MATH. SCAND. 50 (1982), 66–72

AN L^p-ESTIMATE FOR THE GRADIENT OF EXTREMALS

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1. Introduction.

Let $G \subset \mathbf{R}^n$ be a bounded domain. We study variational integrals

$$(1.1) \qquad I(u) = \int_G F(x, \nabla u(x))\, dm(x)\,,$$

where the function u belongs to the Sobolev space $W_n^1(G)$ and the kernel F: $G \times \mathbf{R}^n \to \mathbf{R}$ satisfies the following structure conditions

(1.2). The functions $x \to F(x, \nabla u(x))$ are measurable for all $u \in W_n^1(G)$.

(1.3). For a.e. $x \in G$ the function $z \to F(x,z)$ is convex and $\alpha|z|^n \leqq F(x,z) \leqq \beta|z|^n$ for all $z \in \mathbf{R}^n$, where $\alpha, \beta > 0$.

Fix $\varphi \in W_n^1(G)$, and let $W_{n,0}^1(G)$ denote the closure of $C_0^\infty(G)$-functions in $W_n^1(G)$. We define

$$\mathscr{F}_\varphi(G) = \{u \in W_n^1(G) \mid u - \varphi \in W_{n,0}^1(G)\}\,.$$

A function $u_0 \in \mathscr{F}_\varphi(G)$ is an extremal for the integral (1.1) if $I(u) \geqq I(u_0)$ for all $u \in \mathscr{F}_\varphi(G)$.

In this paper we prove local and global $L^{n+\varepsilon}$-integrability results for the gradient of the extremal u_0. The local version is as follows:

1.4. THEOREM. *The extremal u_0 belongs locally to the space $W_{n+\varepsilon}^1(G)$. The constant $\varepsilon > 0$ depends only on n and α/β.*

The first result of this type has been proved by Bojarski [1]. He studied solutions of two dimensional, first order, uniformly elliptic systems. Linear equations and systems in $\mathbf{R}^n$ have been considered by Meyers [8]. In 1973 Gehring [2] proved the local $L^{n+\varepsilon}$-integrability for the derivatives of quasiconformal mappings in $\mathbf{R}^n$. The corresponding result for quasiregular

Received October 8, 1980; in revised form January 23, 1981.

mappings has been proved by Martio [6] and Meyers-Elcrat [9]. Our proof for Theorem 1.4 is based on a modification of the important lemma of Gehring in [2]. Such a modification has been proved by Giaquinta-Modica [3], see also Stredulinsky [12].

Next we consider global integrability. We show that if the boundary function φ is in $W^1_{n+\varepsilon}(G)$, and if G satisfies certain additional restrictions, then the gradient of the extremal belongs to $L^{n+\varepsilon_0}(G)$ for some $\varepsilon_0>0$. The condition for ∂G is the following.

Suppose $x_0 \in \mathsf{R}^n$ and $r>0$. We consider cubes on R^n.

$$Q(r) = \{x \in \mathsf{R}^n \mid |x_0^i - x^i| < r, \quad i=1,\ldots,n\} .$$

Let $G \subset \mathsf{R}^n$ be a bounded domain. Take an arbitrary cube $Q(\frac{3}{2}r)$. Now either (i) $Q(\frac{3}{2}r) \cap (\mathsf{R}^n \setminus G) = \varnothing$ *or* (ii) $Q(\frac{3}{2}r) \cap (\mathsf{R}^n \setminus G) \neq \varnothing$. We assume that there is a constant $\delta>0$ such that for all cubes in the case (ii)

$$m(Q(2r) \cap (\mathsf{R}^n \setminus G))/m(Q(2r)) \geqq \delta .$$

Clearly all convex domains satisfy this condition.

