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REMARKS ON A C*-DYNAMICAL SYSTEM

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0. Abstract.

Related to the crossed product of a C*-algebra A with a locally compact abelian group G to a co-action δ on it, we introduce the notion of δ-invariantness and using this, we define the essential spectrum $\Gamma(\delta)$ and show that $\Gamma(\hat{\alpha}) = \Gamma(\delta)$. When G is compact, we characterize the Connes spectrum $\Gamma(\hat{\hat{\alpha}})$ by the co-action δ where $\hat{\hat{\alpha}}$ is the bidual action of α. If α and β are exterior equivalent, then $\Gamma(\hat{\hat{\alpha}})$ and $\Gamma(\hat{\hat{\beta}})$ coincide.

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Now take a C*-algebra A and a locally compact group G with a fix left Haar measure dg on G. Suppose that there is a homomorphism α of G into the group $\operatorname{Aut}(A)$ of all *-automorphisms of A such that each function $g \mapsto \alpha_g(a)$ is continuous for $a \in A$. The triple (A, G, α) is called a C*-dynamical system.

Let (A, G, α) be a C*-dynamical system and assume $A \subset B(H)$ for some Hilbert space H. We denote by $K(G, A)$ the space of continuous functions from G to A with compact support. Define a faithful representation of $K(G, A)$ on $L^2(G, H)$ by

$$(x\xi)(g) = \int_G \alpha_{h^{-1}}(x(g))\xi(h^{-1}g)\,dh \tag{1.1}$$

for $x \in K(G, A)$ and $\xi \in L^2(G, H)$. We identify $K(G, A)$ with its image in $B(L^2(G, H))$ and denote by $G \times_\alpha A$ the C*-algebra generated by $K(G, A)$. We say that $G \times_\alpha A$ is a crossed product of G with A. We define a representation ν of G on $L^2(G, H)$ and a faithful representation ι of A on $L^2(G, H)$ by

$$(\nu(h)\xi)(g) = \xi(h^{-1}g), \qquad (\iota(x)\xi)(h) = \alpha_{h^{-1}}(x)\xi(h)$$

for $h, g \in G$, $x \in A$ and $\xi \in L^2(G, H)$. We also use a unitary operator W (called the Kac–Takesaki operator) on $L^2(G \times G)$

$$(W\xi)(g, h) = \xi(g, gh)$$

for $\xi \in L^2(G \times G)$ and we denote by λ a left regular representation of G and by

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$m(G)$ the von Neumann algebra generated by $\{\lambda(g); g \in G\}$. Then we define an isomorphism δ_G of $m(G)$ into $m(G)\bar{\otimes}m(G)$ by $\delta_G(x) = W^*(x\otimes 1)W$ for $x \in m(G)$.

When A and B are C*-algebras, we denote by $M(A)$ its multiplier algebra. If A is a concrete C*-algebra, we may define $M(A) = \{a \in A''; ab + ca \in A$ for $b, c \in A\}$ ([1]). We put

$$\tilde{M}_L(A\otimes B) = \{x \in M(A)\otimes M(b);$$
$$x(1\otimes b) + (1\otimes c)x \in A\otimes B,\ L_\varphi(x) \in A \text{ for } b, c \in B,\ \varphi \in B^*\}$$

where L_φ is the left slice map of φ and the symbol $\otimes$ means the spatial tensor product.

PROPOSITION 1.1. *The map* δ;

$$\delta(x) = (1\otimes W^*)(x\otimes 1)(1\otimes W) \quad \textit{for } x \in G\times_\alpha A,$$

*is a *-isomorphism of* $G\times_\alpha A$ *into* $\tilde{M}_L(G\times_\alpha A\otimes C_r^*(G))$, *where* $C_r^*(G)$ *is the reduced group C*-algebra of G. It satisfies the following relation,*

$$(1.2) \quad \delta(v(g)) = v(g)\otimes\lambda(g),\ (g \in G), \quad \delta(\iota(a)) = \iota(a)\otimes 1,\ (a \in A)$$

$$(1.3) \quad (\delta\otimes i)\delta = (i\otimes\delta_G)\delta$$

where i's are identity mappings of $M(C_r^*(G))$ *or* $M(G\times_\alpha A)$.

