$m'_\sigma\{(\bar{m}, t) \in L \times \mathbf{R}_+ : v(\bar{m}, t) > \alpha\} \leqq C\alpha^{-1}\psi(C(\log_* \alpha)H_0)$$

for $\alpha > 0$, where C depends on L but not on H_0.

PROOF. Again let $s=\sigma(H_0)$ and consider first the case $s>1$. In this proof, we use a measure ν as in the proof of Theorem 3.1, but the construction of ν is much easier this time. In fact, ν is carried by the "lower" boundary of the set $\{v>\alpha\}$ and has an area density there.

Set

$$S(\bar{m}) = \sup\{t : v(\bar{m},t)>\alpha\}$$

when $\bar{m} \in L'$ and L' is the set of $\bar{m} \in L$ for which $S(\bar{m})>0$. Let ν be the measure in $L \times \mathbf{R}_+$ defined by

$$\int \varphi(\bar{m},t)\, d\nu(\bar{m},t) = \int_{L'} e^{sS(\bar{m})}\varphi(\bar{m},S(\bar{m}))\, d\bar{m} .$$

Then clearly,

(ii′) $$v = \alpha \quad \text{in } \operatorname{supp}\nu ,$$

and moreover,

(i′) $$m'_\sigma(L \times \mathbf{R}_+ \cap \{v>\alpha\}) \leqq \int_{L'} d\bar{m} \int_0^{S(\bar{m})} e^{st}\, dt$$
$$\leqq s^{-1} \int_{L'} e^{sS(\bar{m})}\, d\bar{m} = s^{-1}\|\nu\| .$$

Now define U^ν as in the preceding proof ($\lambda=0$). Theorem 3.4 follows if we show

(iii′) $$U^\nu \leqq C\psi(C(\log_* \alpha)H_0) \quad \text{in } L ,$$

cf. (3.4).

LEMMA 3.5. *For any $\bar{n} \in \bar{N}$, the quantity $e^{-\varrho(H(\bar{n}_{-t}))+t}$ increases with t.*

PROOF. The square of this quantity is

$$e^{-2\varrho(H(h_{-t}\bar{n}))+t} = P_{-i\varrho}(\bar{n}^{-1}h_t, e)e^{2\varrho(tH_0)}$$

and $P_{-i\varrho}=P$ is the ordinary Poisson kernel. From the expansion

$$P(\bar{n}^{-1}h_t,e)^{-1} = \sum_s G_s(h_t)D_s(\bar{n}) ,$$

$$G_s(h_t) = \exp \sum_\alpha \pm\alpha(tH_0) ,$$

given in Knapp and Williamson [7, Proposition 5.1, p. 71], the lemma easily follows.

PROOF OF (iii). Since $v \leqq e^{-(s-1)t}$, we need only consider small α, and we have $S(\bar{m}) \leqq t_0$ for all $\bar{m} \in L'$ if t_0 is defined as $C \log_* \alpha$. Any $\bar{n} \in L$ then satisfies

$$U^v(\bar{n}) = \int_{L'} e^{-\varrho(H((\bar{m}^{-1}\bar{n})_{-S(\bar{m})})) + S(\bar{m})} d\bar{m}$$

$$\leqq C \int_{L'} e^{-\varrho(H((\bar{m}^{-1}\bar{n})_{-t_0})) + t_0} d\bar{m} ,$$

where Lemma 3.5 was used. Since $\bar{m}$ and $\bar{n}$ stay in a compact set, we may subtract $\varrho(H(\bar{m}^{-1}\bar{n}))$ in the exponent in the last integral if we change the value of C. Transforming $\bar{m} \to \bar{n}\bar{m}^{-1}$, we obtain

$$U^v(\bar{n}) \leqq C \int e^{-\varrho(H(\bar{m}_{-t_0})) + t_0 - \varrho(H(\bar{m}))} d\bar{m} = \psi(t_0 H_0) ,$$

where the last equality is seen from the proof of Lemma 2.1. This proves (iii′) and Theorem 3.4 for $s > 1$.

When $s < 0$, we may assume α is large since $m'_\sigma(L \times \mathbf{R}_+)$ is finite, and we may neglect the set where $t > t_0 = C \log_* \alpha$, since the m'_σ measure of this set is $O(\alpha^{-1})$. Now $S(\bar{m})$ is defined as $\inf\{t : v(\bar{m}, t) > \alpha\}$ for $\bar{m}$ in the set $L' \subset L$ where this inf is positive but smaller than t_0. The rest goes as for $s > 1$.

