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GENERALIZATIONS OF ANALYTIC AND STANDARD MEASURABLE SPACES

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Summary.

The classical notions of analytic and of standard measurable spaces are generalized to include certain non-separable spaces. These spaces enjoy many of the same properties as the classical spaces and indeed are characterized by some of these properties. Combining these characterizations with the fact that a generalized analytic (respectively generalized standard) space which is countably separated is analytic (respectively standard) yields new characterizations of analytic (respectively standard) spaces. A simplified proof is given of a characterization of standard spaces due to Christensen. A metrizable topological space which is generalized analytic as a measurable space when equipped with its Baire σ-algebra is necessarily separable and is analytic as a topological space.

Introduction.

In this paper we generalize the notions of analytic and of standard measurable spaces to include certain spaces which are not countably generated. The generalized analytic spaces are those we call *smooth* while the generalized standard spaces are those we say have the *extension property* (abbreviated EP). We show that these spaces have many of the same properties that analytic or standard spaces enjoy and indeed are characterized by some of these properties. We show that a countably separated smooth space is analytic and a countably separated space having the extension property is standard. Combining these facts with our characterizations of smooth spaces and of spaces having the extension property gives some new characterizations of analytic and of standard measurable spaces. In 1.15 we give a simplified proof of a characterization of standard spaces due to Christensen [2]. Part of the object of 1.2, 1.7(a), 1.10, and 1.13(b) is to point out the extent to which countability conditions used in certain classical results are necessary.

Our definitions of smooth measurable spaces and of measurable spaces having the extension property and our treatment of these spaces are non-

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topological and so perhaps more measure-theoretically illuminating than the classical topological definitions and treatment of analytic and of standard measurable spaces. In 4.4 we show that an analytic topological space is smooth as a measurable space when equipped with its Baire σ-algebra and in 4.6 we show that a *metrizable* topological space which is smooth as a measurable space when equipped with its Baire σ-algebra is necessarily separable and is analytic as a topological space.

0. Some Notation and Terminology.

0.1. Let $(X, \mathscr{A})$ be a measurable space.

If x, y are elements of X which cannot be separated by an element of $\mathscr{A}$, we say x and y lie in the same *fiber* of $\mathscr{A}$.

We say $(X, \mathscr{A})$ is *countably fibered* iff some countable subset of $\mathscr{A}$ separates the points of X as effectively as $\mathscr{A}$ does. In this case, the fibers of $\mathscr{A}$ belong to $\mathscr{A}$.

We remark that if $(X, \mathscr{A})$ is countably generated, then it is countably fibered. We shall see (in 1.6) that the converse is true if $(X, \mathscr{A})$ is "smooth".

We say $(X, \mathscr{A})$ is *separated* iff $\mathscr{A}$ separates the points of X.

We say $(X, \mathscr{A})$ is *countably separated* iff some countable subset of $\mathscr{A}$ separates the points of X. In this case $\{x\} \in \mathscr{A}$ for all $x \in \mathscr{A}$.

0.2. For any set $\mathscr{H}$ of sets, Borelian $\mathscr{H}$ denotes the smallest set $\mathscr{B}$ of sets such that the union and intersection of any non-empty countable subset of $\mathscr{B}$ are elements of $\mathscr{B}$, and $\mathscr{H} \subseteq \mathscr{B}$.

0.3. If $\mathscr{A}, \mathscr{B}$ are sets of sets, then $\mathscr{A} \odot \mathscr{B}$ denotes $\{A \times B: A \in \mathscr{A}, B \in \mathscr{B}\}$ and $\mathscr{A} \otimes \mathscr{B}$ denotes Borelian $(\mathscr{A} \odot \mathscr{B})$. Note that if $\mathscr{A}$ and $\mathscr{B}$ are rings, then $\mathscr{A} \otimes \mathscr{B}$ is a σ-ring.

0.4. For the basic theory of the Souslin operation and of analytic sets, we refer the reader to Bressler and Sion [1].

$\mathscr{S}$1. Smooth measurable spaces.

1.1. DEFINITION. Let us say that a measurable space $(X, \mathscr{A})$ is *smooth* iff for each countably generated sub-σ-field $\mathscr{B}$ of $\mathscr{A}$, there exists a set Y containing X and a semicompact class $\mathscr{C}$ of subsets of Y such that $\mathscr{B} \subseteq$ Souslin $\mathscr{C}$.

(As $\mathscr{B}$ is countably generated, we can take $\mathscr{C}$ to be countable, so we get an equivalent definition if we replace "semicompact" by "compact".)

1.2. LEMMA. *Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces and let $f: X \to Y$ be measurable.*

Then graph $(f) \in \mathscr{A} \otimes \mathscr{B}$ *iff there exists a countable set $\mathscr{S} \subseteq \mathscr{B}$ such that for* *any* $y \in \text{range}(f)$ *and any* $z \in Y$ *with* $y \neq z$, *there exists* $S \in \mathcal{S}$ *such that* S *separates* y *and* z.

In particular, if f *is onto then* $\text{graph}(f) \in \mathcal{A} \otimes \mathcal{B}$ *iff* $(Y, \mathcal{B})$ *is countable separated.*

REMARK. For the forward implications in Lemma 1.2 it is not important that f be measurable; all that matters is that $\text{graph}(f) \in \mathcal{A} \otimes \mathcal{B}$ and in fact less than this will do. For instance, it would be enough to have $\text{graph}(f) \in \text{Souslin}(\mathcal{A} \otimes \mathcal{B})$. Indeed it is sufficient that $\text{graph}(f)$ belong to the complete field generated by some countable subset of $\mathcal{A} \otimes \mathcal{B}$.

1.3. THEOREM. *For a measurable space* $(X, \mathcal{A})$, *the following are equivalent*:

a) $(X, \mathcal{A})$ *is smooth.*

b) *For any measurable space* $(Y, \mathcal{B})$ *and any* $F \in \text{Souslin}(\mathcal{A} \otimes \mathcal{B})$, *we have* $\pi_Y[F] \in \text{Souslin}\,\mathcal{B}$.

c) *For any measurable space* $(Y, \mathcal{B})$ *and any* $F \in \mathcal{A} \otimes \mathcal{B}$, *we have* $\pi_Y[F] \in \text{Souslin}\,\mathcal{B}$.

d) *For any countably separated measurable space* $(Y, \mathcal{B})$ *and any measurable map* $f: X \to Y$, *we have* $f[X] \in \text{Souslin}\,\mathcal{B}$.

e) *For any measurable map* $f: X \to \mathsf{R}$, $f[X]$ *is analytic.*

(In b) and c), π_Y denotes the projection of $X \times Y$ on Y.)

1.4. THEOREM. *Any product (even uncountable) of smooth measurable spaces is smooth.*

1.5. THEOREM. *Let* $(X, \mathcal{A})$ *and* $(Y, \mathcal{B})$ *be measurable spaces and let* f *be a measurable map of* X *onto* Y.