PROOF. We get (1.2) by an easy calculation. It follows from (1.1) and (1.2) that

$$\delta(x) = \int_G (\iota(x(g))\otimes 1)(v(g)\otimes\lambda(g))\,dg$$

for $x \in K(G, A)$. Since $\iota(A)$ and $v(G)$ are contained in $M(G\times_\alpha A)$ and $\lambda(G)$ is contained in $M(C_r^*(G))$, $\delta(x)$ is an element of $M(G\times_\alpha A)\otimes M(C_r^*(G))$, which implies $\delta(G\times_\alpha A) \subset M(G\times_\alpha A)\otimes M(C_r^*(G))$.

If $x \in K(G, A)$, $f \in K(G) \equiv K(G, \mathbf{C})$, we have

$$\begin{aligned}\delta(x)(1\otimes\lambda(f)) &= \int_G (\iota(x(g))\otimes 1)(v(g)\otimes\lambda(g))(1\otimes\lambda(f))\,dg \\ &= \iint_{G\times G} f(h)(\iota(x(g))\otimes 1)(v(g)\otimes\lambda(gh))\,dg\,dh \\ &= \iint_{G\times G} (\iota(f(g^{-1}h)x(g))\otimes 1)(v(g)\otimes\lambda(h))\,dg\,dh\end{aligned}$$

where $\lambda(f) = \int_G f(h)\lambda(h)\,dh$, therefore $\delta(G\times_\alpha A)(1\otimes C_r^*(G)) \subset G\times_\alpha A\otimes C_r^*(G)$. Similarly we have $(1\otimes C_r^*(G))\delta(G\times_\alpha A) \subset (G\times_\alpha A)\otimes C_r^*(G)$.

Take $x \in K(G, A)$ and $\varphi \in C_r^*(G)^*$, then

$$L_\varphi(\delta(x)) = L_\varphi\left(\int_G (\iota(x(g))\otimes 1)(\nu(g)\otimes\lambda(g))\,dg\right)$$
$$= \int_G \iota(x(g))\varphi(\lambda(g))\nu(g)\,dg$$
$$= \int_G \iota(\varphi(\lambda(g))x(g))\nu(g)\,dg = \varphi x ,$$

where $(\varphi x)(g) = \varphi(\lambda(g))x(g) \in K(G, A)$. We have therefore

$$L_\varphi \circ \delta(G \times_\alpha A) \subset G \times_\alpha A \quad \text{for } \varphi \in C_r^*(G)^* ,$$

because $\|L_\varphi\delta\| \leqq \|\varphi\|$.

The relation (1.3) follows from (1.2) and $\delta_G(\lambda(g)) = \lambda(g)\otimes\lambda(g)$ for $g \in G$.

We will introduce the essential spectrum of a co-action following Y. Nakagami [7] and Y. Nakagami–M. Takesaki [8]. To do this we first recall some definitions.

A co-action δ of G on a C*-algebra A is an isomorphism of A into $\tilde{M}_L(A\otimes C_r^*(G))$ satisfying $(\delta\otimes i)\delta = (i\otimes\delta_G)\delta$. Then we define δ_u by

$$\delta_u(a) = L_u \circ \delta(a) \quad \text{for } u \in B_r(G),\ a \in A ,$$

where $B_r(G)$ is defined in [3] to be regular ring, and we identify $B_r(G)$ with the dual space $C_r^*(G)^*$ of $C_r^*(G)$. It follows from (1.3) that $\delta_{u\cdot v} = \delta_u \cdot \delta_v$ for $u, v \in B_r(G)$.