Theorem 3.4 is proved. Notice that the proof given is based on that of Theorem 2 in [13].

4. Results for $\operatorname{Re} i\lambda \in \mathfrak{a}^*_+$.

If $\lambda \in \mathfrak{a}^*_{\mathbf{C}}$, we define λ' by $i\lambda' = \operatorname{Re} i\lambda$.

THEOREM 4.1. *Let* $\operatorname{Re} i\lambda \in \mathfrak{a}^*_+$ *and* $\sigma \in (2\varrho + {}_+\mathfrak{a}^*) \cup (-{}_+\mathfrak{a}^*)$, *and take* $p \in [1, \infty[$. *For any* $u \in \mathscr{E}_\lambda$, *the following are equivalent*:

(a) $u = P_\lambda f$ *for some* $f \in L^p(K/M)$ *when* $p > 1$, *or* $u = P_\lambda \mu$ *for some Borel measure* μ *on* K/M, *when* $p = 1$.

(b₁) $e^{-\sigma/p} u / \varphi_{\lambda'} \in \Lambda^{r-1}_{p,\sigma}$.

(b₂) $(1 + |.|)^{-(r-1)/p} e^{-\sigma/p} u / \varphi_{\lambda'} \in \Lambda_{p,\sigma}$.

(b₃) $e^{-\sigma/p} u / \varphi_{\lambda'} \in \Lambda^*_{p,\sigma}$.

Observe that for $r = 1$ the (b_j) conditions coincide. Before the proof, we give a lemma.

LEMMA 4.2. Let $L \subset \bar{N}$ be compact. Any nonnegative $\varphi \in \mathscr{E}_\nu$, $\nu \in \overline{\mathfrak{a}^*_+}$, *satisfies*

$$\varphi(\bar{m}h) \sim \varphi(k(\bar{m})h) \tag{4.1}$$

for $\bar{m} \in L$, $h \in A_+$. *The same relation holds when* φ *is replaced by* e^ν, *any* $\nu \in \mathfrak{a}^*$.

PROOF. Let $\bar{m}=kan$ be the Iwasawa decomposition of $\bar{m}$, so that $\bar{m}h=kha\,{}^h n$. If $m \in L$, then also a, n, and ${}^h n$ stay in compact sets, so (4.1) follows from Lemma 2.2. This is true in particular for $\varphi=\varphi_\nu$, $i\nu \in \mathfrak{a}_+^*$. But $\varphi_\nu \sim e^{i\nu-\varrho}$ by Lemma 2.1, so considering quotients $\varphi_{\nu'}/\varphi_{\nu''}$, we see that any e^ν must satisfy (4.1), and the lemma is proved.

PROOF OF THEOREM 4.1. (a) $\Rightarrow$ (b_j). Since $|P_\lambda f| \leqq P_{\lambda'}|f|$, we may assume $i\lambda \in \mathfrak{a}_+^*$. The $p=1$ case then immediately implies the other cases, because of (2.4) and since, by Hölder's inequality, $|P_\lambda f|^p \leqq P_\lambda |f|^p \cdot \varphi_\lambda^{p-1}$.

Now let $i\lambda \in \mathfrak{a}_+^*$ and $u=P_\lambda \mu$, where μ is a probability measure carried by $k(L)M \subset K/M$, and L is as in Lemma 4.2. To begin with, we prove that u satisfies (b_j) in $k(L)A_+ \subset X$, and start with (b_1). Let $w=e^{-\sigma}u/\varphi_\lambda$. Since $dm_\sigma \leqq e^\sigma dkMdH$ and because of (2.3), it suffices to prove that for all $\alpha>0$

$$I \equiv \int_D e^{\sigma(H)}\,dk\,M\,dH \leqq C\alpha^{-1}\log_*^{r-1}\alpha \tag{4.2}$$

where $D=\{(kM,H) \in k(L)M \times \mathfrak{a}_+^* : w(k\exp H)>\alpha\}$.