Suppose $(X, \mathcal{A})$ *is smooth. Then*:

a) $((Y, \mathcal{B})$ *is smooth.*

b) *If* $(Y, \mathcal{B})$ *is countably separated then* f *is a quotient map*; *that is,* $\mathcal{B} = \{B \subseteq Y: f^{-1}[B] \in \mathcal{A}\}$.

c) *If* $(Y, \mathcal{B})$ *is countably separated and* f *is* $1-1$, *then* f *is a Borel isomorphism.*

1.6. COROLLARY. *A countably separated (or just countably fibered) smooth measurably space is countably generated.*

1.7. PROPOSITION. *Let* $(X, \mathcal{A})$ *be a measurable space, let* $Y \subseteq X$, *and let* $\mathcal{B} = \mathcal{A} \mid Y$.

a) *If* $(Y, \mathcal{B})$ *is smooth, then* $Y \in \text{Souslin}\,\mathcal{A}$, *iff there exists a countable set*

$\mathcal{S} \subseteq \mathcal{A}$ such that whenever $y \in Y$ and $z \in X \setminus Y$, there exists $S \in \mathcal{S}$ such that S separates y and z.

b) *If $(X, \mathcal{A})$ is smooth and $Y \in$ Souslin $\mathcal{A}$ then $(Y, \mathcal{B})$ is smooth.*

1.8. Proposition. *Let X be a Polish space and let $\mathcal{A} =$ Borel X. Let $Y \subseteq X$ and let $\mathcal{B} = \mathcal{A} \mid Y$ (which is also equal to* Borel Y*). Then $(Y, \mathcal{B})$ is smooth iff $Y \in$* Souslin $\mathcal{A}$.

1.9. Theorem. *For a measurable space $(X, \mathcal{A})$, the following are equivalent:*

a) *$(X, \mathcal{A})$ is Borel isomorphic to an analytic subset of* $\mathbf{R}$.

b) *$(X, \mathcal{A})$ is countably separated, and there is a set Y containing X and a semicompact class $\mathcal{C}$ of subsets of Y such that $\mathcal{A} \subseteq$* Souslin $\mathcal{C}$.

1.10. Proposition. *Let $(X, \mathcal{A})$ be a measurable space. Let $Y_1, Y_2 \subseteq X$ and let $\mathcal{B}_i = \mathcal{A} \mid Y_i$ $(i = 1, 2)$. Suppose $(Y_i, \mathcal{B}_i)$ is smooth $(i = 1, 2)$. Then the following are equivalent:*

a) *There exist $A_1, A_2 \in \mathcal{A}$ such that $A_1 \cap A_2 = \varnothing$ and $Y_i \subseteq A_i$ $(i = 1, 2)$.*

b) *There exists a countable set $\mathcal{S} \subseteq \mathcal{A}$ such that whenever $y_i \in Y_i$ $(i = 1, 2)$, then there exists $S \in \mathcal{S}$ such that S separates y_1 and y_2.*

1.11 Proposition. *Let $(X, \mathcal{A})$ and $(Y, \mathcal{B})$ be smooth measurable spaces, and suppose $(Y, \mathcal{B})$ is countably separated. Let $\mathcal{C} = \mathcal{A} \otimes \mathcal{B}$. Consider any map $f: X \to Y$. Then the following are equivalent:*

a) *f is measurable.*

b) graph $(f) \in \mathcal{C}$.

c) graph $(f) \in$ Souslin $\mathcal{C}$.

d) For every $B \in \mathcal{B}$, $f^{-1}[B] \in$ Souslin $\mathcal{A}$.

1.12. Definition. Let $(X, \mathcal{A})$ be a measurable space. A sub-σ-field $\mathcal{B}$ of $\mathcal{A}$ will be said to be *determined by its fibers* (relative to $\mathcal{A}$) iff whenever $A \in \mathcal{A}$ and A is a union of fibers of $\mathcal{B}$, then $A \in \mathcal{B}$.

1.13. Proposition. *Let $(X, \mathcal{A})$ be a measurable space and let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$.*

a) *If $(X, \mathcal{A})$ is smooth and $\mathcal{B}$ is the intersection of a family of countably generated sub-σ-fields of $\mathcal{A}$, then $\mathcal{B}$ is determined by its fibers.*

b) *If $\mathcal{A}$ is countably generated and $\mathcal{B}$ is determined by its fibers, then $\mathcal{B}$ is the intersection of a family of countably generated sub-σ-fields of $\mathcal{A}$.*

1.14. One might ask whether we can take $Y=X$ in 1.9. The next result implies that we cannot. It is due to Christensen [2], Theorem 3.5. We have a proof which is significantly simpler than Christensen's. Also, while his proof uses the fact that a space on which ever ultrafilter converges is compact, and thus depends on the full axiom of choice, our proof uses only the principle of dependent choice. This is of some philosophical interest since most of the theory of measurable spaces can be developed using only the principle of dependent choice—even the measurable selection theorems. Now here is the result.

1.15. THEOREM. *Let* $(X, \mathscr{A})$ *be a countably separated measurable space, and suppose there is a semicompact family* $\mathscr{K}$ *of subsets of* X *such that* $\mathscr{A} \subseteqq \text{Souslin}\,\mathscr{K}$.

Let $\mathscr{B} = \{B \subseteqq X : B,\ X \setminus B$ *are both in* $\text{Souslin}\,\mathscr{K}\}$.

Then:

a) $\mathscr{A} = \text{Borelian}\,\mathscr{K} = \mathscr{B}$.
In particular, $\mathscr{K}$ *is necessarily contained in* $\mathscr{A}$.

b) *There is a Polish topology* τ *on* X *such that each* K *in* $\mathscr{K}$ *is* τ*-closed and* $\mathscr{A} = \text{Borel field}\ \tau$. *Thus* $(X, \mathscr{A})$ *is standard.*

2. Spaces with the extension property.

2.1. DEFINITION. Let us say that a measurable space $(X, \mathscr{A})$ has the *extension property* (more briefly, has EP) iff whenever $(Y, \mathscr{B})$ is a measurable space, $Z \subseteqq Y$, and $f\colon Z \to X$ is $(\mathscr{B}\,|\,Z, \mathscr{A})$-measurable, there exists a measurable map $g\colon Y \to X$ such that $g\,|\,Z = f$.

2.2. A two point discrete measurable space has EP by the definition of $\mathscr{B}\,|\,Z$.

2.3. ($\mathbf{R}$, Borel $\mathbf{R}$) has EP since once we have proved something for measurable sets (see 2.2) we can prove it for measurable functions.

2.4. Any product of spaces having EP has EP.

2.5. Any measurable subset of a space having EP has EP.

2.6. Suppose X is a Polish space. Then X is homeomorphic to a $\mathscr{G}_\delta$ subset (even a closed subset) of $\mathbf{R}^{\mathbf{N}}$. Hence (X, Borel X) has EP by 2.3, 2.4, and 2.5.

2.7. Definition. Let $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ be measurable spaces.

We say $(Y, \mathscr{B})$ *is a retract of* $(X, \mathscr{A})$ iff there are measurable maps $f: X \to Y$ and $g: Y \to X$ such that $f \circ g = \mathrm{id}_Y$.

Observe that in this case:

a) $f[X] = Y$ and $\mathscr{B} = \{B \subseteq Y : f^{-1}[B] \in \mathscr{A}\}$.
b) g is $1-1$; indeed g is a Borel isomorphism between $(Y, \mathscr{B})$ and $(g[Y], \mathscr{A} \mid g[Y])$.
c) $g[Y] = \{g \circ f = \mathrm{id}_X\}$.

Hence if $(X, \mathscr{A})$ is countably separated, then $g[Y] \in \mathscr{A}$.

2.8. A measurable space $(X, \mathscr{A})$ is separated and has EP iff it is a retract of $\{0,1\}^{\Lambda}$ for some set Λ. (Here $\{0,1\}^{\Lambda}$ is endowed with the measurable structure generated by the coordinate maps; also, we can take for Λ any set of generators for $\mathscr{A}$.)

2.9. An arbitrary measurable space has EP iff the associated separated measurable space has EP.

2.10. A measurable space which has EP is smooth.

2.11. If $(X, \mathscr{A})$ has EP, $(Y, \mathscr{B})$ is countably separated, and $f: X \to Y$ is $1-1$ and measurable, then $f[X] \in \mathscr{B}$.

