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POSITIVE C_0-SEMIGROUPS ON C*-ALGEBRAS

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Abstract.

We derive various characterizations of the generators of C_0-semigroups of positive operators acting on a C*-algebra $\mathscr{A}$. Subsequently we characterize the generators for which the semigroup τ has the property

$$\tau_t(A^*A) \geqq \tau_t(A)^*\tau_t(A), \qquad A \in \mathscr{A},\ t>0\,,$$

and then we show that this property is also shared by the subordinate semigroups τ^f.

0. Introduction.

In this note we derive various characterizations of generators of C_0-semigroups, i.e., strongly continuous semigroups, acting on a C*-algebra $\mathscr{A}$. In particular we examine positive semigroups, i.e., semigroups which leave the cone $\mathscr{A}_+$ of positive elements of $\mathscr{A}$ invariant. Many examples of such semigroups exist on abelian algebras of continuous functions. For example the heat equation on $\mathbf{R}^\nu$ gives rise to the semigroup

$$(\tau_t f)(x) = (e^{t\nabla^2}f)(x) = (4\pi t)^{-\nu/2}\int d^\nu y e^{-(x-y)^2/4t}f(y)$$

on $C_0(\mathbf{R}^\nu)$. This semigroups is not only positive but also contractive.

A large class of positive contraction semigroups can be constructed by functional analysis of groups of *-automorphisms. If σ is a C_0-group of *-automorphisms of $\mathscr{A}$ with generator δ, then

$$t \geqq 0 \to \tau_t = (4\pi t)^{-\frac{1}{2}}\int_{-\infty}^{\infty} d\lambda e^{-\lambda^2/4t}\sigma_\lambda$$

defines a C_0-semigroup of positive contractions with generator δ^2. The heat equation semigroup on $C_0(\mathbf{R}^\nu)$ arises by this construction from the group of translations. In fact semigroups constructed in this manner are strongly positive in the sense that

$$(*) \qquad \tau_t(A^*A) \geqq \tau_t(A)^*\tau_t(A), \qquad A \in \mathscr{A},\ t \in \mathbf{R}\,.$$

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To see this we note thar

$$\begin{aligned}\tau_t(A^*A)-\tau_t(A)^*\tau_t(A) &= (8\pi t)^{-1}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu e^{-(\lambda^2+\mu^2)/4t}\circ \\ &\circ\{\sigma_\lambda(A^*A)+\sigma_\mu(A^*A)-\sigma_\lambda(A)^*\sigma_\mu(A) \\ &-\sigma_\mu(A)^*\sigma_\lambda(A)\} \\ &= (8\pi t)^{-1}\int_{-\infty}^{\infty} d\lambda \int_{-\infty}^{\infty} d\mu e^{-(\lambda^2+\mu^2)/4t}\circ \\ &\circ(\sigma_\lambda(A)-\sigma_\mu(A))^*(\sigma_\lambda(A)-\sigma_\mu(A)) \\ &\geqq 0\,.\end{aligned}$$

Note also that the domain of the densely defined generator δ^2 of τ is automatically a *-subalgebra of $\mathscr{A}$. This follows from the derivation property of δ. It should, however, be emphasized that the strong positivity and the algebraic property of the domain of the generator are not general features shared by all positive contraction semigroups.

We consider three situations.

First we characterize generators of contraction semigroups. Our result is a slight algebraic reformulation of the Lumer–Phillips theorem [10] on dissipative operators. We have included it for comparison with the second result, a characterization of the generators of positive contraction semigroups.

This second characterization is based on Phillips' notion [13] of a dispersive operator. Phillips examined the generator question on a Banach lattice and we present an analogous disucssion in the algebraic setting. It should be emphasized that the order structure in the two cases is quite distinct; the positive elements of a C*-algebra $\mathscr{A}$ form a lattice if and only if $\mathscr{A}$ is abelian.

Third we examine semigroups which are strongly positive in the sense that they satisfy the inequalities (*). Kadison [7] showed that these inequalities are valid for a positive map of norm one whenever A is normal. Nevertheless there are positive semigroups and non-normal A for which the inequality fails. (An example is quoted by Evans and Hanche-Olsen in [6].)

Finally we note that if $\tau_t=\exp\{t\delta\}$ is strongly positive the same thing is true for the subordinate semigroup $\tau_t^f=\exp\{tf(\delta)\}$ where $x\in[0,\infty)\to -f(-x)$ is a Bernstein function.

1. Contraction semigroups.

Let X denote a Banach space and X^* its dual. A normalized tangent functional at the point $x \in X$ is defined as an element $f_x \in X^*$ satisfying

$$\|f_x\| = 1 \quad \text{and} \quad f_x(x) = \|x\|\,.$$

The Hahn–Banach theorem establishes the existence of at least one such functional for each $x \in X$.

We will need the following well known special property for tangent functionals of C*-algebras.

LEMMA 1. *Let $A \geqq 0$ be a positive element of the* C*-*algebra $\mathscr{A}$ and let ω_A be a normalized tangent functional at the point A.*

It follows that ω_A is a state over $\mathscr{A}$.