We set

$$\mathrm{Sp}_\delta(a) = \{g \in G;\ u(g) = 0 \quad \text{for } \delta_u(a) = 0,\ u \in B_r(G)\} ,$$
$$\mathrm{Sp}(\delta) = \{g \in G,\ u(g) = 0 \quad \text{for } \delta_u = 0,\ u \in B_r(G)\}$$

and

$$\Gamma(\delta) = \bigcap \{\mathrm{Sp}(\delta \mid B);\ B \in \mathscr{H}^\delta(A)\} ,$$

where $\mathscr{H}^\delta(A)$ is the family of non-zero hereditary C*-subalgebras B of A such that $\delta_u(B) \subset B$ for $u \in B_r(G)$, which is called δ-invariant. Let E be a closed subset of G, we set

$$A^\delta(E) = \{a \in A;\ \mathrm{Sp}_\delta(a) \subset E\} .$$

LEMMA 1.2. *If $g \in G$, then $g \in \mathrm{Sp}(\delta)$ if and only if $A^\delta(V) \neq \{0\}$ for every compact neighbourhood V of g.*

PROOF. Let V be a compact neighbourhood of g with $A^\delta(V)=\{0\}$. Take an element $g_0 \in V^c$ and $v \in B_r(G)$ with (the support of v) $\cap V=\emptyset$ and $v(g_0)=1$. If $u \in B_r(G)$ with (the support of u) $\subset V$ and $u(g)=1$, then

$$\delta_v \cdot \delta_u(a) = \delta_{v \cdot u}(a) = 0 \quad \text{for } a \in A ,$$

which implies $g_0 \notin \mathrm{Sp}_\delta(\delta_u(a))$ that is $\delta_u(a) \in A^\delta(V)$. Therefore $\delta_u(a)=0$ for all $a \in A$. As $u(g) \neq 0$, we see $g \notin \mathrm{Sp}(\delta)$.

Suppose that $g \notin \mathrm{Sp}(\delta)$. Take a compact neighbourhood V of g with $V \cap \mathrm{Sp}(\delta)=\emptyset$ and take $a \in A^\delta(V)$, then it follows from $\mathrm{Sp}_\delta(a) \subset \mathrm{Sp}(\delta)$ that $\mathrm{Sp}_\delta(a)=\emptyset$. Since

$$I_a \equiv \{u \in B_r(G),\ \delta_u(a)=0\}$$

is a closed ideal of $B_r(G)$ with $\{g \in G;\ u(g)=0 \text{ for all } u \in I_a\}=\emptyset$ and $B_r(G)$ is a regular ring ([3]), I_a contains the Fourier algebra $A(G)$ of G because it contains $K(G)$ ([3]). For $\omega \in A^*$, we have

$$0 = \langle \delta_u(a), \omega \rangle = \langle \delta(a), \omega \otimes u \rangle \quad \text{for } u \in A(G) \subset B_r(G) .$$

Since, by [1, Proposition 2.4], the algebraic tensor product $A^* \odot A(G)$ of A^* and $A(G)$ is dense in $(M(A) \otimes M(C_r^*(G)))^*$ with respect to the w^*-topology of $M(A) \otimes M(C_r^*(G))$, we have $\delta(a)=0$, that is $a=0$. We have therefore $A^\delta(V)=0$.

LEMMA 1.3. *Let E_i be a compact set in G $(i=1,2)$, then $A^\delta(E_1)A^\delta(E_2) \subset A^\delta(E_1E_2)$.*

This lemma is proved by a usual arguement (See [8, IV, Lemma 1.2]), and we leave its verification to the reader.

PROPOSITION 1.4. *$\Gamma(\delta)$ is a closed subgroup of G.*

PROOF. Since $\mathrm{Sp}(\delta)$ is a closed set, $\Gamma(\delta)$ is a closed set of G. We want to see that $\mathrm{Sp}(\delta)\Gamma(\delta) \subset \mathrm{Sp}(\delta)$. Take $g_1 \in \mathrm{Sp}(\delta)$, $g_2 \in \Gamma(\delta)$ and compact neighbourhoods V, V_1, and V_2 of g_1g_2, g_1, and g_2, respectively such that $V_1V_2 \subset V$. For $a_1 \in A^\delta(V_1)$, $a_1 \neq 0$, B denote the smallest δ-invariant hereditary C*-subalgebra generated by $\{\delta_u(a_1);\ u \in B_r(G)\}$. Then we can find an $a_2 \in B \cap A^\delta(V_2)$, $a_2 \neq 0$. Let I be the closed linear span of $\{a\delta_u(a_1);\ a \in A, u \in B_r(G)\}$, then I is a closed left ideal of A such that $B=I^* \cap I$. Therefore if $\delta_u(a_1)a_2=0$ for any $u \in B_r(G)$, it implies $B\, a_2=0$ that is $a_2=0$. Hence there is a $u \in B_r(G)$ such that $\delta_u(a_1)a_2 \neq 0$. By Lemma 1.3,

$$0 \neq \delta_u(a_1)a_2 \in A^\delta(V_1)A^\delta(V_2) \subset A^\delta(V_1V_2) \subset A^\delta(V) .$$

