MATH. SCAND. 50 (1982), 19–24

A CHARACTERIZATION OF GORENSTEIN ORDERS IN QUATERNION ALGEBRAS

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0. Introduction.

Let Q be a quaternion algebra over the field of fractions F of a Dedekind ring A. Let Q_0 be the F-subspace of Q consisting of the quaternions with trace equal to zero. An A-lattice L on Q_0 is a finitely generated A-module such that $FL=Q_0$, and an A-order Λ in Q is an A-algebra in Q finitely generated as a module over A, containing the identity element of Q and such that $F\Lambda=Q$. Each A-lattice L on Q_0 defines, in a natural way which we recall later, an A-order $O(L)$ in Q. The aim of this note is to characterize the class of orders in Q which can be obtained in this way. The main theorem says that $\Lambda=O(L)$ for an A-lattice L on Q_0 if and only if Λ is a Gorenstein order. Recall that Λ is a Gorenstein order if Λ, considered as a left (or right) Λ-module, is Λ-injective (in the category of left Λ-lattices). Our main objective was to give a proof of this result which was not limited by the assumption that the characteristic of F is different from 2. If char $(F)\neq 2$, it would be possible to deduce a proof from some known results contained in [2], [4], and Proposition (2.3) of the present paper. Unfortunately the relevant proofs in [4] do not generalize to arbitrary characteristic, so we have to give new proofs in the general case. At the same time we give a proof of Brandt's invertibility criterion (Theorem (3.5.)) in which we use orders instead of quadratic forms (as e.g. in [1] or [2, Theorem 10]). This gives a natural generalization of the result to arbitrary characteristic.

1. Some computations.

Let Q be a quaternion algebra over a field F, that is, Q is a central simple F-algebra of dimension 4. Q has an involution $x \mapsto x^*$ such that $T(x)=x+x^*$ and $N(x)=xx^*$ are elements of F equal respectively to the reduced trace and the reduced norm of $x \in Q$. Let $Q_0=\{x \in Q \mid T(x)=0\}$. Define:

$$\{x_1,x_2,x_3\} = T((x_1x_2-x_2x_1)x_3^*) \quad \text{for } x_1,x_2,x_3 \in Q ,$$

and

Received August 11, 1980.

$$\{x_1, x_2, x_3\}_0 = T(x_1 x_2 x_3) \quad \text{for } x_1, x_2, x_3 \in Q_0 .$$

Then $\{x_1, x_2, x_3\}$ and $\{x_1, x_2, x_3\}_0$ are multilinear and alternating. Note that $\{x_1, x_2, x_3\} = 0$ whenever some of the x_i belongs to F.

If $x_1, x_2, \ldots, x_n \in Q$, denote by $d(x_1, x_2, \ldots, x_n)$ the determinant of the matrix $[T(x_i x_j^*)]$. If $x_1, x_2, x_3 \in Q_0$, let $d_0(x_1, x_2, x_3) = (1/2) d(x_1, x_2, x_3)$ which, in the case of char $(F) = 2$, will be understood as:

$$T(x_1 x_2^*)T(x_2 x_3^*)T(x_3 x_1^*) + N(x_1)T(x_2 x_3^*)^2 + N(x_2)T(x_3 x_1^*)^2 + N(x_3)T(x_1 x_2^*)^2$$

(1.1) LEMMA. (a) $\{x_1, x_2, x_3\}^2 = d(1, x_1, x_2, x_3)$.
(b) $\{x_1, x_2, x_3\}_0^2 = d_0(x_1, x_2, x_3) = \{x_1 x_2, x_2 x_3, x_3 x_1\}$.