PROOF. Let E_α be an approximate identity of $\mathscr{A}$. (If $\mathscr{A}$ has a identity **1** one can take $E_\alpha = \mathbf{1}$ and the following proof simplifies accordingly.) It suffices to prove that for some subnet

$$\lim_\alpha \omega_A(E_\alpha^2) = 1$$

(see, for example, [3, Proposition 2.3.11]). Now assume

$$\omega_A(E_\alpha^2) = a_\alpha + ib_\alpha .$$

Since $\|E_\alpha\| \leqq 1$ one has

$$a_\alpha^2 + b_\alpha^2 \leqq 1 .$$

Hence by passing to a subnet we can assume that a_α and b_α converge to a and b respectively and then

$$(*) \qquad a^2 + b^2 \leqq 1 .$$

Now since A is positive

$$\|E_\alpha^2 - 2E_\alpha^2 A/\|A\|\| \leqq 1 .$$

(This can be deduced by addition of an identity **1** and then noting that $\mathbf{1} - 2A/\|A\|$ has spectrum in $[-1, 1]$.) But since ω_A is a tangent functional at A

$$\lim_\alpha \omega_A(E_\alpha^2 - 2E_\alpha^2 A/\|A\|) = a + ib - 2 .$$

Consequently

$$(**) \qquad (a-2)^2 + b^2 \leqq 1 .$$

But (*) and (**) imply that $a = 1$ and $b = 0$.

Throughout the sequel we use $|A|$ to denote the modulus of an element A of a C*-algebra $\mathscr{A}$, that is $|A| = (A^*A)^{\frac{1}{2}}$, and if $A = A^*$ we use $A_\pm$ to denote its positive and negative parts, that is $A_\pm = (|A| \pm A)/2$. Furthermore ω_B will denote some normalized tangent functional at the point $B \in \mathscr{A}$.

The following theorem is a version of a well known result of Lumer and Phillips [10].

THEOREM 2. *Let δ be an operator on a* C*-*algebra $\mathscr{A}$. The following conditions are equivalent*

1. *δ is the generator of a C_0-semigroup τ of contractions.*
2. *δ is densely defined,*

$$R(\iota - \alpha\delta) = \mathscr{A}, \qquad \alpha > 0,$$

and **either**

2a. $$\omega_{|A|}(\delta(A)^* A + A^* \delta(A)) \leqq 0, \qquad A \in D(\delta),$$

or

2b. $$\operatorname{Re} \omega_A(\delta(A)) \leqq 0, \qquad A \in D(\delta).$$

PROOF. The equivalence $1 \Leftrightarrow 2b$ is the Lumer–Phillips theorem. We concentrate on proving $1 \Leftrightarrow 2a$.

$1 \Rightarrow 2a$. The first statements follow as in the Hille–Yosida theorem. The last statement follows by noting that $\omega_{|A|}$ is a state, and τ is contractive. Hence

$$\omega_{|A|}(\tau_t(A)^* \tau_t(A)) \leqq \|\tau_t(A)\|^2 \leqq \|A\|^2 .$$

But using the Cauchy–Schwarz inequality one finds

$$\|A\|^2 = \| |A| \|^2 = \omega_{|A|}(|A|)^2 \leqq \omega_{|A|}(|A|^2) = \omega_{|A|}(A^* A) \leqq \|A\|^2 .$$

Therefore $\omega_{|A|}(A^* A) = \|A\|^2$ and

$$\omega_{|A|}(\tau_t(A)^* \tau_t(A)) - \omega_{|A|}(A^* A) \leqq 0 .$$

Dividing by t and taking the limit $t \to 0$ gives the desired result.

$2 \Rightarrow 1$; Given $A \in \mathscr{A}$ there is a $B \in D(\delta)$ such that $A = (\iota - \varepsilon\delta)(B)$. This follows from the assumption $R(\iota - \varepsilon\delta) = \mathscr{A}$. But then

$$\begin{aligned} \|B\|^2 &= \omega_{|B|}(|B|)^2 \\ &\leqq \omega_{|B|}(B^* B) \\ &\leqq \omega_{|B|}(B^* B - \varepsilon\delta(B)^* B - \varepsilon B^* \delta(B)) \\ &\leqq \omega_{|B|}(A^* A) \leqq \|A\|^2 . \end{aligned}$$

Consequently $(\iota - \varepsilon\delta)^{-1}$ is a bounded operator with $\|(\iota - \varepsilon\delta)\| \leqq 1$. But since

$$\tau_t(A) = \lim_{n \to \infty} \left(\iota - \frac{t}{n}\delta\right)^{-n} (A),$$

one concludes that $\|\tau_t(A)\| \leqq 1$.

2. Positive contraction semigroups.

In the previous section we characterized generators of C_0-semigroups of contractions and in this section we examine the similar problem for semigroups of positive operators, i.e. opeators which map the positive elements $\mathscr{A}_+$ into $\mathscr{A}_+$. The conditions of Theorem 2 reexpressed the notion of dissipativity introduced by Lumer and Phillips [10]. In the following theorem the analogous conditions are related to Phillips' notion [13] of a dispersive operator on a Banach lattice.