By Lemma 1.2, we conclude from this that $\mathrm{Sp}(\delta)\Gamma(\delta) \subset \mathrm{Sp}(\delta)$. As this is true for

$\delta | B$ in place of δ, we see that $\Gamma(\delta)$ is a semi-group and it is easy to prove $\Gamma(\delta) = \Gamma(\delta)^{-1}$. Therefore $\Gamma(\delta)$ is a closed subgroup of G.

From now on, G is supposed to be abelian and we study relations between the co-action δ on $G \times_\alpha A$ and the dual action $\hat{\alpha}$ of the dual group Γ of G on $G \times_\alpha A$ (See [11]).

PROPOSITION 1.5. *Let (A, G, α) be a C^*-dynamical system and B is a C^*-subalgebra of $G \times_\alpha A$. Then B is δ-invariant if and only if B is $\hat{\alpha}_\gamma$-invariant for $\gamma \in \Gamma$ (Γ-invariant).*

PROOF. Take $\xi, \eta \in K(\Gamma)$ and $x \in K(G, A)$,

$$\begin{aligned}
&\int_\Gamma \xi(\gamma)\eta(\gamma)\hat{\alpha}_\gamma(x)\, d\gamma \\
&= \int_\Gamma \xi(\gamma)\eta(\gamma)\hat{\alpha}_\gamma\left(\int_G \iota(x(g))\nu(g)\, dg\right) d\gamma \\
&= \int_\Gamma \int_G \xi(\gamma)\eta(\gamma)\iota(x(g))\nu(g)\overline{\langle g, \gamma\rangle}\, d\gamma \\
&= \int_G \left(\int_\Gamma \xi(\gamma)\eta(\gamma)\overline{\langle g, \gamma\rangle}\, d\gamma\right)\iota(x(g))\nu(g)\, dg \\
&= \int_G \check{\xi} * \check{\eta}(-g)\iota(x(g))\nu(g)\, dg\,,
\end{aligned}$$

where $\check{\xi}$ is the inverse Fourier transform of ξ and the symbol $*$ means the convolution in $L^1(G)$. On the other hand, set

$$\omega(\xi, \eta)(x) = \langle x\check{\xi}, \check{\eta}^b\rangle, \qquad \text{for } x \in C_r^*(G)$$

where $\check{\eta}^b(g) = \overline{\check{\eta}(-g)}$, then we have

$$\omega(\xi, \eta)(\lambda(g)) = \langle \lambda(g)\check{\xi}, \check{\eta}^b\rangle = \int_G \check{\xi}(h-g)\check{\eta}(-h)\, dh = \check{\xi} * \check{\eta}(-g)\,.$$

Then we have,

$$= \int_G \omega(\xi, \eta)(\lambda(g))\iota(x(g))\nu(g)\, dg$$

$$= \delta_{\omega(\xi, \eta)}\left(\int_G \iota(x(g))\nu(g)\, dg\right) = \delta_{\omega(\xi, \eta)}(x)\,.$$

Therefore

$$\int_{\Gamma} \xi(\gamma)\eta(\gamma)\hat{\alpha}_{\gamma}(x)\,d\gamma = \delta_{\omega(\xi,\eta)}(x) \qquad \text{for } x \in G \times_{\alpha} A\,. \tag{1.4}$$

The set $\{\omega(\xi,\eta);\ \xi,\eta \in K(\Gamma)\}$ is dense in $(C_r^*(G))^*$ with respect to $\sigma(C_r^*(G)^*, M(C_r^*(G)))$-topology and the map

$$\varphi \in (C_r^*(G))^* \to \delta_{\varphi}(x) \in G \times_{\alpha} A$$

is norm continuous with respect to $\sigma((C_r^*(G))^*, M(C_r^*(G)))$-topology for each $x \in G \times_{\alpha} A$. Hence if B is Γ-invariant, then B is δ-invariant.