PROOF. Let F_s be a splitting field of Q, that is $Q \otimes_F F_s$ is isomorphic to the matrix algebra $M_2(F_s)$. Let

$$e_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad e_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad e_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad \text{and } e_3 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

If $x_i \otimes 1 = \sum a_{ij} e_j$, $a_{ij} \in F_s$, then

$$\{x_1, x_2, x_3\} = \{x_1 \otimes 1,\ x_2 \otimes 1,\ x_3 \otimes 1\} = (\det [a_{ij}]) \{e_1, e_2, e_3\}$$

and

$$d(1, x_1, x_2, x_3) = (\det [a_{ij}])^2 d(e_0, e_1, e_2, e_3)$$

where $i, j \geqq 1$. Since the equality $\{e_1, e_2, e_3\}^2 = d(e_0, e_1, e_2, e_3)$ is evident, (a) is proved. To prove the first equality in (b), it suffices to repeat similar calculations using e_1, e_2 and $e_3' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. The second equality in (b) can be checked directly.

Note that e_1, e_2, e_3' is a basis of $Q_0 \otimes_F F_s$ and $d_0(e_1, e_2, e_3') = 1$. Hence we get

(1.2) COROLLARY. *If x_1, x_2, x_3 is a basis of Q_0, then $d_0(x_1, x_2, x_3) \neq 0$.*

2. Locally principal lattices.

Let L be an A-lattice on Q, where A is a Dedekind ring with the field of fractions F. Denote by $N(L)$ the A-ideal in F generated by the norms $N(x)$ for $x \in L$. By $D(L)$ we denote the discriminant of L, that is, the A-ideal in F generated by $d(x_1, x_2, x_3, x_4)$, where $x_i \in L$. If A is a discrete valuation ring and $N(L) = (N(x))$, where $x \in L$, then $Lx^{-1} = \langle 1, x_1, x_2, x_3 \rangle$ ($\langle a_1, a_2, \ldots, a_n \rangle$ denotes the A-submodule of Q generated by $a_i \in Q$), and $D(Lx^{-1}) = (\{x_1, x_2, x_3\}^2)$ by

(1.1) (a). Since $D(Lx^{-1}) = N(x)^{-4}D(L)$, $D(L)$ is a square of an A-ideal in F. We denote by $\mathfrak{d}(L)$ the square-root of $D(L)$.

If L and L' are A-lattices on Q, then $[L:L']$ denotes the product of the invariant factors of L' in L, that is, if e_i is a basis of Q over F such that $L = \oplus\, \mathfrak{a}_i e_i$, $L' = \oplus\, \mathfrak{a}'_i e_i$, where $\mathfrak{a}_i, \mathfrak{a}'_i$ are A-ideals in F, then $[L:L'] = \prod \mathfrak{a}'_i \mathfrak{a}_i^{-1}$. Note that $L \supseteqq L'$ and $[L:L'] = A$ imply $L = L'$.

An A-lattice L on Q is (left) Λ-principal if $L = \Lambda x$, where Λ is an order and $x \in Q$. L is locally (Λ-)principal if for each prime ideal $\mathfrak{p}$ in A the localization $L_\mathfrak{p}$ is $\Lambda_\mathfrak{p}$-principal, where $\Lambda = \{x \in Q \mid xL \subseteqq L\}$.

(2.1) PROPOSITION. *Let L be a left Λ-lattice. L is locally Λ-principal if and only if* $[\Lambda:L] = N(L)^2$.

PROOF. We may assume that A is local. For each $x \in L$, we have the equalities

$$(2.2) \qquad [\Lambda:L][L:\Lambda x] = [\Lambda:\Lambda x] = (N(x)^2),$$

which prove the Proposition if we choose $x \in L$ such that $N(L) = (N(x))$.

Let $\Lambda^{\#} = \{x \in Q \mid T(x\Lambda) \subseteqq A\}$. Λ is a Gorenstein order if and only if $\Lambda^{\#}$ is Λ-projective as a left (or right) Λ-module ([5, p. 252]).

(2.3) PROPOSITION. *Λ is a Gorenstein order if and only if* $N(\Lambda^{\#})\mathfrak{d}(\Lambda) = A$.