1.5. THEOREM. *Assume that the above boundary condition is satisfied with a constant $\delta>0$. There is a constant $t=t(n,\delta,\alpha/\beta)>0$ such that if $\varphi \in W^1_{n+\varepsilon}(G)$, then the gradient of the extremal u_0 belongs to $L^{n+\varepsilon_0}(G)$, where $\varepsilon_0=\min\{\varepsilon, t\}$.*

2. Proof for the integrability results.

2.1. *Auxiliary lemmas.*

We need three lemmas on Sobolev functions defined in cubes $Q(r)$ in R^n. The last lemma is the essential tool and it is due to Giaquinta-Modica [3].

2.2. LEMMA. *Let $u \in W^1_n(Q(r))$ and $\int_{Q(r)} u\,dm=0$. Then the following inequality is valid:*

$$\left(\int_{Q(r)} |u|^n\,dm\right)^{\frac{1}{n}} \leqq c_0(n)\left(\int_{Q(r)} |\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}} \tag{2.3}$$

PROOF. See [5, p. 45] and [4, pp. 148–151, p. 164].

2.4. LEMMA. *Suppose that $u \in W^1_n(Q(r))$. Write*

$$S = \{x \in Q(r) \mid u(x) = 0\} .$$

If there is a constant $\mu>0$ *such that* $m(S)\geqq\mu m(Q(r))$, *then*

$$\left(\int_{Q(r)}|u|^n\,dm\right)^{\frac{1}{n}}\leqq c_1(n,\mu)\left(\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}. \tag{2.5}$$

PROOF. First observe that the following inequality is valid

$$\int_{Q(r)}|u|^{\frac{n}{2}}\,dm\leqq c_2(n,\mu)r^{\frac{n}{2}}\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm. \tag{2.6}$$

For a proof of (2.6) see [5, p. 54, Lemma 3.4]. Now write

$$h=\frac{1}{m(Q(r))}\int_{Q(r)}u\,dm.$$

We use Minkowski's inequality and Lemma 2.2

$$\left(\int_{Q(r)}|u|^n\,dm\right)^{\frac{1}{n}}\leqq\left(\int_{Q(r)}|u-h|^n\,dm\right)^{\frac{1}{n}}+m(Q(r))^{\frac{1}{n}}|h|$$

$$\leqq c_0(n)\left(\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}+\frac{1}{m(Q(r))^{1-\frac{1}{n}}}\int_{Q(r)}|u|\,dm$$

$$\leqq c_0(n)\left(\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}+\frac{1}{m(Q(r))^{1-\frac{1}{n}}}m(Q(r))^{1-\frac{2}{n}}\left(\int_{Q(r)}|u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}$$

$$\leqq c_0(n)\left(\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}+\frac{1}{m(Q(r))^{\frac{1}{n}}}c_2(n,\mu)^{\frac{2}{n}}r\left(\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}$$

$$\leqq c_1(n,\mu)\left(\int_{Q(r)}|\nabla u|^{\frac{n}{2}}\,dm\right)^{\frac{2}{n}}.$$

2.7. LEMMA. *Let* $Q(2a)$ *be a cube in* $\mathbf{R}^n$. *Assume that* g *and* f *are non-negative functions in* $Q(2a)$ *and that* $g\in L^q(Q(2a))$, $q>1$, $f\in L^s(Q(2a))$, $s>q$. *Suppose that for every* $x\in Q(2a)$ *and* $r<\frac{1}{2}\operatorname{dist}(x,\partial Q(2a))$ *the following estimate holds*

$$\frac{1}{m(Q(r))}\int_{Q(r)}g^q\,dm\leqq b\left\{\left(\frac{1}{m(Q(2r))}\int_{Q(2r)}g\,dm\right)^q+\right.$$
$$\left.+\frac{1}{m(Q(2r))}\int_{Q(2r)}f^q\,dm\right\}, \tag{2.8}$$

where $b>0$. *Then there exist constants* $\varepsilon_1>0$, $c>0$ *such that for* $p\in[q,q+\varepsilon_1]$, $g\in L^p_{\mathrm{loc}}(Q(2a))$ *and*

$$(2.9)\quad \left(\frac{1}{m(Q(a))}\int_{Q(a)} g^p\,dm\right)^{\frac{1}{p}} \leqq c\left\{\left(\frac{1}{m(Q(2a))}\int_{Q(2a)} g^q\,dm\right)^{\frac{1}{q}} + \right.$$

$$\left. + \left(\frac{1}{m(Q(2a))}\int_{Q(2a)} f^p\,dm\right)^{\frac{1}{p}}\right\}.$$