THEOREM 3. *Let δ be a densely defined operator on a* C*-*algebra whose domain $D(\delta)$ is closed under the star operation and which is symmetric,* i.e. $\delta(A)^* = \delta(A^*)$ *for all $A \in D(\delta)$. The following conditions are equivalent*

1. *δ is the generator of a C_0-semigroup τ of positive contractions.*
2. *$(\iota - \alpha\delta)^{-1}$ is a positive contraction operator for all $\alpha > 0$.*
3. *δ is densely defined,*

$$R(\iota - \alpha\delta) = \mathscr{A}, \qquad \alpha > 0\,,$$

and **either**

3a. $$\omega_{A_+}(\delta(A)A + A\delta(A)) \leqq 0, \qquad A = A^* \in D(\delta),\ A_+ \neq 0\,,$$

or

3b. $$\omega_{A_+}(\delta(A)) \leqq 0, \qquad A = A^* \in D(\delta),\ A_+ \neq 0\,,$$

or

3c. $$B^*\delta(A)B \leqq 0, \qquad A = A^* \in D(\delta)$$

whenever $B \in \mathscr{A}^{**}$ satisfies $AB = \|A_+\|B$

or

3d. $$P_A\delta(A)P_A \leqq 0, \qquad A = A^* \in D(\delta)\,,$$

where $P_A \in \mathscr{A}^{**}$ is the maximal projection such that $AP_A = \|A_+\|P_A$.

PROOF. $1 \Leftrightarrow 2$; This is a standard result which follows from

$$(\iota - \alpha\delta)^{-1}(A) = \int_0^\infty dt e^{-t}\tau_{\alpha t}(A), \qquad A \in \mathscr{A}$$

and

$$\tau_t(A) = \lim_{n\to\infty}\left(\iota - \frac{t}{n}\delta\right)^{-n}(A), \qquad A \in \mathscr{A}\,.$$

$1 \Rightarrow 3$; The fact that the generator of a contraction semigroup is densely defined and satisfies $R(\imath - \alpha\delta) = \mathscr{A}$ is part of the Hille–Yosida theorem.

$1 \Rightarrow 3b$; If $A = A^* \in D(\delta)$ and ω_{A_+} is a normalized tangent functional at A_+ it follows that the restriction of ω_{A_+} to the C*-algebra generated by A is a point measure on $\sup(\mathrm{Sp}\, A) = \sup(\mathrm{Sp}\, A_+)$, and hence $\omega_{A_+}(A_-) = 0$. Thus

$$\omega_{A_+}(A) = \omega_{A_+}(A_+) = \|A_+\| .$$

But then

$$\begin{aligned}\omega_{A_+}(\tau_t(A)) &\leqq \omega_{A_+}(\tau_t(A_+)) \\ &\leqq \|A_+\| = \omega_{A_+}(A) .\end{aligned}$$

Hence

$$\omega_{A_+}(\delta(A)) = \lim_{t \to 0} \omega_{A_+}(\tau_t(A) - A)/t \leqq 0 .$$

$3b \Rightarrow 2$; For $A \in \mathscr{A}_+$ choose $B = B^* \in D(\delta)$ such that $(\imath - \alpha\delta)(B) = A$ and let $\omega_{(-B)_+}$ be a normalized tangent functional at $(-B)_+$. Once again

$$\omega_{(-B)_+}(-B) = \omega_{(-B)_+}((-B)_+) = \|B_-\| ,$$

and

$$\omega_{(-B)_+}(\delta(-B)) \leqq 0$$

by assumption. Thus

$$\begin{aligned}\|B_-\| &= -\omega_{(-B)_+}(B) \\ &\leqq -\omega_{(-B)_+}((\imath - \alpha\delta)(B)) \\ &= -\omega_{(-B)_+}(A) \leqq 0 ,\end{aligned}$$

because $\omega_{(-B)_+}$ is a state by Lemma 1. Consequently $B_- = 0$ and $B \geqq 0$. Next let ω_B be a normalized tangent functional at B. Since $B \geqq 0$, we have

$$\omega_B(\delta(B)) \leqq 0$$

by assumption. Therefore

$$\begin{aligned}\|B\| &= \omega_B(B) \\ &\leqq \omega_B((\imath - \alpha\delta)(B)) \\ &= \omega_B(A) \leqq \|A\| .\end{aligned}$$

We have thus shown that if $A \geqq 0$ and $B = B^*$ is an element such that $(\imath - \alpha\delta)(B)$ $(\imath - \alpha\delta)(B) = A$, then $B \geqq 0$ and $\|B\| \leqq \|A\|$. Thus if $B_1, B_2 \in \mathscr{A}_{\mathrm{sa}}$ (the self adjoint elements in $\mathscr{A}$) are elements with

$$(\iota-\alpha\delta)(B_1) = (\iota-\alpha\delta)(B_2),$$

then

$$(\iota-\alpha\delta)(B_1-B_2) = (\iota-\alpha\delta)(B_2-B_1) = 0,$$

and hence $B_1-B_2 \geqq 0$, $B_2-B_1 \geqq 0$, that is $B_1=B_2$, and $(\iota-\alpha\delta)$ is injective on $\mathscr{A}_{sa}$. It is surjective since $\mathscr{A}_+ \subseteqq (\iota-\alpha\delta)(D(\delta)_{sa})$, and hence $(\iota-\alpha\delta)^{-1}$ exists as an operator on $\mathscr{A}_{sa}$. But as $(\iota-\alpha\delta)$ commutes with the involution it follows that $(\iota-\alpha\delta)^{-1}$ exists on $\mathscr{A}$, and from the positivity of $(\iota-\alpha\delta)^{-1}$ we find

$$\begin{aligned}\|(\iota-\alpha\delta)^{-1}\| &= \sup_{A\in\mathscr{A}} \|(\iota-\alpha\delta)^{-1}A\|/\|A\| \\ &\leqq \sup_{A\in\mathscr{A}} \|(\iota-\alpha\delta)^{-1}|A|\|/\||A|\| \\ &\leqq 1.\end{aligned}$$