PROOF. In the case of quaternion algebras $\Lambda^{\#}$ is Λ-projective if and only if $\Lambda^{\#}$ is locally Λ-principal ([2, Theorem 2]). Hence by Proposition (2.1), Λ is a Gorenstein order if and only if $[\Lambda:\Lambda^{\#}] = N(\Lambda^{\#})^2$. But $[\Lambda:\Lambda^{\#}] = \mathfrak{d}(\Lambda)^{-2}$, which proves our assertion.

(2.4) REMARK. Note that for each A-order Λ in Q, $N(\Lambda^{\#})\mathfrak{d}(\Lambda) \subseteqq A$ which follows from (2.2) for $L = \Lambda^{\#}$.

3. Orders defined by ternary lattices.

Let (V, q) be a quadratic space over a field F, that is, $q\colon V \to F$ is a mapping such that $q(ax) = a^2 q(x)$ for $a \in F$, and

$$b(x,y) = q(x+y) - q(x) - q(y)$$

is bilinear. The Clifford algebra $C(V,q)$ of (V,q) is the F algebra $T(V)/I$, where $T(V)=\oplus_{i\geqq 0}T^i(V)$ is the tensor algebra of V, and I is the ideal of $T(V)$ generated by $x\otimes x-q(x)$, $x\in V$. We denote by $C_0(V,q)$ the subalgebra of $C(V,q)$ generated by $1\in T^0(V)=F$ and the images in $C(V,q)$ of the products $x_1\otimes\ldots\otimes x_{2r}\in T^{2r}(V)$, for $r>0$, which we denote by $[x_1,\ldots,x_{2r}]$.

As earlier, assume that F is the field of fractions of a Dedekind ring A. If L is an A-lattice on V, denote by $N(L)$ the norm of L, that is, the A-ideal in F generated by $q(x)$, $x\in L$. We denote by $O(L)$ the A-order in $C_0(V,q)$ corresponding to L, which is generated as an A-module by the elements 1 and $a[x_1,\ldots,x_{2r}]$, where $x_i\in L$, $a\in F$, and $aN(L)^r\subseteqq A$.

If (V,q) is a half-regular ([3, (2.14)]) ternary quadratic space, then $C_0(V,q)=Q$ is a quaternion algebra over F ([3, (5.21) and (6.11)]), and (V,q) is similar to the quadratic space (Q_0,N), where N is the reduced norm on Q ([3, (5,20)]). Moreover, $Q\cong C_0(Q_0,N)$, where an isomorphism is given by $[x,y]\mapsto xy^*$. Since the order $O(L)$ is the same if we replace the lattice L by a similar one, we may restrict our investigation of orders $O(L)$, where L is a lattice on (V,q), to orders $O(L)$ where L is a lattice on (Q_0,N).

(3.1) PROPOSITION. *If L is an A-lattice on (Q_0,N), then $\Lambda=O(L)$ is a Gorenstein order in Q.*

PROOF. We may assume that A is a discrete valuation ring (see e.g. (2.3)). Let $L=\langle x_1,x_2,x_3\rangle$ and $N(L)=(n)$. Then

$$\Lambda = O(L) = \langle 1,(1/n)x_1x_2,(1/n)x_2x_3,(1/n)x_3x_1\rangle .$$

Hence by (1.1)

$$\mathfrak{d}(\Lambda) = N(L)^{-3}\{x_1x_2,x_2x_3,x_3x_1\} = N(L)^{-3}\{x_1,x_2,x_3\}_0^2 .$$

One checks directly that

$$\Lambda^{\#} = \{x_1,x_2,x_3\}_0^{-1}\langle x_1x_2x_3,nx_3,nx_1,nx_2\rangle .$$

Since $N(\Lambda^{\#})=N(\Lambda^{\#}\cap Q_0)$ ([4, p. 341]), we get $N(\Lambda^{\#})=\{x_1,x_2,x_3\}_0^{-2}N(L)^3$. Hence $N(\Lambda^{\#})\mathfrak{d}(\Lambda)=A$, so Λ is a Gorenstein order by (2.3).

Following [2, p. 222], L is called a semi-order if $1\in L$ and $N(L)=A$. Note that each element of a semi-order L is integral over A, since $N(x)\in A$ and $N(1+x)\in A$ imply that $T(x)\in A$.