Conversely suppose that B is δ-invariant. Take $\gamma \in \Gamma$, the positive definite function $\langle \cdot, \gamma \rangle$ is an element of $B_r(G)$. Then by an easy calculation, we have $\delta_{\langle \cdot, \gamma \rangle} = \hat{\alpha}_{\gamma}$, therefore B is Γ-invariant.

Given a C*-dynamical system (A, G, α), we denote by $\mathscr{H}^{\alpha}(A)$ the family of non-zero, G-invariant, hereditary C*-subalgebra of A. The Connes spectrum $\Gamma(\alpha)$ of α is defined as

$$\Gamma(\alpha) = \bigcap \{\mathrm{Sp}\,(\alpha \mid B),\ B \in \mathscr{H}^{\alpha}(A)\}\,,$$

cf. [9].

THEOREM 1.6. *Let (A, G, α) be a C*-dynamical system with an abelian group G. Starting from this, we have a C*-dynamical system $(G \times_{\alpha} A, \Gamma, \hat{\alpha})$ and a dual system $(G \times_{\alpha} A, \delta)$. Then $\Gamma(\delta)$ and $\Gamma(\hat{\alpha})$ coincide.*

PROOF. At first we prove $A^{\hat{\alpha}}(V) = A^{\delta}(-V)$ for every compact neighbourhood V of $g \in G$, where $A^{\hat{\alpha}}(V) = \{x \in G \times_{\alpha} A,\ \mathrm{Sp}_{\hat{\alpha}}(x) \subset V\}$. Take $x \in A^{\hat{\alpha}}(V)$, $g_0 \notin V$ and compact neighbourhood V_0 of g_0 with $V_0 \cap V = \emptyset$, we can find $\xi, \eta\ K(\Gamma)$ with $\check{\xi} * \tilde{\eta}(g_0) = 1$ and $\check{\xi} * \tilde{\eta} \equiv 0$ on V_0^c. The inverse Fourier transform of $\xi(\gamma)\eta(\gamma)$ is $\check{\xi} * \tilde{\eta}$, so

$$\delta_{\omega(\xi,\eta)}(x) = \int_{\Gamma} \xi(\gamma)\eta(\gamma)\hat{\alpha}_{\gamma}(x)\,d\gamma = 0, \qquad \text{as } x \in A^{\hat{\alpha}}(V)\,.$$

As $\omega(\xi,\eta)(\lambda(-g_0)) = \check{\xi} * \tilde{\eta}(g_0) = 1$, we have $-g_0 \in \mathrm{Sp}\,\delta(x)$ that is $\mathrm{Sp}_{\delta}(x) \subset -V$.

Conversely, take $x \in A^{\delta}(-V)$, $g_0 \notin V$ and a compact neighbourhood V_0 of g_0 with $V_0 \cap (-V) = \emptyset$, and take $\xi, \eta \in K(\Gamma)$ as above. Put $y = \delta_{\omega(\xi,\eta)}(x)$, then we have $\mathrm{Sp}_{\delta}(y) = \emptyset$, since $\mathrm{Sp}_{\delta}(\delta_u(y)) \subset$ (the support of u) $\cap\, \mathrm{Sp}_{\delta}(y)$, hence we get $y = 0$. By (1.4) we have

$$\int_{\Gamma} \xi(\gamma)\eta(\gamma)\hat{\alpha}_{\gamma}(x)\,d\gamma = 0\,,$$

which implies $g_0 \notin \mathrm{Sp}_{\hat{\alpha}}(x)$ that is $\mathrm{Sp}_{\hat{\alpha}}(x) \subset V$. As $g \in \mathrm{Sp}(\hat{\alpha})$ if and only if $A^{\hat{\alpha}}(V) \neq \{0\}$ for each compact neighbourhood V of g (See [9]), by Lemma 1.2, we have $\mathrm{Sp}(\hat{\alpha}) = -\mathrm{Sp}(\delta)$. We conclude that $\Gamma(\hat{\alpha}) = \Gamma(\delta)$.