The first inequality above follows from the fact that the norm $\|\varphi\|$ of a positive linear map φ equals

$$\|\varphi\| = \sup\{\|\varphi(A)\|;\ A\geqq 0,\ \|A\|\leqq 1\}.$$

This is known if $\mathscr{A}$ has an identity $\mathbf{1}$ and $\varphi(\mathbf{1})=\mathbf{1}$, see [3, Corollary 3.2.6] or [14], but the next lemma implies it is true in more general cases. We are indepted to T. Digernes, C. F. Skau, and R. Wong for a discussion of this point.

LEMMA. *Let φ be a linear positive map from a C*-algebra $\mathscr{A}$ into a C*-algebra $\mathscr{B}$.*

It follows that φ is bounded, and if $\{E_\alpha\}$ is an approximate identity for $\mathscr{A}$, then

$$\|\varphi\| = \lim_\alpha \|\varphi(E_\alpha)\|$$

PROOF: If φ had been unbounded, there would exist a sequence A_n of positive elements in $\mathscr{A}$ such that $\|A_n\|\leqq 1$ and $\|\varphi(A_n)\|\geqq n^3$. Define

$$A = \sum_{n=1}^{\infty} \frac{1}{n^2} A_n.$$

Then $A \geqq (n^2)^{-1}A_n \geqq 0$, hence $\varphi(A) \geqq (n^2)^{-1}\varphi(A_n) \geqq 0$ and so

$$\|\varphi(A)\| \geqq \frac{1}{n^2}\|\varphi(A_n)\| \geqq n, \quad \text{for all } n.$$

But this contradicts $\|\varphi(A)\| < +\infty$.

Let $\mathscr{A}^{**}$ and $\mathscr{B}^{**}$ be the von Neumann enveloping algebras of $\mathscr{A}$ and $\mathscr{B}$ and let φ also denote the normal extension of φ to $\mathscr{A}^{**}$. It follows from

Kaplansky's density theorem, [3], that the extension has the same norm as the original φ. Furthermore, by [14] we have $\|\varphi\| = \|\varphi(\mathbf{1})\|$.

If $\{E_\alpha\}$ is an approximate identity for $\mathscr{A}$, then $E_\alpha \to \mathbf{1}$ in the w^*-topology in $\mathscr{A}^{**}$, and hence $\varphi(E_\alpha) \to \varphi(\mathbf{1})$ in the w^*-topology.

As $\varphi(E_\alpha)$ is increasing it follows that

$$\|\varphi(E_\alpha)\| \to \|\varphi(\mathbf{1})\| = \|\varphi\| .$$

We now resume the proof of Theorem 3:

3a $\Leftrightarrow$ 3b; Since

$$\omega_{A_+}(A) = \omega_{A_+}(A_+) = \|A_+\| ,$$

it follows that

$$\pi(A)\Omega = \|A_+\|\Omega ,$$

where $(\mathscr{H}, \pi, \Omega)$ is the cyclic representation associated with ω_{A_+}. Therefore

$$\omega_{A_+}(\delta(A)A + A\delta(A)) = 2\|A_+\|\omega(\delta(A)) ,$$

and the equivalence is immediate.

3b $\Rightarrow$ 3c; Let ω be a state on $\mathscr{A}$ and let ω also denote its normal extension to the bidual $\mathscr{A}^{**}$. Assume $A = A^* \in D(\delta)$ and let $B \in \mathscr{A}^{**}$ be such that

$$AB = \|A_+\|B .$$

Define a positive linear functional ω^B on $\mathscr{A}$ by

$$\omega^B(C) = \omega(B^*CB), \quad C \in \mathscr{A} .$$

Then

$$\begin{aligned} \omega^B(A) &= \omega(B^*AB) \\ &= \|A_+\|\omega(B^*B) = \|A_+\| \, \|\omega^B\| \end{aligned}$$

Hence

$$\omega^B(\delta(A)) \leqq 0$$

by 3b. But this means that

$$\omega(B^*\delta(A)B) \leqq 0 ,$$

for all normal states ω on $\mathscr{A}^{**}$, and hence

$$B^*\delta(A)B \leqq 0 .$$

3c $\Rightarrow$ 3d; This follows by choosing $B = P_A$.

3d $\Rightarrow$ 3b; If $A=A^* \in D(\delta)$ and ω_{A_+} is a normalized tangent functional at A_+, then

$$\omega_{A_+}(P_A) = 1$$

and hence $\omega_{A_+}(1-P_A)=0$. It follows that

$$\omega_{A_+}(C) = \omega_{A_+}(P_A C P_A),$$

for all $C \in \mathscr{A}$ by the Cauchy–Schwarz inequality. Consequently

$$\omega_{A_+}(\delta(A)) = \omega_{A_+}(P_A \delta(A) P_A) \leqq 0.$$

Next we derive a set of characterizations of generators of positive semigroups which are similar to the results obtained for uniformly continuous semigroups by Evans and Hanche-Olsen [6]. Their conditions were either

$$\delta(A^2)+A\delta(1)A \geqq \delta(A)A+A\delta(A),$$

for all $A=A^* \in \mathscr{A}$, or

$$\delta(1)+U^*\delta(1)U \geqq \delta(U^*)U+U^*\delta(U),$$

for all unitaries U in $\mathscr{A}$. These can be viewed as the first order terms in our conditions 3–6.