(3.2) PROPOSITION. *If Λ is a semi-order such that $N(\Lambda^{\#})\mathfrak{d}(\Lambda)\subseteqq A$, then Λ is an order. Moreover, $\Lambda=\langle 1,\mathfrak{d}(\Lambda)\Lambda^{\#}\Lambda^{\#}\rangle$.*

PROOF. It suffices to prove the last equality, since $N(\Lambda^{\#})^{-1}\Lambda^{\#}\Lambda^{\#}$ is always an order ([2, Theorem 6]), and the assumption $N(\Lambda^{\#})\mathfrak{d}(\Lambda)\subseteqq \Lambda$ implies that $\langle 1, \mathfrak{d}(\Lambda)\Lambda^{\#}\Lambda^{\#}\rangle$ must be an order as well.

As earlier, we may assume that A is local. Let $\Lambda=\langle 1, x_1, x_2, x_3\rangle$. Then $\Lambda^{\#}=\langle y_0, y_1, y_2, y_3\rangle$, where $\sum T(x_i x_j^*)y_j = x_i$ with $x_0=1$. It is easy to check that

$$x_i = \{y_1, y_2, y_3\}_0^{-1}(T(y_j y_k y_0^*) - y_j y_k) \tag{3.3}$$

where (i, j, k) is a cyclic permutation of $(1,2,3)$. By Lemma (1.1)

$$d = \{x_1, x_2, x_3\} = \{y_1, y_2, y_3\}_0^{-3}\{y_1 y_2, y_2 y_3, y_3 y_1\} = \{y_1, y_2, y_3\}_0^{-1}$$

is a generator of $\mathfrak{d}(\Lambda)$. From (3.3) we get $dy_j y_k = x_i - dT(y_j y_k y_0^*)$. Since $\mathfrak{d}(\Lambda)\Lambda^{\#}\Lambda^{\#}\subseteqq N(\Lambda^{\#})^{-1}\Lambda^{\#}\Lambda^{\#}$, the products $dy_j y_k$ are integral over A, so $dT(y_i y_k y_0^*) \in A$. Hence

$$\Lambda = \langle 1, dy_1 y_2, dy_2 y_3, dy_3 y_1\rangle . \tag{3.4}$$

But all the remaining products $dy_j y_k$, $j, k \in \{0,1,2,3\}$ are also in Λ, which follows easily from the equalities $y_j^2 = -N(y_j)$ for $j=1,2,3$, $y_0^2 = y_0 - N(y_0)$, and

$$y_j y_k = \sum_{i=0}^{3} T(y_j y_k y_i^*)x_i, \qquad x_0 = 1 .$$

Hence $\Lambda=\langle 1, \mathfrak{d}(\Lambda)\Lambda^{\#}\Lambda^{\#}\rangle$.

(3.4) THEOREM. *Λ is a Gorenstein order if and only if $\Lambda=O(L)$, where L is an A-lattice on Q_0.*

PROOF. If Λ is a Gorenstein order, then by Proposition (2.3), $\mathfrak{d}(\Lambda)=N(\Lambda^{\#})^{-1} = N(L)^{-1}$, where $L=\Lambda^{\#}\cap Q_0$. Hence (3.4) says that $\Lambda=O(L)$. The converse was proved in Proposition (3.1).

(3.5) THEOREM. (Brandt's invertibility criterion). *Let L be an A-lattice on Q. L is locally principal if and only if $N(L^{\#})\mathfrak{d}(L)\subseteqq N(L)$.*

PROOF. We may assume that A is discrete valuation ring. If $N(L)=(N(x))$, $x \in L$, we replace L by Lx^{-1}. Then $\Lambda=Lx^{-1}$ is a semi-order such that $N(\Lambda^{\#})\mathfrak{d}(\Lambda)\subseteqq \Lambda$. Hence Λ is an order by (3.2), and $L=\Lambda x$. The converse follows from (2.1).

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