2.12. A measurable space is Borel isomorphic to a Borel subset of $\mathbf{R}$ iff it is countably separated and has EP.

2.13. Example. Let $X = \mathbf{R}$ and let $\mathscr{A}$ be the σ-field of Lebesgue measurable subsets of $\mathbf{R}$. Then $(X, \mathscr{A})$ does not have EP.

2.14. For a measurable space $(Y, \mathscr{B})$, the following are equivalent:

a) $(Y, \mathscr{B})$ is separated and has EP.
b) Whenever $(X, \mathscr{A})$ is a measurable space, Ω a set, $\varphi: \Omega \to X$, $\mathscr{A}' = \varphi^{-1}(\mathscr{A})$, and $\psi: \Omega \to Y$ is $\mathscr{A}'$-measurable, there exists a measurable map $f: X \to Y$ such that $\psi = f \circ \varphi$.

2.15. Let $(\Omega, \mathscr{F})$ be a smooth measurable space. $(X, \mathscr{A})$ a countably separated measurable space, and $(Y, \mathscr{B})$ a measurable space having EP.

Let $\varphi: \Omega \to X$ and $\psi: \Omega \to Y$ be measurable maps. Suppose $\omega, \omega' \in \Omega$, $\varphi(\omega) = \varphi(\omega') \Rightarrow \psi(\omega) = \psi(\omega')$. Then there is a measurable map $f: X \to Y$ such that $\psi = f \circ \varphi$.

3. Varadarajan spaces.

A *logic* is a σ-complete orthomodular lattice. If $\mathscr{L}$ is a logic and $(X,\mathscr{A})$ is a measurable space, then an $(X,\mathscr{A})$-valued *observable* of $\mathscr{L}$ is an orthocomplementation-preserving σ-homomorphism of $\mathscr{A}$ into $\mathscr{L}$. Varadarajan [12] has shown that if (x_i) is a countable family of "commuting" real-valued observables of a logic $\mathscr{L}$ then the x_i's are all "functions" of a single real-valued observable of $\mathscr{L}$. This generalizes a theorem of von Neumann concerning commuting self-adjoint operators.

The main mathematical reason for restricting attention to real-valued observables is that, as a measurable space, R has a certain nice property. We abstract this property in the following definition.

3.1. Definition. Let us say that a measurable space $(X,\mathscr{A})$ is a Varadarajan space (V-space, for short) iff whenever $\mathscr{L}$ is a logic, x an $(X,\mathscr{A})$-valued observable of $\mathscr{L}$, $(Y,\mathscr{B})$ another measurable space, and y a $(Y,\mathscr{B})$-valued observable of $\mathscr{L}$ such that range $(x)\subseteq$range (y), then x is a function of y; that is, there is a measurable function $f\colon Y\to X$ such that $x(A)=y(f^{-1}[A])$ for all $A\in\mathscr{A}$.

3.2. Theorem. *Let $(X,\mathscr{A})$ be a measurable space. Then the following equivalent*:

a) *$(X,\mathscr{A})$ has* EP.
b) *$(X,\mathscr{A})$ is a* V-*space.*
c) *$\mathscr{A}$ is a projective object in the category of Boolean σ-algebras, and every two-valued probability measure on $\mathscr{A}$ is a Dirac measure δ_x for some $x\in X$.*

Remark on part c. The first part of c) refers only to the algebraic structure of $\mathscr{A}$. Note also that any projective object in the category of Boolean σ-algebras is necessarily isomorphic to a σ-field of sets by the theorem of Loomis [8]. As for the second part of c), it refers only to how $\mathscr{A}$ sits on the space X; we can always make this part hold by adding points to X, and if $\mathscr{A}$ is countably generated, it holds automatically.

3.3. Lemma. *Let $\mathscr{L}$ be a logic, let $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ be measurable spaces, let x be an $(X,\mathscr{A})$-valued observable of $\mathscr{L}$, and let $f,g\colon X\to Y$ be measurable maps.*

Suppose $x(f^{-1}[B])=x(g^{-1}[B])$ for all $B\in\mathscr{B}$, and $(Y,\mathscr{B})$ is countably separated.

Let $E=\{f\neq g\}$. Then:

a) $E \in \mathscr{A}$.
b) x *is not in* E; *that is,* $x(E)=0$.
In other words, $f=g$ *almost everywhere with respect to* x.

It is now easy to prove the following result of von Neumann [9].

3.4. THEOREM. *Let* $(X, \mathscr{A})$ *and* $(Y, \mathscr{B})$ *be standard measurable spaces.* (*By* 2.12, *standard = countably separated* + EP.)

Let $\mathscr{I}, \mathscr{J}$ *be* σ*-ideals in* $\mathscr{A}, \mathscr{B}$, *respectively.*
Let Φ *be a* σ*-isomorphism of* $\mathscr{A}/\mathscr{I}$ *onto* $\mathscr{B}/\mathscr{J}$.
Then there exist sets $I \in \mathscr{I}$, $J \in \mathscr{J}$ *and a Borel isomorphism* φ *of* $Y \setminus J$ *onto* $X \setminus I$ *such that for all* $A \in \mathscr{A}$,

$$\Phi(A/\mathscr{I}) = \varphi^{-1}[A]/\mathscr{J} .$$

If $\mathscr{A}/\mathscr{I}$ *(and hence* $\mathscr{B}/\mathscr{J}$*) is* σ*-finite* (i.e., *every orthogonal family in* $\mathscr{A}/\mathscr{I}$ is *countable) and not purely atomic, then we may take* I *and* J *to be empty.*

REMARK. von Neumann actually proved 3.3 in the case where $\mathscr{I}, \mathscr{J}$ are the collections of null sets of finite measures μ, ν on $\mathscr{A}, \mathscr{B}$, respectively and he even assumed that Φ carried μ onto ν. Next we give an example of a particular case of 3.4 which cannot be deduced from the version involving measures, since if Z is a separable topological space which has no isolated points and in which each point is a $\mathscr{G}_\delta$ set, then every finite Borel measure on Z lives on a set of the first category (see Oxtoby [10], Theorem 1.6).

3.5. COROLLARY. *Let* X *and* Y *be uncountable Polish spaces. Let* $\mathscr{A} =$ Borel X *and let* $\mathscr{B} =$ Borel Y.
Let $\mathscr{I}$ (*respectively* $\mathscr{J}$) *be the collection of Borel sets of the first category in* X (*respectively in* Y).
Suppose Φ *is a* σ*-isomorphism of* $\mathscr{A}/\mathscr{I}$ *onto* $\mathscr{B}/\mathscr{J}$.
Then there is a Borel isomorphism φ *of* Y *onto* X *such that for all* $A \in \mathscr{A}$, $\Phi(A/\mathscr{I}) = \varphi^{-1}[A]/\mathscr{I}$.

4. Some topological results.

4.1. NOTATION. For any topological space X, Baire X will denote the σ-field generated by the continuous real-valued functions on X.

Now here is the topological analogue of 1.10.

4.2. THEOREM. *Let A, B be analytic subsets of a topological space X. Then the following are equivalent*:

a) *For every $a \in A$ and every $b \in B$, there exists $f \in C(X, \mathbf{R})$ such that $f(a) \neq f(b)$.*

(b) *A and B can be separated by disjoint Baire sets.*

c) *There exists $\varphi \in C(X, \mathbf{R}^{\mathbf{N}})$ such that $\varphi[A] \cap \varphi[B] = \varnothing$.*

REMARK. a) $\Rightarrow$ b) is a slight generalization of a result of Frolík [3].