Setting $k=k(\bar{m})$, we know that dkM corresponds to $e^{-2\varrho(H(\bar{m}))}d\bar{m}$ which is majorized by $d\bar{m}$, so

$$I \leqq \int_{D'} e^{\sigma H}\,d\bar{m}\,dH\ ,$$

with $D'=\{(\bar{m},H) \in L \times \mathfrak{a}_+ : w(k(\bar{m})\exp H)>\alpha\}$. Because of Lemma 4.2,

$$D' \subset D'' = \{(\bar{m},H) \in L \times \mathfrak{a}_+ : w(\bar{m}\exp H)>\alpha/C\}\ .$$

The inverse image of μ under $\bar{n} \to k(\bar{n})M$ is a measure in L which is also called μ. For $H_0 \in S$, let v be as in Theorem 3.1. Because of (2.1), we have $w(\bar{m}\exp tH_0) \leqq Cv(\bar{m},t)$. When $r=1$, we see that $I \leqq C$ times the m'_σ measure of that part of $L \times \mathbf{R}_+$ where $v>\alpha/C$. So by Theorem 3.1, $I \leqq C\alpha^{-1}$ which is (4.2). For $r>1$, we get

$$I \leqq \int_{D''} t^{r-1}e^{t\sigma(H_0)}\,d\bar{m}\,dt\,dH_0 = \int dH_0 \int d\bar{m}\,dt \ldots . \tag{4.3}$$

As in the proof of Theorem 3.4, we may neglect the subset E of $L \times \mathfrak{a}_+$ where $t=|H|>C\log_*\alpha$, either because v is small in E or because the measure of E is small. This means that t^{r-1} can be estimated by $C\log_*^{r-1}\alpha$ in (4.3). Hence, the inner integral in (4.3) is $O(\alpha^{-1}\log_*^{r-1}\alpha)$, uniformly in H_0, and (4.2) follows again.

To obtain (b_2) in $k(L)A_+$, notice that we may also neglect the set where $t<(\log_*\alpha)/C$ for similar reasons. But when $t \sim \log_*\alpha$, the factor $1+|H|$ behaves like a constant and (b_2) follows from (b_1). Finally, (b_3) is a consequence of Theorem 3.1 in a similar way.

To complete the proof of (a) $\Rightarrow$ (b_j), we must get rid of L. Since $k(\bar{N})M$ is open and dense in K/M, it is easy to find a compact set L and finitely many points $k_1,\ldots,k_n$ so that the sets $k_jk(L)M$ together cover K/M and their intersection is a neighborhood U of eM. Decomposing a given measure μ in K/M into parts carried by the $k_jk(L)M$, we see that $u=P_\lambda\mu$ satisfies (b_j) in UA_+. Hence by translation, (b_j) holds in all of X, $j=1,2,3$.

(b_j) $\Rightarrow$ (a). Assume $u\in\mathscr{E}_\lambda$ satisfies some (b_j) and that the associated quasinorm is at most 1. We start with a crude preliminary estimate.

LEMMA 4.3. *$|u|\leqq Ce^{C\varrho}$ in X.*

PROOF. Because of the mean value theorem (see Helgason [3, p. 438]), we have for any $g\in G$ and $x\in X$

$$\int_K u(gkx)\,dk \;=\; \lambda_x u(g)\,, \tag{4.4}$$

where $\lambda_x\to 1$ as $x\to o$ and dk is the normalized Haar measure in K. The use of the mean value theorem at this point was suggested by T. Rychener. Let B_R denote the geodesic ball in X with center o and radius R. Now integrate (4.4) with respect to $dm(x)$ over B_R, when $R>0$ is small. We get

$$\begin{aligned}|u(g)| \;&\leqq\; Cm(B_R)^{-1}\int_K dk\int_{B_R}|u(gkx)|\,dm(x)\\ &=\; Cm(B_R)^{-1}\int_{B_R}|u(gx)|\,dm(x)\end{aligned} \tag{4.5}$$

because of the K-invariance of m and B_R. Fix $g\in G$. By Lemmas 2.1 and 2.2, the functions e^ν, $\nu\in\mathfrak{a}^*$, are approximately constant in gB_1, so $dm_\sigma/dm\sim\beta\equiv e^{\sigma(g)-2\varrho(g)}$ in gB_1. For some C, the function $v=e^{-C\varrho}|u|$ is in $\Lambda^{r-1}_{p,\sigma}$, with a quasinorm $<C$, when (b_1) or (b_2) is satisfied. In the (b_3) case, we replace σ by a slightly smaller σ', and reason in the same way. Clearly, $v\sim e^{-C\varrho(g)}|u|$ in gB_1. Now let v^* be the decreasing rearrangement of the restriction of v to gB_1 with respect to m. Considering distribution functions with respect to m_σ and m, we get