2. C*-dynamical system with the action of a compact group.

Throughout this section, we assume that G is compact and dg is the normalized Haar measure on G. Let (A, G, α) be a dynamical system and $\hat{G}$ be the space of isomorphism classes of all irreducible representation of G. If $\pi \in \hat{G}$, we denote by χ_π the associated "modified character" $\chi_\pi(g) = (\dim \pi)^{-1} \mathrm{Tr}(\pi(g))$, and $u(i, j, \pi)$ the associated "coordinate functions" $u(i, j, \pi)(g) = \langle \pi(g)\xi_i, \xi_j \rangle$, where $\{\xi_i\}$ is a normalized orthogonal basis for H_π. By definition $\pi \in \mathrm{Sp}(\alpha)$, iff $\alpha(\chi_\pi)(A) \neq \{0\}$, where

$$\alpha(\chi_\pi)(a) = \int_G \chi_\pi(g)\alpha_g(a)\,dg \qquad \text{for } a \in A ,$$

([2]), and $\Gamma(\alpha) = \bigcap \{\mathrm{Sp}(\alpha \mid B);\ B \in \mathscr{H}^\alpha(A)\}$.

LEMMA 2.1. *If $\pi \notin \Gamma(\alpha)$, then there is a non-zero closed ideal I of $G \times_\alpha A$ such that*

$$\bigvee_{i,j} \delta_{u(i,j,\pi)}(I) \cap I = \{0\} ,$$

where $\bigvee_{i,j} \delta_{u(i,j,\pi)}(I)$ denotes the closed ideal of $G \times_\alpha A$ generated by $\delta_{u(i,j,\pi)}(I)$, $i, j = 1, 2, \ldots, \dim \pi$.

PROOF. If $\pi \notin \Gamma(\delta)$, there is a $B \in \mathscr{H}^\alpha(A)$ such that $\alpha(\chi_\pi)(B) = \{0\}$. Take a non-zero G-invariant positive element b of B and put a non-zero element $y = \int_G \iota(b)v(g)\,dg \in G \times_\alpha A$, then we have, for $a \in A$, $g \in G$,

$$\begin{aligned}
&\delta_{u(i,j,\pi)}(y)\iota(a)v(g)y \\
&= \iint_{G\times G} u(i,j,\pi)(h)\iota(b\alpha_h(a)b)v(hgk)\,dk\,dh \\
&= \int_G \iota\left(\int_G u(i,j,\pi)(h)\alpha_h(bab)\,dh\right)v(k)\,dk ,
\end{aligned}$$

put

$$z = \int_G u(i,j,\pi)(h)\alpha_h(bab)\,dh .$$

Since B is hereditary, the element bab is in B. Therefore we have $\alpha(\chi_\pi)(bab) = 0$.

By the relation $\alpha(u(i,j,\pi))[(u(k,l,\pi))(c)]=\alpha(u(i,j,\pi)*u(k,l,\pi))(c)$ for any c in A and the orthogonality relations for compact groups, $\alpha(\chi_\pi)(c)=0$ is equivalent to

$$\int_G u(i,j,\pi)(h)\alpha_h(c)\,dh = 0, \quad \text{for } i,j=1,2,\ldots,\dim\pi .$$

Therefore $z=0$ and so $\delta_{u(i,j,\pi)}(y)\iota(a)v(g)y=0$, that is

$$\delta_{u(i,j,\pi)}(y)G\times_\alpha Ay = \{0\}, \quad \text{for } i,j=1,2,\ldots,\dim\pi .$$

Let I be the non-zero closed ideal of $G\times_\alpha A$ generated by y. By easy calculation, we have $\bigvee_{i,j}\delta_{u(i,j,\pi)}(I)\cap I=\{0\}$.

We use the definition of the crossed product $G\times_\delta(G\times_\alpha A)$ with the co-action δ, the dual action $\hat{\delta}$ of δ, and Takesaki's duality (See [5], [6], and [8]).

LEMMA 2.2. *If there is a non-zero closed ideal I of $G\times_\alpha A$ such that $\bigvee_{i,j}\delta_{u(i,j,\pi)}(I)\cap I=\{0\}$, then π does not belong to $\Gamma(\hat{\hat{\alpha}})$ where $\hat{\hat{\alpha}}$ is the bidual action of* α (See [8] or [6]).