THEOREM 4. *Let $\mathscr{A}$ be a C*-algebra with identity* 1, *and τ a C_0-semigroup on $\mathscr{A}$ with generator δ such that $\tau_t(A)^*=\tau_t(A^*)$ for all $A \in \mathscr{A}$ and $t \geqq 0$. The following conditions are equivalent*

1. *The semigroup τ_t is positive for all $t \geqq 0$.*
2. *The resolvent $(\iota-\varepsilon\delta)^{-1}$ is positive for all small $\varepsilon>0$.*
3. $(\iota-\varepsilon\delta)^{-1}(A^2)+A(\iota-\varepsilon\delta)^{-1}(1)A \geqq (\iota-\varepsilon\delta)^{-1}(A)A+A(\iota-\varepsilon\delta)^{-1}(A)$ *for all $A=A^* \in \mathscr{A}$, and all small $\varepsilon>0$.*
4. $(\iota-\varepsilon\delta)^{-1}(1)+U^*(\iota-\varepsilon\delta)^{-1}(1)U \geqq (\iota-\varepsilon\delta)^{-1}(U^*)U+U^*(\iota-\varepsilon\delta)^{-1}(U)$ *for all unitaries $U \in \mathscr{A}$, and all small $\varepsilon>0$.*
5. $\tau_\varepsilon(A^2)+A\tau_\varepsilon(1)A \geqq \tau_\varepsilon(A)A+A\tau_\varepsilon(A)$ *for all $A=A^* \in \mathscr{A}$, and all small $\varepsilon>0$.*
6. $\tau_\varepsilon(1)+U^*\tau_\varepsilon(1)U \geqq \tau_\varepsilon(U^*)U+U^*\tau_\varepsilon(U)$ *for all unitaries $U \in \mathscr{A}$, and all small $\varepsilon>0$.*

PROOF. The equivalence 1 $\Leftrightarrow$ 2 is proved as in Theorem 3.

If δ is bounded it is known, [6], that $\tau_t=e^{t\delta}$ is positive if, and only if, one of the following two equivalent conditions is fulfilled

a. $\delta(A^2)+A\delta(1)A \geqq \delta(A)A+A\delta(A)$, for all $A=A^* \in \mathscr{A}$.
b. $\delta(1)+U^*\delta(1)U \geqq \delta(U^*)U+U^*\delta(U)$, for all unitaries U in $\mathscr{A}$.

Hence, to establish the theorem, it is enough to show that 1 or 2 is equivalent to each of the following two conditions

7. $e^{t(\iota-\varepsilon\delta)^{-1}} \geqq 0$ for all $t \geqq 0$ and all small $\varepsilon > 0$.
8. $e^{te^{\varepsilon\delta}} \geqq 0$ for all $t \geqq 0$ and all small $\varepsilon > 0$.

We have to show four implications.

2 ⇒ 7. This is evident from the expansion

$$e^{t(\iota-\varepsilon\delta)^{-1}} = \sum_{n=0}^{\infty} \frac{t^n}{n!}(\iota-\varepsilon\delta)^{-n} .$$

7 ⇒ 1. We have

$$e^{t\delta(\iota-\varepsilon\delta)^{-1}} = e^{t(-(\iota/\varepsilon)+(1/\varepsilon)(\iota-\varepsilon\delta)^{-1})} = e^{-t/\varepsilon}e^{(t/\varepsilon)(\iota-\varepsilon\delta)^{-1}} ,$$

and hence $e^{t\delta(\iota-\varepsilon\delta)^{-1}}$ is positive whenever $e^{t(\iota-\varepsilon\delta)^{-1}}$ is positive. But

$$e^{t\delta}(A) = \lim_{\varepsilon\to 0} e^{t\delta(\iota-\varepsilon\delta)^{-1}}(A) ,$$

for all $A \in \mathscr{A}$ by [3, Theorem 3.1.10]. Hence $\tau_t = e^{t\delta}$ is positive.

1 ⇒ 8. This is evident from the expansion

$$e^{te^{\varepsilon\delta}} = \sum_{n=0}^{\infty} \frac{t^n}{n!} e^{n\varepsilon\delta} .$$

8 ⇒ 1. We have

$$e^{t((e^{\varepsilon\delta}-\iota)/\varepsilon)} = e^{-(t/\varepsilon)}e^{(t/\varepsilon)e^{\varepsilon\delta}} ,$$

and hence $e^{t((e^{\varepsilon\delta}-\iota)/\varepsilon)}$ is positive whenever $e^{te^{\varepsilon\delta}}$ is positive. But

$$\delta(A) = \lim_{\varepsilon\to 0}\left(\frac{e^{\varepsilon\delta}-\iota}{\varepsilon}\right)(A) ,$$

for $A \in D(\delta)$, and it follows that

$$e^{t\delta}(A) = \lim_{\varepsilon\to 0} e^{t((e^{\varepsilon\delta}-\iota)/\varepsilon)}(A) ,$$

for all $A \in \mathscr{A}$ by [3, Theorem 3.1.28]. Hence $\tau_t = e^{t\delta}$ is positive.