4.3. COROLLARY. *Let X be a topological space on which $C(X, \mathbf{R})$ separates points. Assume X is analytic. Then*

$$\text{Baire}\, X = \{B \subseteq X : B \text{ and } X \setminus B \text{ are analytic}\}\ .$$

4.4. PROPOSITION. *Let X be an analytic topological space. Then $(X, \text{Baire}\, X)$ is a smooth measurable space.*

4.5. Frolík [4] has shown that a metrizable space, which is the image of an analytic topological space under a Baire measurable map, is analytic (topologically).

By similar methods, one can prove the following more general result.

4.6. THEOREM. *Let X be a metrizable space such that $(X, \text{Baire}\, X)$ $(= (X, \text{Borel}\, X))$ is a smooth measurable space.*

Then X is homeomorphic to an analytic subset of $\mathbf{R}^{\mathbf{N}}$. (In particular, X is necessarily separable.)

5. Proofs.

Some of the simpler proofs are omitted or only commented on.

PROOF OF 1.2 ($\Leftarrow$). We may assume that $S \in \mathscr{S} \Rightarrow Y \setminus S \in \mathscr{S}$. Then graph $(f) = (X \times Y) \setminus \bigcup_{S \in \mathscr{S}} f^{-1}[S] \times (Y \setminus S)$.

($\Rightarrow$) There are countable sets $\mathscr{C}, \mathscr{D}$ contained in $\mathscr{A}, \mathscr{B}$, respectively, such that graph $(f) \in$ Borel field $(\mathscr{C} \odot \mathscr{D})$. We can take $\mathscr{S} = \mathscr{D}$.

PROOF OF 1.3 (a) $\Rightarrow$ b)). $\mathscr{C} = $ Borelian $(\mathscr{A} \odot \mathscr{B})$, so by the idempotence of the Souslin operation, $F \in$ Souslin $(\mathscr{D} \odot \mathscr{E})$, where $\mathscr{D}, \mathscr{E}$ are countable subsets of $\mathscr{A}, \mathscr{B}$, respectively. As $(X, \mathscr{A})$ is smooth, there is a set $X' \supseteq X$ and a semicompact class $\mathscr{K}$ of subsets of X' such that $\mathscr{D} \subseteq$ Souslin $\mathscr{K}$. Then

$F \in \text{Souslin}\,(\mathscr{K} \odot \mathscr{E})$, by the idempotence of the Souslin operation. Now we may assume $\mathscr{K}$ is closed under finite intersections. Thus there is a nested $\mathscr{K}$-valued Souslin scheme (K_s) and an $\mathscr{E}$-valued Souslin scheme (E_s) such that

$$F = \bigcup_{\sigma} \bigcap_{s<\sigma} K_s \times E_s \, .$$

(Here σ runs over infinite sequences of natural numbers, s runs over finite sequences of natural numbers, and $s<\sigma$ means s is a finite initial segment of σ.) Let π'_Y denote the projection of $X' \times Y$ on Y. Then

$$\begin{aligned} \pi_Y[F] &= \bigcup_{\sigma} \pi'_Y\left[\bigcap_{s<\sigma} K_s \times E_s\right] \\ &= \bigcup_{\sigma} \bigcap_{s<\sigma} \pi'_Y[K_s \times E_s] \, , \end{aligned}$$

where the second step follows from the fact that for any σ, if $K_s \neq \varnothing$ for all $s < \sigma$, then $\bigcap_{s<\sigma} K_s \neq \varnothing$, by the semicompactness of $\mathscr{K}$. Thus $\pi[F] \in \text{Souslin}\,\mathscr{B}$.

(b) $\Rightarrow$ c)). $\mathscr{A} \otimes \mathscr{B} \subseteq \text{Souslin}\,(\mathscr{A} \otimes \mathscr{B})$.

(c) $\Rightarrow$ d)). graph $(f) \in \mathscr{A} \otimes \mathscr{B}$ by 1.2, and $f[X] = \pi_Y[\text{graph}\,(f)]$.

(d) $\Rightarrow$ e)). $(\mathbf{R}, \text{Borel}\,\mathbf{R})$ is countably separated, and a subset of $\mathbf{R}$ is analytic iff it belongs to Souslin (Borel $\mathbf{R}$).

(e) $\Rightarrow$ a)). Let $\mathscr{B}$ be a countably generated sub-σ-field of $\mathscr{A}$. Then there is a map $f\colon X \to \mathbf{R}$ such that $\mathscr{B} = f^{-1}(\text{Borel}\,\mathbf{R})$. Now choose a pair (R, γ), where R is a set, and γ is a $1-1$ map of R onto $\mathbf{R}$, and $R \cap X = \varnothing$. (We remark that such a pair can be constructed without using any choice axioms, as follows. Let I be a set such that there is no map of X onto I; say I = power set of X. For each $i \in I$, let $R_i = \mathbf{R} \times \{i\}$. Then for at least one $i \in I$, $R_i \cap X = \varnothing$. Let $R = R_i$ and let $\gamma((t, i)) = t$.) Next let $Y = X \cup \gamma^{-1}[\mathbf{R} \setminus f[X]]$ (note this is a disjoint union) and define $g\colon Y \to \mathbf{R}$ by

$$g(y) = \begin{cases} f(y) & \text{if } y \in X \\ \gamma(y) & \text{if } y \in Y \setminus X \, . \end{cases}$$

Let $\mathscr{C} = \{g^{-1}[K]\colon K \in \mathscr{K}\}$, where $\mathscr{K}$ is the collection of compact subsets of $\mathbf{R}$. Then $\mathscr{C}$ is a compact family of subsets of Y (since $g[Y] = \mathbf{R}$) and $\mathscr{B} \subseteq \text{Souslin}\,\mathscr{C}$ (since if $B \in \mathscr{B}$, then $B = g^{-1}[f[B]]$ and $f[B] \in \text{Souslin}\,\mathscr{K}$).

Note also that the $\mathscr{C}$ we have obtained has an additional property not required in the definition of smoothness; namely, for any $C \in \mathscr{C}$, $C \cap X \in \mathscr{B}$.

Proof of 1.4. If $(X, \mathscr{A}) = \otimes_i (X_i, \mathscr{A}_i)$, where each $(X_i, \mathscr{A}_i)$ is smooth, and if $\mathscr{B}$ is a countably generated sub-σ-field of $\mathscr{A}$, then $\mathscr{B} \subseteq \otimes_i \mathscr{B}_i$, where each $\mathscr{B}_i$ is a countably generated sub-σ-field of $\mathscr{A}_i$ and $\mathscr{B}_i = \{\varnothing, X_i\}$ for all but countably many i. The proof may now be easily concluded by working directly with the definition of smoothness.

PROOF OF 1.5 a). Apply (e) $\Rightarrow$ a)) of 1.3.

b). As $(Y, \mathscr{B})$ is countably separated, there is a $1-1$ measurable map g: $Y \to \mathsf{R}$. Let $h = g \circ f$. Suppose $B \subseteq Y$ and $f^{-1}[B] \in \mathscr{A}$. Let $C = Y \setminus B$. Then $g[B] = h[f^{-1}[B]]$ and $g[C]$ are disjoint analytic sets in R, so by the classical separation theorem there are disjoint Borel sets $D, E \subseteq \mathsf{R}$ with $g[B] \subseteq D$ and $g[C] \subseteq E$. But then $B = g^{-1}[D] \in \mathscr{B}$.

c) follows from b).

PROOF OF ($\Leftarrow$) OF a) OF 1.7. Arrange the elements of $\mathscr{S}$ in a sequence (S_i) and let $f = \sum_i (2/3^i) 1_{S_i}$. Then f is measurable, $f[Y] \in$ Souslin (Borel R), and $Y = f^{-1}[f[Y]]$.