PROOF. Take a non-zero positive element y in I. For $\sigma\in\hat{G}$, put $z=\delta(y)(1\otimes u(i,j,\sigma))\delta(x)\delta(y)$ for $x\in G\times_\alpha A$. We then have

$$\begin{aligned}
&\int_G \chi_\pi(g)\hat{\hat{\alpha}}_g(z)\,dg\\
&= \int_G \chi_\pi(g)\delta(y)\hat{\hat{\alpha}}_g(1\otimes u(i,j,\sigma))\delta(x)\delta(y)\,dg\\
&= \int_G \chi_\pi(g)\delta(y)\sum_{k,m=1}^{\dim\sigma}\overline{u(k,i,\sigma)(g)}\,\delta(\delta_{u(m,j,\sigma)}(xy))(1\otimes u(k,m,\sigma))\,dg\\
&= \sum_{k,m=1}^{\dim\sigma}\int_G \chi_\pi(g)\overline{u(k,i,\sigma)(g)}\,dg\,\delta(y\delta_{u(m,j,\sigma)}(xy))(1\otimes u(k,m,\sigma))\\
&= \begin{cases} 0, & \text{when } \pi\neq\sigma, \text{ by [4, Theorem 27.19]},\\ 0, & \text{when } \pi=\sigma, \text{ by } y\delta_{u(m,j,\sigma)}(xy)=0 .\end{cases}
\end{aligned}$$

Then we get $\int_G\chi_\pi(g)\hat{\hat{\alpha}}_g(\delta(y)a\delta(y))\,dg=0$ for $a\in G\times_\delta(G\times_\alpha A)$ because the vector space generated by $\{(1\otimes u(i,j,\sigma))\delta(x);\ x\in G\times_\alpha A,\ \sigma\in\hat{G}\}$ is norm-dense in $G\times_\delta(G\times_\alpha A)$. Let B be the non-zero hereditary C*-subalgebra of $G\times_\delta(G\times_\alpha A)$ generated by $\delta(y)$. Then B is $\hat{\hat{\alpha}}_g$-invariant for each $g\in G$ because $\hat{\hat{\alpha}}_g(\delta(y))=\delta(y)$, therefore

$$\int_G \chi_\pi(g)\hat{\hat{\alpha}}_g(B)\,dg = \{0\} ,$$

which implies $\pi\notin\Gamma(\hat{\hat{\alpha}})$.

THEOREM 2.3. *Let (A, G, α) be a C^*-dynamical system, then*

$$\Gamma(\hat{\hat{\alpha}}) = \{\pi \in \hat{G}, \bigvee_{i,j} \delta_{u(i,j,\pi)}(I) \cap I \neq \{0\}, \text{ for each non-zero ideal } I \text{ of } G \times_\alpha A\}$$

and $\Gamma(\alpha) \supset \Gamma(\hat{\hat{\alpha}})$.

PROOF. By Takesaki's duality (See [6]), $G \times_{\hat{\alpha}} (G \times_\delta (G \times_\alpha A))$ is isomorphic to $(G \times_\alpha A) \otimes C(L^2(G))$, therefore each closed ideal I' of $(G \times_\alpha A) \otimes C(L^2(G))$ is of the form $I \otimes C(L^2(G))$, where I is a closed ideal of $G \times_\alpha A$, moreover

$$\bigvee_{i,j} \hat{\hat{\delta}}_{u(i,j,\pi)}(I') = \bigvee_{i,j} \delta_{u(i,j,\pi)}(I) \otimes C(L^2(G)) .$$

Hence, if $\pi \notin \Gamma(\hat{\hat{\alpha}})$, we have a non-zero closed ideal I of $G \times_\alpha A$ such that

$$\bigvee_{i,j} \delta_{u(i,j,\pi)}(I) \cap I = \{0\}$$

by Lemma 2.1.

REMARK 2.4. Let (A, G, α) and (A, G, β) be a C^*-dynamical system. If α are exterior equivalent to β (See [10, 4.2]), then $\Gamma(\hat{\hat{\alpha}}) = \Gamma(\hat{\hat{\beta}})$, but $\Gamma(\alpha)$ is not always equal to $\Gamma(\beta)$ (See [2]).

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