$$v^*(t) \;\leqq\; C(\beta t)^{-1/p}\log_*^{(r-1)/p}(\beta t)\,. \tag{4.6}$$

When $p>1$, the lemma follows at once from (4.5–4.6) and (2.2), so assume

$p=1$. Let s_j be the sup of v in $gB_{1-2^{-j}}$, $j=1,2,\ldots$. Set $n=\dim X$, so that $m(B_R) \sim R^n$ for $R<1$. For $x \in gB_{1-2^{-j}}$, (4.5) implies

$$v(x) \leqq C2^{nj} \int_{xB_{2^{-j-1}}} v\,dm .$$

Now $xB_{2^{-j-1}} \subset gB_{1-2^{-j-1}}$, so $v \leqq s_{j+1}$ there. Applying (4.6) and (2.2), we therefore have

$$\begin{aligned} v(x) &\leqq C2^{nj} \int_0^{C2^{-nj}} \min\,(s_{j+1}, (\beta t)^{-1} \log_*^{r-1}(\beta t))\,dt \\ &\leqq C2^{nj}\beta^{-1} + C2^{nj} \int_{1/\beta s_{j+1}}^{C2^{-nj}} (\beta t)^{-1} \log_*^{r-1}(\beta t)\,dt . \end{aligned}$$

Transforming $t \to t/\beta$ in the last integral, we see that

$$v(x) \leqq C2^{nj}\beta^{-1} + C2^{nj}\beta^{-1} (\log_*^r s_{j+1} + \log_*^r \beta) .$$

It is possible to assume that all the s_j are $>\beta^{\pm 2}$, so that

$$\log_*^r s_{j+1} + \log_*^r \beta \sim \log_*^r (\beta s_{j+1}) ,$$

since otherwise the lemma follows at once. Letting x vary, we have proved

$$s_j \leqq C2^{nj}\beta^{-1} + C2^{nj}\beta^{-1} \log_*^r (\beta s_{j+1}) .$$

It is elementary to see from this inequality that if $A>0$ is large enough, and if the inequality

$$2^{-nj}\beta s_j > A2^j \tag{4.7}$$

holds for $j=1$, then it holds for all j. But this would mean that v is unbounded in gB_1, which is false. Hence, (4.7) cannot hold for $j=1$, and this gives the desired estimate for $v(g)$ and $u(g)$. The lemma is proved.

Continuing the proof of (b$_j$) $\Rightarrow$ (a), we shall show that

$$\liminf_{H\to\infty} I_\lambda(H) < \infty, \quad \text{where } I_\lambda(H) = \int_{K/M} \left|\frac{u(k\exp H)}{\varphi_\lambda(\exp H)}\right|^p dk\,M . \tag{4.8}$$

If (b$_1$) is satisfied, take a compact set $S' \subset S$. For $T>1$, clearly

$$\int_1^T t^{r-1}\,dt \int_{S'} I_{\lambda'}(tH_0)\,dH_0 = \int_{D_T} |e^{-\sigma(H)/p} u/\varphi_{\lambda'}|^p e^{\sigma(H)}\,dk\,M\,dH \tag{4.9}$$

when $D_T = \{k\exp H \in X : k \in K,\ H \in R_+S',\ 1 \leqq |H| \leqq T\}$. Notice that $e^{\sigma(H)}dkdH \leqq C dm_\sigma$ in D_T and that $m_\sigma(D_T) \leqq Ce^{CT}$. Lemma 4.3 and (b$_1$) give two estimates for $|e^{-\sigma(H)/p}u/\varphi_{\lambda'}|^p$. From (2.4) and (2.2), applied to m_σ, it follows that both sides of (4.9) are dominated by

$$C\int_0^{e^{CT}} \min(e^{CT}, t^{-1}\log_*^{r-1} t)\,dt \leqq CT^r .$$

But then necessarily

$$\liminf_{t\to\infty} \int_{S'} I_{\lambda'}(tH_0)\,dH_0 < \infty ,$$

so $\liminf_{t\to\infty} I_{\lambda'}(tH_0) < \infty$ for some $H_0 \in S'$ by Fatou's lemma. From this (4.8) follows, since $|\varphi_\lambda|$ and $\varphi_{\lambda'}$ have the same asymptotic behavior on a ray $\{tH_0\}$, as proved by Harish-Chandra [1, p. 291].