PROOF OF 1.8. By 1.7 we need only prove that $(X, \mathscr{A})$ is smooth. As X is separable and metrizable, X is homeomorphic to a set Z in a compact metrizable space Y. As X is completely metrizable, Z must be a $\mathscr{G}_\delta$-set.

PROOF OF (b) $\Rightarrow$ a)) OF 1.10. Let f be as in the proof of 1.7. Then $f[Y_1] \cap f[Y_2] = \varnothing$. Now apply the classical separation theorem, and pull back into X using f^{-1}.

PROOF OF 1.11 (a) $\Rightarrow$ b)). Apply 1.2.

(b) $\Rightarrow$ c)). Trivial.

(c) $\Rightarrow$ d)). $f^{-1}[B] = \pi[\text{graph}(f) \cap (X \times B)]$, where π is the projection of $X \times Y$ on X. Now apply (a) $\Rightarrow$ b)) of 1.3.

(d) $\Rightarrow$ a)). Let $B \in \mathscr{B}$. Then $A = f^{-1}[B]$ and $A' = f^{-1}[Y \setminus B]$ are disjoint elements of Souslin $\mathscr{A}$. Now $(A, \mathscr{A} \mid A)$ and $(A', \mathscr{A} \mid A')$ are smooth, by 1.7(b)). Also, the hypothesis 1.10(b)) is satisfied. (Consider a Souslin scheme for A.) It follows that $A, A' \in \mathscr{A}$, by (b) $\Rightarrow$ a)) of 1.10.

PROOF OF 1.13(b)). Let $I = X \times X$. We shall find countably generated sub-σ-fields $\mathscr{C}_i$ of $\mathscr{A}$ (for $i \in I$) such that $\mathscr{B} = \bigcap_i \mathscr{C}_i$. Consider $i = (x, y) \in I$. If x and y can be separated by some element of $\mathscr{B}$, let $\mathscr{C}_i = \mathscr{A}$. On the other hand, suppose x and y cannot be so separated. Let α, β be the fibers of $\mathscr{A}$ containing x, y, respectively, and let $\mathscr{C}_i$ be the σ-field generated by $\alpha \cup \beta$ and the sets of the form

$A \cap (X \setminus (\alpha \cup \beta))$ for $A \in \mathscr{A}$. Then x and y cannot be separated by an element of $\mathscr{C}_i$. Note $\alpha, \beta \in \mathscr{A}$ as $\mathscr{A}$ is countably fibred.

Now in either case, $\mathscr{C}_i$ is a countably generated sub-σ-field of $\mathscr{A}$, and $\mathscr{B} \subseteqq \mathscr{C}_i$. Suppose $B \in \bigcap_i \mathscr{C}_i$. If $x, y \in X$ and x, y cannot be separated by an element of $\mathscr{B}$, then B does not separate x and y since $B \in \mathscr{C}_i$, where $i = (x, y)$. Thus B is a union of fibers of $\mathscr{B}$, whence $B \in \mathscr{B}$. Thus $\mathscr{B} = \bigcap_i \mathscr{C}_i$.

PROOF OF 1.15. Clearly $\mathscr{A} \subseteqq \mathscr{B}$, $(X, \mathscr{B})$ is smooth, and $(X, \mathscr{A})$ is countably separated. Hence $\mathscr{A} = \mathscr{B}$ and $\mathscr{A}$ is countably generated, by 1.5 and 1.6.

It then follows that there is a countable set $\mathscr{L} \subseteqq \mathscr{K}$ such that $\mathscr{A} \subseteqq \text{Souslin}\, \mathscr{L}$.

Now if $x \in X$, then $\{x\}$ belongs to $\mathscr{A}$ and hence to Souslin $\mathscr{L}$; therefore $\{x\}$ is the intersection of a sequence of elements of $\mathscr{L}$.

Let $\mathscr{M} = \{\bigcap \mathscr{H} : \varnothing \neq \mathscr{H} \text{ finite } \subseteqq \mathscr{L}\}$.

If $K \in \mathscr{K}$ and $x \in X \setminus K$, then $x \in M \subseteqq X \setminus K$ for some $M \in \mathscr{M}$, since $\{x\} \in \mathscr{L}_\delta$ and $\mathscr{K}$ is a semicompact family.

Thus, as $\mathscr{M}$ is countable, $X \setminus K$ is the union of a sequence of elements of $\mathscr{M}$. (Note that this is true even if $K = X$: $\varnothing \in \mathscr{A}$, so $\varnothing \in \text{Souslin}\, \mathscr{L}$; hence $\varnothing \in \mathscr{L}_\delta$; but then $\varnothing \in \mathscr{M}$ as $\mathscr{K}$ is semi-compact.)

In particular we have shown that for every $K \in \mathscr{K}$, $X \setminus K \in \text{Borelian}\, \mathscr{K}$. Thus $\mathscr{K} \subseteqq \mathscr{B}$, so a) is completely proved.

Let $\tau = \mathscr{M}_\sigma$. Then τ is a topology on X and, as we showed above, each element of $\mathscr{K}$ is τ-closed. Note that τ is the weakest topology on X with respect to which the elements of $\mathscr{M}$ are both open and closed.

Let $Y = \{0, 1\}^{\mathscr{M}}$, equipped with its usual produet topology. Then Y is a compact metrizable space. A compatible metric d on Y may be defined by

$$d(y, y') = \sum_{M \in \mathscr{M}} w(M) |y_M - y'_M| ,$$

where w is a strictly positive function on $\mathscr{M}$ with finite sum over $\mathscr{M}$.

Define $f\colon X \to Y$ by $f(x) = (1_M(x))_{M \in \mathscr{M}}$.

If x, x' are distinct elements of X then, since $\{x\} \in \mathscr{L}_\delta$, there exists $M \in \mathscr{M}$ such that $x \in M$ and $x' \notin M$; hence f is $1-1$.

Clearly then, f is a homeomorphism of (X, τ) onto a subspace of Y.

Thus to show that (X, τ) is Polish, it is sufficient (and necessary) to show that $f[X]$ is a $\mathscr{G}_\delta$-set in Y.

Let ϱ be the metric (compatible with τ) on X defined by $\varrho(x, x') = d(f(x), f(x'))$.

Then as $\mathscr{M}$ is a base for τ, we can find a double sequence $(M(i, j))$ in $\mathscr{M}$ such that for each i, $\{M(i, j) : j = 0, 1, \ldots\}$ is a covering of X by sets of ϱ-diameter less than 2^{-i}. For each i, j choose $U(i, j)$ open in Y such that $M(i, j) = f^{-1}[U(i, j)]$.

Let $T = Z \cap (\bigcap_i \bigcup_j U(i,j))$, where Z is the closure of $f[X]$ in Y. Then T is a $\mathscr{G}_\delta$-set in Y.

We claim $T = f[X]$.

As $X = \bigcup_j M(i,j)$ for each i, $f[X] \subseteq T$. Let us prove the other inclusion. Let $y \in T$. Then since $y \in Z$, there is a sequence (x_k) in X such that $f(x_k) \to y$.

Now for each i, $y \in U(i, j_i)$ for some j_i; then for some k_i, we have $f(x_k) \in U(i, j_i)$ for all $k \geqq k_i$.

Consider any natural number n; let $k = \max\{k_0, \ldots, k_n\}$. Then $x_k \in \bigcap_{i=0}^n M(i, j_i)$; in particular, the intersection is non-empty.

Thus there exists $x \in \bigcap_i M(i, j_i)$, since $\mathscr{K}$ is semicompact. Since ϱ-diameter $(M(i, j_i)) \to 0$, (x_k) τ-converges to x. Hence $f(x) = y$.