The constants c and ε_1 depend only on b, q, s, and n.

PROOF. See [3, p. 164, Proposition 5.1].

2.10. PROOF FOR TEOREM 1.4. Let $u \in W_n^1(G)$ be an extremal for the integral (1.1) and $Q(2r) \subset G$ a cube. We first prove the inequality

$$(2.11)\qquad \int_{Q(r)} |\nabla u|^n\,dm \leqq c_2(n, \alpha/\beta)\frac{1}{r^n}\int_{Q(2r)} |u|^n\,dm\,.$$

Let $\xi \in C_0^\infty(Q(2r))$ be non-negative and such that $\xi(x)=1$ for $x \in Q(r)$ and $0 \leqq \xi(x) \leqq 1$, $|\nabla\xi(x)| \leqq c_3(n)/r$. The function $v = u - \xi^n u$ belongs to the class $\mathscr{F}_u(Q(2r))$ and it has the gradient

$$\nabla v = (1-\xi^n)\nabla u - n\xi^{n-1}u\nabla\xi\,.$$

Suppose $x \in Q(2r)$ is such that $\xi(x) > 0$. It follows from the convexity condition

$$(2.12)\quad F(x, \nabla v) \leqq (1-\xi^n)F(x, \nabla u) + \xi^n F\left(x, \frac{nu}{\xi}\nabla\xi\right)$$

$$\leqq (1-\xi^n)F(x, \nabla u) + \beta n^n |u|^n |\nabla\xi|^n\,.$$

If $\xi(x)=0$ the inequality is trivially valid. Then (2.12) is valid for a.e. $x \in Q(2r)$. Since $v \in \mathscr{F}_u(Q(2r))$ we obtain by integration

$$\int_{Q(2r)} F(x, \nabla u)\,dm(x) \leqq \int_{Q(2r)} F(x, \nabla v)\,dm(x)$$

$$\leqq \int_{Q(2r)} (1-\xi^n)F(x, \nabla u)\,dm(x) + n^n \int_{Q(2r)} |\nabla\xi|^n |u|^n\,dm\,.$$

The inequality (2.11) follows from the condition (1.3).

Next we combine the inequality (2.11) and the result of Lemma 2.2. Write

$$h = \frac{1}{m(Q(2r))}\int_{Q(2r)} u\,dm\,,$$

then

$$\left(\int_{Q(r)} |\nabla u|^n\, dm\right)^{\frac{1}{n}} \leqq \left(\frac{c_1}{r^n}\int_{Q(2r)} |u-h|^n\, dm\right)^{\frac{1}{n}}$$

$$\leqq \frac{c}{m(Q(2r))^{\frac{1}{n}}}\left(\int_{Q(2r)} |\nabla u|^{\frac{n}{2}}\, dm\right)^{\frac{2}{n}}.$$

It follows that

$$\int_{Q(r)} |\nabla u|^n\, dm \leqq \frac{c^n}{m(Q(2r))}\left(\int_{Q(2r)} |\nabla u|^{\frac{n}{2}}\, dm\right)^2. \tag{2.13}$$

Choose $g=|\nabla u|^{\frac{n}{2}}$. Then we get from (2.13)

$$\frac{1}{m(Q(r))}\int_{Q(r)} g^2\, dm \leqq b\left(\frac{1}{m(Q(2r))}\int_{Q(2r)} g\, dm\right)^2. \tag{2.14}$$

Now Lemma 2.7 yields $g \in L^{2+\varepsilon_1}_{\mathrm{loc}}(G)$.