3. Strongly positive semigroups.

Next we examine semigroups that are strongly positive in the sense that

$$\tau_t(A^*A) \geqq \tau_t(A)^*\tau_t(A), \quad A \in \mathscr{A},\ t > 0 .$$

Such semigroups are automatically contractive. For example if $\mathscr{A}$ possesses an identity **1**, then it follows from positivity that

$$\|\tau_t\| = \|\tau_t(\mathbf{1})\| .$$

But by strong positivity

$$\tau_t(\mathbf{1}) \geqq \tau_t(\mathbf{1})^*\tau_t(\mathbf{1}) \geqq 0 ,$$

and hence

$$\|\tau_t(\mathbf{1})\| \geqq \|\tau_t(\mathbf{1})\|^2 .$$

Thus

$$\|\tau_t\| \leqq 1 .$$

Lindblad [9], Evans [5], and Evans and Hanche-Olsen [6] have studied strongly positive semigroups with bounded generators δ. In particular it is shown in [6] that strong positivity of τ is equivalent to the dissipation property

$$\delta(A^*A) \geqq \delta(A^*)A + A^*\delta(A), \qquad A \in \mathscr{A} ,$$

whenever δ is bounded. No direct analogy of this result is generally possible in the case of unbounded δ for domain reasons. To formulate the dissipation property for unbounded δ one would require at least $A \in D(\delta)$ implies $A^*A \in D(\delta)$. But by polarization this would mean that $D(\delta)$ is an algebra. This is often not the case. Counter examples are easily constructed from the heat equation on a finite region $\Lambda \subset \mathbf{R}^\nu$ with smooth boundary $\partial\Lambda$. If we consider the Laplacian ∇^2 with boundary condition $\partial f/\partial n = \sigma f$ on $\partial\Lambda$, then this generates a strongly positive semigroup on $C(\Lambda)$ by Kadison's result which we mentioned in the introduction. But since $\partial f/\partial n = \sigma f$ and $\partial g/\partial n = \sigma g$ imply that $\partial fg/\partial n = 2\sigma fg$, the domain of ∇^2 is not an algebra for non-zero σ. If, however, $\sigma = 0$, then the domain has a core which is an algebra. Finally if we consider Dirichlet boundary conditions $f = 0$ on $\partial\Lambda$, then the domain is in fact an algebra. Thus all possibilities occur.

We avoid this domain difficulty by using functional analysis of generators. Thus we pass from $\tau_t = \exp\{t\delta\}$ to the semigroup $\tau_t^\varepsilon = \exp\{t(e^{\varepsilon\delta} - \iota)/\varepsilon\}$ or the semigroup $\tau_t^\alpha = \exp\{t\delta(\iota - \alpha\delta)^{-1}\}$ and then pass back to τ by taking the limit $\varepsilon \to 0$ or $\alpha \to 0$. By this artifice we can exploit the results of [6], [9], and [5].

THEOREM 5. *Let $\mathscr{A}$ be a C*-algebra, and let τ be a C_0-semigroup on $\mathscr{A}$ with generator δ such that $\tau_t(A^*) = \tau_t(A)^*$ for all $A \in \mathscr{A}$.*

The following conditions are equivalent

1. $\tau_t(A^*A) \geqq \tau_t(A)^*\tau_t(A)$ *for all $A \in \mathscr{A}$ and all $t \geqq 0$.*
2. $(\iota - \varepsilon\delta)^{-1}(A^*A) \geqq (\iota - \varepsilon\delta)^{-1}(A^*)(\iota - \varepsilon\delta)^{-1}(A)$ *for all $A \in \mathscr{A}$ and all small $\varepsilon > 0$.*
3. $\tau_t(A^*A) + A^*A \geqq \tau_t(A^*)A + A^*\tau_t(A)$ *for all $A \in \mathscr{A}$ and all small $t > 0$.*
4. $(\iota - \varepsilon\delta)^{-1}(A^*A) + A^*A \geqq (\iota - \varepsilon\delta)^{-1}(A^*)A + A^*(\iota - \varepsilon\delta)^{-1}(A)$ *for all $A \in \mathscr{A}$ and all small $\varepsilon > 0$.*

PROOF. When δ is bounded, condition 1 is equivalent to

5. $\delta(A^*A) \geqq \delta(A^*)A + A^*\delta(A)$, see [6, Corollary 3].

It follows immediately that 3 $\Leftrightarrow$ 6 and 4 $\Leftrightarrow$ 7, where

6. $e^{t(e^{\varepsilon\delta}-\iota)/\varepsilon}(A^*A) \geqq e^{t(e^{\varepsilon\delta}-\iota)/\varepsilon}(A^*)e^{t(e^{\varepsilon\delta}-\iota)/\varepsilon}(A)$

and

7. $e^{t\delta(\iota-\varepsilon\delta)^{-1}}(A^*A) \geqq e^{t\delta(\iota-\varepsilon\delta)^{-1}}(A^*)e^{t\delta(\iota-\varepsilon\delta)^{-1}}(A)$.