PROOF OF 2.3. It is good enough to prove this with $\mathbf{R}$ replaced by $[0, \infty]$. Let $(Y, \mathscr{B})$ be a measurable space, let $Z \subseteq Y$, let $\mathscr{C} = \mathscr{B} \mid Z$, and let $f\colon Z \to [0, \infty]$ be $\mathscr{C}$-measurable. Then $f = \sum_{i=0}^{\infty} c_i 1_{C_i}$ for suitable numbers $c_i \in [0, \infty)$ and sets $C_i \in \mathscr{C}$. For each i, let $B_i \in \mathscr{B}$ such that $B_i \cap Z = C_i$. Let $g = \sum_{i=1}^{\infty} c_i 1_{B_i}$. Then g is $\mathscr{B}$-measurable, and $g \mid Z = f$.

PROOF OF ($\Rightarrow$) OF 2.8. Let Λ be any set of generators for $\mathscr{A}$. Define $g\colon X \to \{0,1\}^\Lambda$ by

$$g(x) = (1_A(x))_{A \in \Lambda} .$$

Then g is a Borel isomorphism onto its range. (g is $1-1$ as $(X, \mathscr{A})$ is separated.) Thus g^{-1} extends to a measurable map $f\colon \{0,1\}^\Lambda \to X$.

REMARK ON THE PROOF OF 2.9. The proof of this trivial result depends, annoyingly, on the full axiom of choice. The rest of the results of this paper, with the exception of 4.6, require no stronger choice axiom than the principle of dependent choice, provided those results which depend on 2.9 are reformulated so as to talk only about *separated* measurable spaces.

PROOF OF 2.10. Let $(X, \mathscr{A})$ have EP. First suppose $(X, \mathscr{A})$ is separated. Then for some set Λ, there is a measurable map of $Y \equiv \{0,1\}^\Lambda$ onto X, by 2.8. Now Y is smooth, by 1.4. Hence $(X, \mathscr{A})$ is smooth, by 1.5(a)). Now if $(X, \mathscr{A})$ is not separated, then conclude by applying 2.9 and then (e) $\Rightarrow$ a)) of 1.3.

PROOF OF 2.11. By 2.10 and 1.5(c)), f is a Borel isomorphism onto its range. Thus as $(X, \mathscr{A})$ has EP, f^{-1} extends to a measurable map $g\colon Y \to X$. Now $f[X] = \{f \circ g = \mathrm{id}_Y\}$, which belongs to $\mathscr{B}$ as $(Y, \mathscr{B})$ is countably separated.

PROOF OF 3.2. First let us emphasize that by "projective" in c) we mean that whenever we have a morphism of $\mathscr{A}$ into a quotient, we can lift it to the numerator. This may not be the same as the abstract categorical definition of projectivity, for I do not know if the categorical epimorphisms in the category of Boolean σ-algebras are really onto maps.

Now it is easy to show that each of a), b), and c) of 3.2 holds for $(X,\mathscr{A})$ iff it holds for the separated measurable space associated with $(X,\mathscr{A})$, Hence let us assume that $(X,\mathscr{A})$ is separated.

(a) $\Rightarrow$ b)). By 2.8, $(X,\mathscr{A})$ is a retract of $\{0,1\}^{\Lambda}$, for some set Λ. Now both products and retracts of V-spaces are V-spaces. Thus $(X,\mathscr{A})$ is a V-space.

(b) $\Rightarrow$ c)). First let us check that $\mathscr{A}$ is projective. Let $\mathscr{B},\mathscr{C}$ be Boolean σ-algebras, let β be a morphism of $\mathscr{B}$ onto $\mathscr{C}$, and let γ be a morphism of $\mathscr{A}$ into $\mathscr{C}$. (Note morphisms preserve 0 and 1, by definition.) By Loomis [8], there is a measurable space $(Y,\mathscr{D})$ and a morphism δ of $\mathscr{D}$ onto $\mathscr{B}$. But then $\beta\circ\delta$ is a $(Y,\mathscr{D})$-valued observable of the logic $\mathscr{C}$, γ is an $(X,\mathscr{A})$-valued observable of $\mathscr{C}$, and range $(\gamma)\subseteqq$ range $(\beta\circ\delta)$. Thus, as $(X,\mathscr{A})$ is a V-space, there is a measurable function $f\colon Y\to X$ such that $\gamma(A)=\beta\circ\delta(f^{-1}[A])$ for all $A\in\mathscr{A}$. Let $\alpha(A)=\delta(f^{-1}[A])$ for all $A\in\mathscr{A}$. Then α is a morphism of $\mathscr{A}$ into $\mathscr{B}$, and $\gamma=\beta\circ\alpha$.

Next, let μ be a two-valued probability measure on $\mathscr{A}$, and let us show that μ is "fixed". Let Y be a one-point set; say $Y=\{p\}$. Let $\mathscr{B}$ be the power set of Y. Let $y=\mathrm{id}_{\mathscr{B}}$ and let $x\colon\mathscr{A}\to\mathscr{B}$ be defined by

$$x(A)=\begin{cases}\varnothing & \text{if } \mu(A)=0\\ Y & \text{if } \mu(A)=1\,.\end{cases}$$

Then x and y are observables of the logic $\mathscr{B}$, and range $(x)=$ range (y). Thus, as $(X,\mathscr{A})$ is a V-space, there is a map $f\colon Y\to X$ such that $x(A)=y(f^{-1}[A])$ for all $A\in\mathscr{A}$. But then $\mu=\delta_{f(p)}$.

(c) $\Rightarrow$ a)). Let $(Y,\mathscr{B})$ be a measurable space, let $Z\subseteqq Y$, let $\mathscr{C}=\mathscr{B}\,|\,Z$, and let $f\colon Z\to X$ be $(\mathscr{C},\mathscr{A})$-measurable. Define $\gamma\colon\mathscr{A}\to\mathscr{C}$ and $\beta\colon\mathscr{B}\to\mathscr{C}$ by $\gamma(A)=f^{-1}[A]$, $\beta(B)=B\cap Z$. Then γ and β are morphisms of Boolean σ-algebras, and β is onto. Hence there is a morphism $\alpha\colon\mathscr{A}\to\mathscr{B}$ such that $\gamma=\beta\circ\alpha$. For each $y\in Y$, $\delta_y\circ\alpha$ is a two-valued probability measure on $\mathscr{A}$, and so is of the form δ_x for some $x\in X$; moreover this x is unique since we are assuming $(X,\mathscr{A})$ is separated. Thus we obtain a map $g\colon Y\to X$ such that $\delta_y\circ\alpha=\delta_{g(y)}$ for each $y\in Y$. Then for any $A\in\mathscr{A}$, $g^{-1}[A]=\alpha(A)$. Thus g is measurable, and

$$(g\,|\,Z)^{-1}[A]=g^{-1}[A]\cap Z=\beta(\alpha(A))=\gamma(A)=f^{-1}[A]\qquad\text{for all } A\in\mathscr{A}\,.$$

As $(X,\mathscr{A})$ is separated, it follows that $g\,|\,Z=f$.