2.15. Proof for Theorem 1.5. Let $Q_1(2r_0)\subset \mathbf{R}^n$ be a cube such that $\bar{G}\subset Q_1(r_0)$. Suppose $Q(2r)\subset Q_1(2r_0)$ is arbitrary. If $Q(\frac{3}{2}r)\subset G$, then we proceed as in the proof of Theorem 1.4 and obtain the estimate

$$\int_{Q(r)} |\nabla u|^n\, dm \leqq \frac{c^n}{m(Q(2r))}\left(\int_{Q(\frac{3}{2}r)} |\nabla u|^{\frac{n}{2}}\, dm\right)^2.$$

The inequality (2.8) follows by choosing $g=|\nabla u|^{\frac{n}{2}}$ and $f=|\nabla \varphi|^{\frac{n}{2}}$ in G and equal to zero outside of G.

Next we suppose $Q(\frac{3}{2}r)\cap(\mathbf{R}^n\setminus G)\neq\emptyset$. The boundary condition implies

$$m(Q(2r)\cap(\mathbf{R}^n\setminus G))/m(Q(2r)) \geqq \delta\,.$$

Choose $\xi \in C_0^\infty(Q(2r))$ non-negative and such that $\xi(x)=1$ for $x\in Q(r)$ and $0\leqq\xi(x)\leqq 1$, $|\nabla\xi(x)|\leqq c_3(n)/r$ for $x\in Q(2r)$. Define

$$h(x) = \begin{cases} u(x)-\varphi(x) & \text{for } x\in G \\ 0 & \text{for } x\in \mathbf{R}^n\setminus G. \end{cases}$$

The function $v=u-\xi^n h$ belongs to the class $\mathscr{F}_\varphi(G)$ and it has the gradient

$$\nabla v = (1-\xi^n)\nabla u+\xi^n\nabla\varphi - n\xi^{n-1}(u-\varphi)\nabla\xi\,.$$

As before the convexity condition (1.3) yields for a.e. $x\in Q(2r)\cap G$

$$F(x,\nabla v) \leqq (1-\xi^n)F(x,\nabla u)+\lambda\beta(|u-\varphi|^n|\nabla\xi|^n+|\nabla\varphi|^n)\,,$$

where λ depends only on n. For a.e. $x\in G\setminus Q(2r)$ we have $F(x,\nabla v)=F(x,\nabla u)$. Since $v\in\mathscr{F}(G)$, we get by integration

$$\int_G F(x,\nabla u)\, dm(x) \leqq \int_G F(x,\nabla v)\, dm(x)$$

$$\leqq \int_G (1-\xi^n)F(x,\nabla u)\,dm\,(x)+\frac{\lambda c_3}{r^n}\beta\int_{Q(2r)\cap G}|u-\varphi|^n\,dm$$

$$+\ \lambda\beta\int_{Q(2r)\cap G}|\nabla\varphi|^n\,dm\,.$$

Then the condition (1.3) yields

$$\text{(2.16)}\quad \int_{Q(r)\cap G}|\nabla u|^n\,dm \leqq \frac{\lambda c_3}{r^n}\frac{\beta}{\alpha}\int_{Q(2r)\cap G}|u-\varphi|^n\,dm\,+$$

$$+\lambda\frac{\beta}{\alpha}\int_{Q(2r)\cap G}|\nabla\varphi|^n\,dm\,.$$

Define $g\colon Q_1(2r_0)\to \mathbf{R}$ and $f\colon Q_1(2r_0)\to \mathbf{R}$

$$g(x) = \begin{cases}|\nabla u(x)|^{\frac{n}{2}} & \text{for } x\in G\\ 0 & \text{for } x\in Q_1(2r_0)\setminus G\end{cases}$$

$$f(x) = \begin{cases}|\nabla \varphi(x)|^{\frac{n}{2}} & \text{for } x\in G\\ 0 & \text{for } x\in Q_1(2r_0)\setminus G\,.\end{cases}$$