Since

$$e^{t\delta}(A) = \lim_{\varepsilon\to 0} e^{t(e^{\varepsilon\delta}-\iota)/\varepsilon}(A) = \lim_{\varepsilon\to 0} e^{t\delta(\iota-\varepsilon\delta)^{-1}}(A)\,,$$

for all $A \in \mathscr{A}$, it follows that 3 $\Rightarrow$ 1 and 4 $\Rightarrow$ 1.

1 $\Rightarrow$ 3. If 1 holds, we have

$$\tau_t(A^*A) + A^*A - \tau_t(A^*)A - A^*\tau_t(A) \geqq (\tau_t(A)-A)^*(\tau_t(A)-A) \geqq 0\,,$$

and hence 3 holds.

2 $\Rightarrow$ 4. This is proved in the same manner as the previous implication.

1 $\Rightarrow$ 2. Define

$$\delta_\alpha = \frac{e^{\alpha\delta}-\iota}{\alpha}\,,$$

for $\alpha > 0$. From the implication 1 $\Rightarrow$ 3 it follows that

$$\delta_\alpha(A^*A) \geqq \delta_\alpha(A^*)A + A^*\delta_\alpha(A)\,,$$

for all $A \in \mathscr{A}$, and $(\iota-\varepsilon\delta_\alpha)^{-1}$ is positivity preserving by Theorem 4.
If $A \in \mathscr{A}$ is arbitrary, define $B = (\iota-\varepsilon\delta_\alpha)^{-1}A$. It follows that

$$\begin{aligned}(\iota-\varepsilon\delta_\alpha)^{-1}(A^*A) &= (\iota-\varepsilon\delta_\alpha)^{-1}\big(B^*B - \varepsilon(\delta_\alpha(B^*)B + B^*\delta_\alpha(B)) \\ &\quad + \varepsilon^2\delta_\alpha(B^*)\delta_\alpha(B)\big) \\ &\geqq (\iota-\varepsilon\delta_\alpha)^{-1}\big(B^*B - \varepsilon(\delta_\alpha(B^*)B + B^*\delta_\alpha(B))\big) \\ &\geqq (\iota-\varepsilon\delta_\alpha)^{-1}(B^*B - \varepsilon\delta_\alpha(B^*B)) \\ &= B^*B = (\iota-\varepsilon\delta_\alpha)^{-1}(A^*)(\iota-\varepsilon\delta_\alpha)^{-1}(A)\,.\end{aligned}$$

But

$$\lim_{\alpha\to 0}(\iota-\varepsilon\delta_\alpha)^{-1}(C) = (\iota-\varepsilon\delta)^{-1}(C)$$

by [3, Theorems 3.1.28 and 3.1.26].

It follows that

$$(\iota-\varepsilon\delta)^{-1}(A^*A) \geqq (\iota-\varepsilon\delta)^{-1}(A^*)(\iota-\varepsilon\delta)^{-1}(A)\,,$$

for all $A \in \mathscr{A}$ and all small $\varepsilon>0$.

2 ⇒ 1. By iteration of 2 it follows that

$$\left(\iota-\frac{t}{n}\delta\right)^{-n}(A^*A) \geqq \left(\iota-\frac{t}{n}\delta\right)^{-n}(A^*)\left(\iota-\frac{t}{n}\delta\right)^{-n}(A)\,,$$

and in the limit $n \to \infty$ one obtains

$$\tau_t(A^*A) \geqq \tau_t(A^*)\tau_t(A)\,.$$

The equivalence of conditions 1 and 2 in Theorem 5 is at first sight not surprising. It is a general rule of semigroup theory that the semigroup τ and its resolvent $(\iota-\varepsilon\delta)^{-1}$ have analogous properties and one usually demonstrates this by using Laplace transforms to pass from the semigroup to the resolvent. This technique can also be used to prove that 1 ⇒ 2 by a calculation similar to that given in the introduction:

$$\begin{aligned}(\iota-\alpha\delta)^{-1}(A^*A) &= \tfrac{1}{2}\int_0^\infty ds \int_0^\infty dt\; e^{-(s+t)}(\tau_{\alpha t}(A^*A)+\tau_{\alpha s}(A^*A))\\ &\geqq \tfrac{1}{2}\int_0^\infty ds \int_0^\infty dt\; e^{-(s+t)}(\tau_{\alpha t}(A)^*\tau_{\alpha t}(A)+\tau_{\alpha s}(A)^*\tau_{\alpha s}(A))\\ &\geqq \tfrac{1}{2}\int_0^\infty ds \int_0^\infty dt\; e^{-(s+t)}(\tau_{\alpha t}(A)^*\tau_{\alpha s}(A)+\tau_{\alpha s}(A)^*\tau_{\alpha t}(A))\\ &= (\iota-\alpha\delta)^{-1}(A)^*(\iota-\alpha\delta)^{-1}(A)\,.\end{aligned}$$

This implication can be extended in this manner to fractional powers of the resolvent. In fact it is a simple corollary of the theory of subordinate semigroups which we consider in the next section.

4. Subordinate semigroups.

In this section we demonstrate that if a C_0-semigroup $\tau_t=\exp\{t\delta\}$ of maps of a C*-algebra is strongly positive then the subordinate semigroups $\tau_t^f=\exp\{tf(\delta)\}$ are also strongly positive.