Proof of 3.4. By 3.2, $(X,\mathscr{A})$ and $(Y,\mathscr{B})$ are V-spaces. From this it follows easily that there are measurable maps $f\colon X\to Y$, $g\colon Y\to X$ such that

$$\Phi(A/\mathscr{I}) = g^{-1}[A]/\mathscr{I} \quad \text{and} \quad \Phi^{-1}[B/\mathscr{I}] = f^{-1}[B]/\mathscr{I}$$

for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$. Let $I = \{g \circ f \neq \mathrm{id}_X\}$, $J = \{f \circ g \neq \mathrm{id}_Y\}$. Let $\varphi = g \mid Y \setminus J$. Then φ is a $1-1$ map of $Y \setminus J$ onto $X \setminus I$, and $\varphi^{-1} = f \mid X \setminus I$. Also, as $(X, \mathscr{A})$ and $(Y, \mathscr{B})$ are countably separated, it follows from 3.3 that $I \in \mathscr{I}$ and $J \in \mathscr{I}$. Now if $\mathscr{A}/\mathscr{I}$ is not purely atomic, then X must be uncountable so $(X, \mathscr{A})$ is Borel isomorphic to $\{0,1\}^{\mathsf{N}}$ by 2.6 of Christensen [2]. But then, letting $\Sigma = \{0, 2, 4, \ldots\}$, $T = \{1, 3, 5, \ldots\}$, we can write X as $\bigcup_{\alpha \in \{0,1\}^\Sigma} X_\alpha$, where for each α, $X_\alpha \in \mathscr{A}$ and $(X_\alpha, \mathscr{A} \mid X_\alpha)$ is Borel isomorphic to $\{0,1\}^T$. Thus if we also assume that $\mathscr{A}/\mathscr{I}$ is σ-finite, then for some α, $X_\alpha/\mathscr{I} = 0$. Then let $I' = I \cup X_\alpha$, $J' = J \cup \varphi^{-1}[X_\alpha]$. Then I' and J' are uncountable, so again by 2.6 of Christensen [2], there is a Borel isomorphism ψ of J' onto I'. To conclude, let

$$\varphi' = \begin{cases} \varphi & \text{on } Y \setminus J' \\ \psi & \text{on } J' . \end{cases}$$

Then φ' is a Borel isomorphism of Y onto X, and $\Phi(A/\mathscr{I}) = \varphi'^{-1}[A]/\mathscr{I}$ for all $A \in \mathscr{A}$.

Proof of 3.5. By 20 (F) of Sikorski [11], $\mathscr{A}/\mathscr{I} = \{U/\mathscr{I} : U \text{ open } \subseteq X\}$. Suppose U is open in X and $U/\mathscr{I}$ is an atom of $\mathscr{A}/\mathscr{I}$. If p, q are distinct points of U then there are disjoint open sets $V, W \subseteq U$ with $p \in V$, $q \in W$. But then $0 \neq V/\mathscr{I} \neq U/\mathscr{I}$, contradicting the assumption that $U/\mathscr{I}$ is an atom. Thus $U = \{p\}$ for some p which is an isolated point of X. As there are only countably many such isolated points, $\mathscr{A}/\mathscr{I}$ is not purely atomic. Next suppose (U_i) is a family of open sets such that $(U_i/\mathscr{I})$ is an orthogonal family in $\mathscr{A}/\mathscr{I}$. For each i, let

$$V_i = U_i \setminus \overline{\bigcup_{i' \neq i} U_{i'}} .$$

Then (V_i) is a disjoint family of open sets in X, so $V_i = \emptyset$ for all but countably many i. But by 6.35 of Kelley [5], $U_i \setminus V_i$ is of the first category in X for each i. Thus for every i, $U_i/\mathscr{I} = V_i/\mathscr{I}$. Hence $\mathscr{A}/\mathscr{I}$ is σ-finite.

Proof of 4.2.. (a) $\Rightarrow$ b)). As A and B are analytic, there are topological spaces Y and Z, sets $M \subseteq Y$ and $N \subseteq Z$, and continuous maps $g\colon M \to X$ and $h\colon N \to X$ such that

$$g[M] = A ,$$

$$h[N] = B ,$$

$$M = \bigcap_{i \in \mathsf{N}} \bigcup_{j \in \mathsf{N}} K_j^i \text{ where each } K_j^i \text{ is a closed compact subset of } Y,$$

$$N = \bigcap_{i \in \mathbf{N}} \bigcup_{j \in \mathbf{N}} L_j^i \text{ where each } L_j^i \text{ is a closed compact subset of } Z\,.$$

Suppose A and B cannot be separated by disjoint Baire sets. Then by induction we can construct $j(0), j'(0), j(1), j'(1), \ldots \in \mathbf{N}$ such that for each $k \in \mathbf{N}$, $g[K(k) \cap M]$ and $h[L(k) \cap N]$ cannot be separated by disjoint Baire sets, where

$$K(k) = \bigcap_{i=0}^{k} K_{j(i)}^i$$

$$L(k) = \bigcap_{i=0}^{k} L_{j'(i)}^i\,.$$

Let

$$K = \bigcap_{k \in \mathbf{N}} K(k), \qquad L = \bigcap_{k \in \mathbf{N}} L(k)\,.$$

Then K is a compact subset of M and L is a compact subset of N. It follows readily from the hypothesis of a) that there are open Baire sets $U, V \subseteq X$ such that $g[K] \subseteq U$, $h[L] \subseteq V$, and $U \cap V = \varnothing$. Now $g^{-1}[U] = G \cap M$ for some open set $G \subseteq Y$. Next, for some $k_1 \in \mathbf{N}$, $K(k_1) \subseteq G$, as $(K(k))$ is a decreasing sequence of closed compact subsets of Y whose intersection is contained in G. Then $g[K(k_1) \cap M] \subseteq U$. Similarly, for some $k_2 \in \mathbf{N}$, $h[L(k_2) \cap N] \subseteq V$. Let $k = \max\{k_1, k_2\}$. Then $g[K(k) \cap M]$ and $h[L(k) \cap N]$ are separated by the disjoint Baire sets U and V, which is a contradiction.

(b) $\Rightarrow$ c)). Let E, F be disjoint Baire sets which separate A, B. Then there is a sequence (f_i) in $C(X, \mathbf{R})$ such that $E, F \in \sigma(f_0, f_1, \ldots)$. We can take φ to be the element of $C(X, \mathbf{R}^{\mathbf{N}})$ defined by $\varphi(x) = (f_0(x), f_1(x), \ldots)$.

(c) $\Rightarrow$ a)). Clear.

Proof of 4.4. There is a topological space Y, a $\mathscr{K}_{\sigma\delta}$ set $X' \subseteq Y$, where $\mathscr{K}$ is the collection of closed compact subsets of Y, and a continuous map f of X' onto X. Now Baire X is equal to the Borelian family generated by the zero sets in X. Thus if we let $\mathscr{A} = f^{-1}$ (Baire X), then $\mathscr{A} \subseteq$ Borelian $\mathscr{K}$. Thus $(X', \mathscr{A})$ is smooth. Hence $(X, \text{Baire } X)$ is smooth by 1.5.

Proof of 4.6. Let us begin by showing that X must be separable. Suppose not. Then (provided we work in a set theory in which we can make aleph$_1$ choices successively, each depending on all the preceding ones) we can construct a closed discrete set S in X, with card $(S) = \text{aleph}_1$. The metrizability of X is used in this construction. Let f be a $1-1$ map of S into $\mathbf{R}$. As each subset of S is closed in X, f is measurable. Thus $f[S]$ is analytic, by 1.3. But an uncountable analytic subset of $\mathbf{R}$ contains a homeomorph of $\{0,1\}^{\mathbf{N}}$, by the theorem in III.36.V of Kuratowski [6]. (We remark that the proof of this fact given by Kuratowski appears to depend on the full axiom of choice, but this can be avoided by using a (measurable) selection theorem depending only on the principle of dependent choice; e.g., see 6. of Kuratowski [7].) Thus card $(f[S])=c$, where $c=\text{card}\,(\mathbf{R})$. But every subset of $f[S]$ is the image, under f, of a closed subset of S, and so is analytic. This implies that $\text{card}(\text{Souslin}(\text{Borel}\,\mathbf{R}))=2^c$, which is a contradiction. Thus X must be separable. Hence X is homeomorphic to a subset Y of $\mathbf{R}^{\mathbf{N}}$. As $(X, \text{Baire}\,X)$ is smooth, Y is analytic, by 1.8.