When u satisfies (b_2), we write instead

$$\int dt \int I_{\lambda'}(tH_0)\,dH_0 = \int \|H\|^{-(r-1)/p} e^{-\sigma(H)/p} u/\varphi_{\lambda'}|^p e^{\sigma(H)}\,dk\,M\,dH ,$$

where the integrals are taken over the same sets as before. Then this is estimated by $O(T)$ in the same way. The details, as well as the (b_3) case, are left to the reader.

Finally, we must show that (4.8) yields the representation of u as a Poisson integral. For each irreducible representation δ of K, let $\alpha_\delta = d_\delta \bar{\chi}_\delta$, where d denotes dimension, χ character, and the bar complex conjugate. As in Helgason [4, p. 138], we expand u in

$$u = \sum_\delta \alpha_\delta * u ,$$

where the convolution is performed in K.

Harish-Chandra [2, Corollary 1, p. 13] has proved that this series converges in $C^\infty(X)$. Now every $\alpha_\delta * u$ is a K-finite function in $\mathscr{E}_\lambda$, so by [5, Corollary 7.4, p. 207], $\alpha_\delta * u = P_\lambda f_\delta$ for some K-finite function f_δ in K/M. Because of (4.8), we may take a sequence $H_j \to \infty$ for which $u(k \exp H_j)/\varphi_\lambda(\exp H_j)$ converges weakly to a measure μ in K/M, and μ is an L^p function if $p > 1$. Then

$$\alpha_\delta * u(k \exp H_j)/\varphi_\lambda(\exp H_j) \to \alpha_\delta * \mu(kM), \quad j \to \infty ,$$

uniformly for $k \in K$. Michelson [10, Theorem 1.3] has proved that $P_\lambda f_\delta/\varphi_\lambda \to f_\delta$ as $H \to \infty$, so we conclude $f_\delta = \alpha_\delta * \mu$. Thus,

$$u = \sum_\delta P_\lambda(\alpha_\delta * \mu) ,$$

and it remains to prove that this last sum equals $P_\lambda \mu$. And this follows from a direct calculation since $P_\lambda(x, \cdot)$ is smooth and thus has a convergent α_δ expansion. Theorem 4.1 is completely proved.

REMARK 1. As to the last part of this proof, cf. also the general representation theorem in [6].

REMARK 2. In the case when $r=1$ and $i\lambda \in \mathfrak{a}_+^*$, we sketch a proof that (4.8) implies the desired representation for u which does not use any general representation theorem. Let the measure μ on the boundary be a weak* accumulation point of $u(\cdot \exp H)/\varphi_\lambda$ as $H \to \infty$, and regularize by convolving in K by a smooth approximate identity ψ_ε. Then $\psi_\varepsilon * u(\cdot \exp H_j)/\varphi_\lambda$ will converge uniformly to $\psi_\varepsilon * \mu$ for some sequence $H_j \to \infty$. Now if $v \in \mathscr{E}_\lambda$, then v/φ_λ must assume its maximum in the domain $\{k \exp H : H \in \mathfrak{a}_+, H < H_j\}$ on the boundary $K \exp H_j$. This follows from Hopf's maximum principle applied to v/φ_λ and the operator $w \to \Delta(\varphi_\lambda w) - w\Delta\varphi_\lambda$, where Δ is the Laplacian of X. Applying this with $v = \pm(\psi_\varepsilon * u - P_\lambda(\psi_\varepsilon * \mu))$ and letting $j \to \infty$ gives $\psi_\varepsilon * u = P_\lambda(\psi_\varepsilon * \mu)$ and thus $u = P_\lambda \mu$.

5. Results for $\lambda = 0$.

THEOREM 5.1. *If $r=1$, Theorem 4.1 holds when $\lambda=0$. For $r>1$ and $\lambda=0$, let σ and p be as in Theorem 4.1, and assume $u \in \mathscr{E}_0$. Then u has a representation as in condition* (a) *of Theorem* 4.1 *if and only if* (b_j) *holds in some (or every) restricted domain. Here j is* 1, 2, *or* 3.

We do not know whether Theorem 4.1 holds for $\lambda=0$, $r>1$, although this seems plausible in view of Theorem 3.4. However, conditions like

$$e^{-\sigma/p}u/e^{-\varrho} \in \Lambda^{qp+r-1}_{p,\sigma} \quad \text{in all of } X$$

also characterize the Poisson integrals of L^p functions or measures for $\lambda=0$. The proof of this is left to the reader.