The theory of subordinate semigroups has been developed for Bernstein functions, [2], [12]. In these papers the semigroup generator δ is defined by the convention $\tau_t=\exp\{-t\delta\}$ and the Bernstein functions f are defined as C^∞-functions from $(0,\infty)$ to $[0,\infty)$ with the property that $(-1)^{n+1}f^{(n)}\geqq 0$, for all $n \geqq 1$. Since we are using the convention $\tau_t=\exp\{t\delta\}=\exp\{-t(-\delta)\}$, it is necessary to replace $f(x)$ by $-f(-x)$ and hence we consider Bernstein functions as C^∞-functions from $(-\infty, 0)$ to $(-\infty, 0]$ such that $f^{(n)} \geqq 0$ for all $n \geqq 1$.

The class of Bernstein functions occurs naturally in the Laplace transform theory of convolution semigroups supported by the half line [1]. It is also the largest class of functions for which the replacement of τ by τ^f respects order properties [4]. Following Phillips [12] one can use the Laplace transform theory to define the subordinate semigroup τ^f. The basic result is a one-to-one correspondence between vaguely continuous convolution semigroups of positive measures μ supported by the half line $[0, \infty)$ and Bernstein functions f; this correspondence is such that

$$e^{tf(x)} = \int_0^\infty d\mu^t(\lambda) e^{\lambda x} .$$

One can then use this relation to define τ^f from τ by setting

$$\tau_t^f = \int_0^\infty d\mu^t(\lambda) \tau_\lambda .$$

It is readily checked that τ^f is a C_0-semigroup but it remains to identify its generator. But using Bernsteins theorem one can argue that each Bernstein function [1] has a representation

$$(*) \qquad f(x) = -\alpha + \beta x + \int_0^\infty d\mu(t)(e^{tx} - 1) ,$$

where $\alpha, \beta \geqq 0$ and μ is a positive measure satisfying

$$\int_0^\infty d\mu(t) \frac{t}{1+t} < +\infty .$$

Nelson [11] used this formula to complete the work of Phillips [12] and show that τ^f has generator

$$f(\delta) = -\alpha\iota + \beta\delta + \int_0^\infty d\mu(t)(\tau_t - \iota) ,$$

where the integral is understood in the strong sense and the strong closure is taken.

THEOREM 6. *Let $\mathscr{A}$ be a C*-algebra, τ a C_0-semigroup on $\mathscr{A}$, f a Bernstein function, and τ^f the corresponding subordinate C_0-semigroup.*

If

$$\tau_t(A^*A) \geqq \tau_t(A)^*\tau_t(A) ,$$

for all $A \in \mathscr{A}$ *and* $t \geqq 0$, *then*

$$\tau_t^f(A^*A) \geqq \tau_t^f(A^*A)e^{tf(0)} \geqq \tau_t^f(A)^*\tau_t^f(A),$$

for all $A \in \mathscr{A}$ *and* $t \geqq 0$.

PROOF. First since $f(0) \leqq 0$ one has $\exp\{tf(0)\} \leqq 1$ for $t \geqq 0$. Second one has

$$\begin{aligned}\tau_t^f(A^*A)e^{tf(0)} &= \tfrac{1}{2}\int_0^\infty d\mu^t(\lambda)\int_0^\infty d\mu^t(\varrho)(\tau_\lambda(A^*A)+\tau_\varrho(A^*A))\\ &\geqq \tfrac{1}{2}\int_0^\infty d\mu^t(\lambda)\int_0^\infty d\mu^t(\varrho)(\tau_\lambda(A)^*\tau_\lambda(A)+\tau_\varrho(A)^*\tau_\varrho(A))\\ &\geqq \tfrac{1}{2}\int_0^\infty d\mu^t(\lambda)\int_0^\infty d\mu^t(\varrho)(\tau_\lambda(A)^*\tau_\varrho(A)+\tau_\varrho(A)^*\tau_\lambda(A))\\ &= \tau_t^f(A)^*\tau_t^f(A).\end{aligned}$$

One can exploit Theorem 6 to obtain a generalization of part of Theorem 5.

COROLLARY 7. *Under the assumptions of Theorem* 5 *the following conditions are equivalent*

1. $\tau_t(A^*A) \geqq \tau_t(A)^*\tau_t(A)$, *for all* $A \in \mathscr{A}$ *and* $t \geqq 0$.
2. $(\iota-\varepsilon\delta)^{-t}(A^*A) \geqq (\iota-\varepsilon\delta)^{-t}(A)^*(\iota-\varepsilon\delta)^{-t}(A)$, *for all* $A \in \mathscr{A}$ *and* $\varepsilon, t \geqq 0$.

PROOF. Condition 2 implies Condition 1 by Theorem 5. But the converse follows by choosing

$$f(x) = -\log(1-\varepsilon x), \qquad x \leqq 0.$$

Thus

$$\tau_t^f = e^{tf(\delta)} = (\iota-\varepsilon\delta)^{-t}.$$

Finally we note that if $\beta > 0$ in the representation (*) of the Bernstein function f, then $f(\delta)$ and δ have the same domain. But if $\beta = 0$, then $D(\delta)$ is only a core of $f(\delta)$. Thus if $D(\delta)$ is a *-algebra, then $f(\delta)$ always has a core which is a *-algebra, but its domain does not necessarily have this property.

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