6. Concluding remarks.

First I would like to say that I discussed this paper with J. Hoffman-Jørgensen and he informed me that he has proved the following result:

THEOREM. *For a measurable space $(X, \mathscr{A})$, consider the following statements:*

a) *For every measurable space $(Y, \mathscr{B})$ and every $F \in \mathscr{A} \otimes \mathscr{B}$, we have $\pi_Y[F] \in \text{Souslin}\,\mathscr{B}$, where π_Y is the projection of $X \times Y$ on Y.*
b) *For every $\mathscr{A}$-valued Souslin scheme (A_s), the set $\{\sigma \in \mathbf{N}^{\mathbf{N}} \colon \bigcap_{s<\sigma} A_s \neq \varnothing\}$ is an analytic subset of $\mathbf{N}^{\mathbf{N}}$, where $\mathbf{N}^{\mathbf{N}}$ is given its usual product topology.* (*See the proof of* 1.3 *for the notation.*)
c) *Whenever $(Y, \mathscr{B})$ is a measurable space, $Z \subseteq Y$, and $f \colon Z \to X$ is $\mathscr{B}\,|\,Z$-measurable, there exists $W \in \text{Souslin}\,\mathscr{B}$ and a $\mathscr{B}\,|\,W$-measurable map $g \colon W \to X$ such that $Z \subseteq W$ and $g\,|\,Z = f$.*

Then a) *and* b) *are equivalent and are implied by* c).

In view of (a) $\Leftrightarrow$ c)) of 1.3, this theorem of Hoffmann-Jørgensen yields an additional characterization of smooth measurables spaces, namely b), and also generalizes 2.10. More recently, Hoffmann-Jørgensen has found examples of smooth measurable spaces which do not satisfy his condition c); in particular, he showed that an uncountable set with its countable-cocountable Borel structure is a smooth measurable space which does not satisfy c).

Some readers may be unfamiliar with the notion of a semicompact class; for this they may consult Christensen [2].

I would like to emphasize that the point of view taken in section 1 of this paper makes it possible to develop the theory of analytic measurable spaces without first developing the theory of classical analytic topological spaces. To make this clear, let me mention two points. First, we have used the fact that a subset of $\mathbf{R}$ is analytic iff it is an element of Souslin(Borel $\mathbf{R}$), but we could have just taken this as the definition of analyticity for subsets of $\mathbf{R}$ for the purposes of section 1. Second, we have used the fact that any disjoint pair of analytic subsets of R can be separated by disjoint Borel sets. This can be proved topologically, as in 4.2, but it is also a consequence of the following "abstract" separation theorem:

THEOREM. *Let* $\mathscr{K}$ *be a semicompact class of subsets of a set* X, *and assume* $K_1, K_2 \in \mathscr{K} \Rightarrow K_1 \cap K_2 \in \mathscr{K}$. *Let* $\mathscr{B}$ *be a Borelian family of subsets of* X *such that* $K_1, K_2 \in \mathscr{K}, K_1 \cap K_2 = \varnothing \Rightarrow$ *there exist* $B_1, B_2 \in \mathscr{B}$ *with* $K_1 \subseteqq B_1$, $K_2 \subseteqq B_2$, *and* $B_1 \cap B_2 = \varnothing$. *Suppose* $A_1, A_2 \in \text{Souslin}\,\mathscr{K}$ *with* $A_1 \cap A_2 = \varnothing$. *Then there exist* $B_1, B_2 \in \mathscr{B}$ *with* $A_1 \subseteqq B_1$, $A_2 \subseteqq B_2$, *and* $B_1 \cap B_2 = \varnothing$.

This theorem can be proved in the same manner as 5.8 of Bressler and Sion [1]. Also, note that one suitable choice of $\mathscr{B}$ is $\mathscr{B} = \text{Borelian}\,\mathscr{K}$.

Perhaps a little more explanation of the statement of proposition 1.10 would be appropriate. Suppose $(X, \mathscr{A})$ is a smooth measurable space and $Y_1, Y_2 \in \text{Souslin}\,\mathscr{A}$ with $Y_1 \cap Y_2 = \varnothing$. Then $(Y_1, \mathscr{A} \mid Y_1)$ and $(Y_2, \mathscr{A} \mid Y_2)$ are smooth and b) of 1.10 is satisfied (we can take $\mathscr{S}$ to consist of the collection of sets occurring in a pair of $\mathscr{A}$-valued Souslin schemes defining Y_1 and Y_2, respectively). Thus we can conclude that a) of 1.10 holds; that is Y_1 and Y_2 can be separated by disjoint members of $\mathscr{A}$.

Finally, here is a typical application of a) of proposition 1.13. Let $(\Omega, \mathscr{F})$ be a measurable space, let $(E, \mathscr{E})$ be a separated measurable space, and let $(X_t)_{0 \leqq t < \infty}$ be a process in $(E, \mathscr{E})$ over $(\Omega, \mathscr{F})$. For $0 \leqq t < \infty$ let

$$\mathscr{F}_t = \sigma(X_s\colon 0 \leqq s \leqq t)$$
$$\mathscr{F}_{t+} = \bigcap_{s>t} \mathscr{F}_s .$$

If ω and ω' are elements of Ω, then ω and ω' lie on the same fiber of $\mathscr{F}_t$, iff $X_s(\omega) = X_s(\omega')$ for $0 \leqq s \leqq t$. Also, ω and ω' lie in the same fiber of $\mathscr{F}_{t+}$ iff there exists $\varepsilon > 0$ such that $X_s(\omega) = X_s(\omega')$ for $0 \leqq s \leqq t + \varepsilon$. (Note that ε depends on ω and ω' and that as ω and ω' vary, arbitrarily small values of ε will in general occur.) Now suppose in addition that E is a separable metrizable space, that $\mathscr{E} = \text{Borel}\,E$, and that (X_t) is either right-continuous or left-continuous. Then for each t, $\mathscr{F}_t$ is countably generated and $\mathscr{F}_{t+}$ is thus an intersection of countably generated sub-σ-fields of $\mathscr{F}$. Finally, suppose $(\Omega, \mathscr{F})$ is smooth. Then by 1.13(a)), $\mathscr{F}_t$ and $\mathscr{F}_{t+}$ are determined by their fibers. From this is obtain the so-called:

GALMARINO'S TEST. Let $T\colon \Omega \to [0, \infty]$ be $\mathscr{F}$-measurable. Then:

a) T is an $(\mathscr{F}_t)$-stopping time iff $[t \in [0,\infty)$, $\omega, \omega' \in \Omega$, $X_s(\omega) = X_s(\omega')$ for $0 \leqq s \leqq t \Rightarrow (T(\omega) \leqq t$ and $T(\omega') \leqq t)$ or $(T(\omega) > t$ and $T(\omega') > t)]$.

b) T is an $(\mathscr{F}_{t+})$-stopping time iff $[t \in [0,\infty)$, $\omega, \omega' \in \Omega$, $X_s(\omega) = X_s(\omega')$ for $0 \leqq s \leqq t + \varepsilon$ for some $\varepsilon > 0 \Rightarrow (T(\omega) \leqq t$ and $T(\omega') \leqq t)$ or $(T(\omega) > t$ and $T(\omega') > t)]$.

In words, T is an $(\mathscr{F}_t)$-stopping time iff T depends only on the past and present; T is an $(\mathscr{F}_{t+})$-stopping time iff T depends only on the past, the present, and the "infinitesimal future". We caution the reader not to overlook the hypotheses made in the paragraph preceding the statement of Galmarino's test.

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