In the preceding section, we already used a convergence result of type $P_\lambda f/\varphi_\lambda \to f$ at the boundary, for $\operatorname{Re} i\lambda \in \mathfrak{a}_+$. Michelson [10] obtains such results by proving that $P_\lambda(\exp H, kM)/\varphi_\lambda(\exp H)$ is an approximate identity in K/M as $H \to \infty$. Since this expression has integral 1 and bounded L^1 norm in K/M, it defines an approximate identity if and only if its L^1 norm in $K/M \setminus U$ tends to 0 as $H \to \infty$ for any neighborhood U of eM in K/M. Whether this is true for $\lambda=0$ and $r>1$ seems to be unknown. The following weaker result will be needed in the proof of Theorem 5.1.

THEOREM 5.2. *Let $H_0 \in S$ and $\varepsilon>0$, and set $h_t = \exp tH_0$. There exists a Lebesgue measurable set $F \subset \mathbf{R}_+$ such that for any $T>1$ the measure of $F \cap [T, 2T]$ is larger than $(1-\varepsilon)T$ and such that for any neighborhood U of eM in K/M*

$$\frac{1}{\varphi_0(h_t)} \int_{K/M \setminus U} P_0(h_t, kM)\, dkM \to 0 \quad \text{as } t \to \infty,\ t \in F. \tag{5.1}$$

PROOF. As usual, we transform the integral to $\bar{N}$. Assume $U=k(B)M$, for a compact neighborhood B of $e \in \bar{N}$. Writing B_t as in Section 3, we have

$$I(B,t) \equiv e^{\varrho(tH_0)} \int_{K/M \setminus U} P_0(h_t, kM)\, dkM$$
$$= \int_{\bar{N} \setminus B} e^{-\varrho(H(\bar{n}_{-t}))+t-\varrho(H(\bar{n}))}\, d\bar{n}\,.$$

Now

$$\bar{N} \setminus B = \bigcup_{j=1}^{\infty} (B_{-j} \setminus B_{-j+1})\,,$$

and

$$e^{-\varrho(H(\bar{n}))} \leqq Ce^{-\delta j} \quad \text{for } \bar{n} \notin B_{-j+1} \text{ and some } \delta>0\,,$$

by [7, Proposition 5.5]. Thus,

$$I(B,t) \leqq C \sum_{j=1}^{\infty} e^{-\delta j} \int_{B_{-j} \setminus B_{-j+1}} e^{-\varrho(H(\bar{n}_{-t}))+t}\, d\bar{n}\,. \tag{5.2}$$

Notice that the quantity in (5.1) is $I(B,t)/\psi(tH_0)$ and that $\psi(tH_0) \sim t^q$, $t \to \infty$. We must thus determine F so that $I(B,t)=o(t^q)$, $t \to \infty$, $t \in F$.

In the terms with $j>t$ in (5.2), we transform $\bar{n} \to \bar{n}_{-j}$, getting

$$\sum_{j>t} \cdots \leqq C \sum_{j>t} e^{-\delta j} \int_{B \setminus B_1} e^{-\varrho(H(\bar{n}_{-j-t}))+j+t}\, d\bar{n}\,.$$

Since $H(\bar{n})$ is bounded in $B \setminus B_1$, the integral in the last sum is dominated by

$$C \int_{\bar{N}} e^{-\varrho(H(\bar{n}_{-j-t}))+j+t-\varrho(H(\bar{n}))}\, d\bar{n} = C\psi((j+t)H_0) \leqq C(j+t)^q\,.$$

Hence,

$$\sum_{j>t} \cdots \leqq C \sum_{j>t} e^{-\delta j}(j+t)^q \to 0 \quad \text{as } t \to \infty\,.$$

As to the other terms in (5.2), we have

$$\sum_{t=1}^{T} \sum_{j=1}^{t} e^{-\delta j} \int_{B_{-j} \setminus B_{-j+1}} e^{-\varrho(H(\bar{n}_{-t}))+t}\, d\bar{n} \leqq \sum_{j=1}^{T} e^{-\delta j} \sum_{t=j}^{T} \int_{B_{-j-t} \setminus B_{-j-t+1}} e^{-\varrho(H(\bar{n}))}\, d\bar{n}$$
$$\leqq \sum_{j=1}^{T} e^{-\delta j} \int_{B_{-2T}} e^{-\varrho(H(\bar{n}))}\, d\bar{n}\,,$$

since $B_{-j-t} \setminus B_{-j-t+1}$ are disjoint for distinct t, fixed j. Transforming $\bar{n} \to$

$\bar{n}_{-2T}$, one can estimate the last integral by $\psi(2TH_0) \leqq CT^q$ as before, and so the last sum is also $O(T^q)$.

Altogether then, we conclude

$$\sum_{t=1}^{T} I(B,t) \leqq CT^q .$$

Because of Lemma 2.2, this estimate remains valid if we replace summation in t by integration dt. If we let $F = \{t : I(B,t) \leqq Ct^{q-1}\}$ and choose C large enough, it is clear that $I(B,t) = o(t^q)$ in F and that F is as dense at ∞ as claimed.

Finally, to find an F which works for all U simultaneously, we repeat this construction as U describes a neighborhood basis at eM, choosing the values of ε suitably. The proof of Theorem 5.2 is complete.

Notice that we actually proved that

$$\int_0^T \psi(tH_0)\varphi_0(h_t)^{-1} P_0(h_t, kM)\, dt \Big/ \int_0^T \psi(tH_0)\, dt$$

is an approximate identity as $T \to \infty$. Since this makes it possible to reconstruct f from $P_0 f$, we incidentally also get a proof of the fact that the value $\lambda = 0$ is simple without using the general criterion of Helgason [5, Theorem 6.1].

Proof of Theorem 5.1. We only indicate at which points this proof differs from that of Theorem 4.1, leaving the details to the reader. Assume first $u = P_0\mu$, $\mu \geqq 0$ a measure, and take $\alpha > 0$. As before, we need only care about the region where $|H| \sim \log_* \alpha$. If, further, H is in a restricted cone, we know that $\psi(H) \sim |H|^q \sim \log_*^q \alpha$. Now the (b_j) conditions are proved as in Section 4, by means of Theorem 3.4 instead of Theorem 3.1.

Conversely, let u satisfy (b_1), say, in the restricted domain corresponding to $S' \subset S$. As in the deduction of (4.8), we have

$$\int_1^T t^{r-1}\, dt \int_{S'} I_0(tH_0)\, dH_0 \leqq CT^r .$$

This implies that $I_0(tH_0) \leqq C$ in "most of" the set $\{(t, H_0) : T \leqq t \leqq 2T,\ H_0 \in S'\}$ for every large T and some C. But then one can find an $H_0 \in S'$ for which the same inequality holds for most t in $[2^j, 2^{j+1}]$ for infinitely many values of j. Hence, there is a sequence $t_j \to \infty$ contained in the set F of Theorem 5.2 and such that $I(t_j H_0)$ is bounded as $j \to \infty$. This is all we need to apply the reasoning at the end of the proof of Theorem 4.1, and the proof is complete.

6. The case when σ is between 0 and 2ϱ.

We say that the maximum theorem holds for a $p>1$ and a λ, $\operatorname{Re} i\lambda \in \mathfrak{a}_+^*$ or $\lambda=0$, if

$$u^*(kM) \equiv \sup\{|u(k\exp H)|/\varphi_{\lambda'}(\exp H) : H \in \mathfrak{a}_+\} \in L^p(K/M)$$

whenever $u=P_\lambda f$ and $f \in L^p(K/M)$. This is true for all such p and λ when $r=1$ (see Michelson [10, Sec. 3]). For $r>1$, the maximum theorem holds for p large enough, at least when $i\lambda=\varrho$ (see Lindahl [8]). The following result generalizes a theorem of Lohoué and Rychener [9, Proposition 1].

THEOREM 6.1. *Let $p>1$ and $\operatorname{Re} i\lambda \in \mathfrak{a}_+^*$ or $\lambda=0$, and assume $\sigma \in {}_+\mathfrak{a}^* \cup (-{}_+\mathfrak{a}^*)$. If the maximal theorem holds for these p and λ, then conditions* (a), (b$_2$), *and* (b$_3$) *are equivalent.*

To prove (a) $\Rightarrow$ (b$_j$), one estimates u by means of u^*. The details are left to the reader (see also [9]). For the converse implications, the corresponding proofs given in Sections 4 and 5 carry over without change.

However, (a) does not in general imply (b$_1$) under the hypotheses of Theorem 6.1. To get a counterexample, consider a bi-disk U^2, U being the noneuclidean unit disk, and write each coordinate $z_i \in U$ as $(r_i\cos\theta_i, r_i\sin\theta_i)$, $-\pi<\theta_i\leqq\pi$, $i=1,2$. Then dm_σ is essentially the product of the measures $r_i(1-r_i)^{-1-s_i}dr_i d\theta_i$, $i=1,2$, and we let $0<s_i<1$, which means choosing σ strictly "between" 0 and 2ϱ. Given $p\geqq 1$ and $\varepsilon>0$, choose

$$f(\theta_1,\theta_2) = f(\theta_1) = |\theta_1|^{-1/p}\log_*^{-(1+\varepsilon)/p}|\theta_1| ,$$

which is an L^p function on the boundary $\partial U \times \partial U$. If

$$v(z_1,z_2) = (1-r_1)^{s_1/p}(1-r_2)^{s_2/p}P_\lambda f/\varphi_\lambda$$

and $i\lambda \in \mathfrak{a}_+^*$, it is easily seen that

$$v \geqq (1-r_1)^{s_1/p}(1-r_2)^{s_2/p}f(\max(|\theta_1|, 1-r_1))/C . \tag{6.1}$$

Let $0<\varepsilon'<1-s_1$ and $\alpha>0$. Suppose

$$(1-r_1)^{1-s_1-\varepsilon'} < (1-r_2)^{s_2}\alpha^{-p} < 1 \tag{6.2}$$

so that

$$\log_*(1-r_1)^{s_1}(1-r_2)^{s_2}\alpha^{-p} \sim \log_*(1-r_1) .$$

If in addition

$$1-r_1 < |\theta_1| < (1-r_1)^{s_1}(1-r_2)^{s_2}\alpha^{-p}\log_*^{-1-\varepsilon}(1-r_1)/C , \tag{6.3}$$

it follows from (6.1) that $v>\alpha$. For r_2 fixed, we integrate $r_1(1-r_1)^{-1-s_1}dr_1 d\theta_1$ over the set of (r_1, θ_1) defined by (6.2) and (6.3), getting at least

$$(1-r_2)^{s_2}\alpha^{-p}\log_*^{-\varepsilon}((1-r_2)^{s_2}\alpha^{-p})/C .$$

Integrating now in r_2 and θ_2, we see that $m_\sigma\{v>\alpha\}=\infty$, so that $v \notin \Lambda^1_{p,\sigma}$.

Next, we give examples showing that Theorem 6.1 is false for $p=1$ and σ "between" 0 and 2ϱ. When $\sigma=0$, the function $u=P_\lambda 1=\varphi_\lambda$ does not satisfy any (b_j). For other σ, we consider only the ordinary Poisson kernel P in the unit disk, or, more conveniently, the upper half-plane $\mathsf{R}^2_+=\{(x,t): t>0\}$. We choose measures in $0\leqq x\leqq 1$ and estimate their Poisson integrals in R^2_+ near this interval. If $\sigma=s\cdot 2\varrho$, $0<s\leqq 1$, we have $e^{-\sigma}\sim t^s$ and $dm_\sigma \sim t^{-s-1}dx\,dt$ here. For $s=1$, $\sigma=2\varrho$, consider the Dirac measure δ_0. It is easily verified that $tP\delta_0\sim t^2/(x^2+t^2)$ is not in $\Lambda_{1,\sigma}$. And when $0<s<1$, we use measures of Cantor type, carried by Cantor sets of ration $2^{-\varkappa}$, $\varkappa=1/(1-s)>1$, constructed as follows. Choose two 1st step intervals $[0,2^{-\varkappa}]$ and $[1-2^{-\varkappa},1]$, thus situated at the ends of $[0,1]$, and then four 2nd step intervals, each of length $2^{-2\varkappa}$, at the ends of the two 1st step intervals. Continuing in this way, we get at the nth step 2^n intervals of length $2^{-n\varkappa}$. There exists a measure μ such that each of these nth step intervals has measure 2^{-n}. It is easily verified that at points (x,t) with $t\sim 2^{-\varkappa n}$ and x in an nth step interval, we have $t^sP\mu(x,t)\sim 1$. Hence, $m_\sigma\{t^sP\mu(x,t)\sim 1\}=\infty$, and we are done.

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