Clearly $g\in L^2(Q_1(2r_0))$, $f\in L^{2+2\varepsilon}(Q_1(2r_0))$. We estimate the right side of the inequality (2.16) by using Lemma 2.4

$$\int_{Q(r)\cap G}|\nabla u|^n\,dm$$

$$\leqq \frac{\gamma}{m(Q(2r))}\left(\int_{Q(2r)\cap G}|\nabla u-\nabla\varphi|^{\frac{n}{2}}\,dm\right)^2+\gamma\int_{Q(2r)\cap G}|\nabla\varphi|^n\,dm$$

$$\leqq \frac{\gamma_1}{m(Q(2r))}\left(\int_{Q(2r)\cap G}|\nabla u|^{\frac{n}{2}}\,dm\right)^2+\frac{\gamma_1}{m(Q(2r))}\left(\int_{Q(2r)\cap G}|\nabla \varphi|^{\frac{n}{2}}\,dm\right)^2+$$

$$+\ \gamma_1\int_{Q(2r)\cap G}|\nabla\varphi|^n\,dm$$

$$\leqq \frac{\gamma_1}{m(Q(2r))}\left(\int_{Q(2r)\cap G}|\nabla u|^{\frac{n}{2}}\,dm\right)^2+2\gamma_1\int_{Q(2r)\cap G}|\nabla\varphi|^n\,dm\,.$$

Hence

$$\frac{1}{m(Q(r))}\int_{Q(r)}g^2\,dm \leqq b\left\{\left(\frac{1}{m(Q(2r))}\int_{Q(2r)}g\,dm\right)^2+\frac{1}{m(Q(2r))}\int_{Q(2r)}f^2\,dm\right\}.$$

The constant b depends on n, α/β, and δ. Let $t=t(n,\delta,\alpha/\beta)=\frac{n}{2}\varepsilon_1$, where $\varepsilon_1>0$ is the constant of Lemma 2.7. Lemma 2.7 gives $g \in L^{2+\varepsilon_1}(Q_1(r_0))$ and our theorem is proved.

2.17. REMARK. Suppose that $f=(f_1,\ldots,f_n)$: $G \to \mathbf{R}^n$ is a quasiregular mapping, see [7], [10], [11]. Theorems 1.4 and 1.5 can be applied to f. By using Theorem 1.4 a new proof for the result of Martio [6] and Meyers-Elcrat [9] is obtained.

2.18. REMARK. Let us consider a domain G for which Sobolev's imbedding theorem is valid. By using Theorem 1.5 and an imbedding theorem we obtain uniform Hölder-constants in G for the extremal u_0.

REFERENCES

1. B. V. Bojarski, *Generalized solutions of a system of differential equations of first order and elliptic type with discontinuous coefficients* (in Russian), Mat. Sb. (N.S.) 43 (85) (1957), 451–503.
2. F. W. Gehring, *The L^p-integrability of the partial derivatives of a quasiconformal mapping*, Acta Math. 130 (1973), 265–277.
3. M. Giaquinta and G. Modica, *Regularity results for some classes of higher order non-linear elliptic systems*, J. Reine Angew. Math. 311/312 (1979), 145–169.
4. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Grundlehren der Mathematischen Wissenschaften 224), Springer-Verlag, Berlin, Heidelberg, New York, 1977.
5. O. A. Ladyzenskaja and N. N. Ural'ceva, *Linear and quasilinear elliptic equations* (Mathematics in Science and Engineering 46), Academic Press, New York, London, 1968.
6. O. Martio, *On the integrability of the derivative of a quasiregular mapping*, Math. Scand. 35 (1974), 43–48.
7. O. Martio, S. Rickman, and J. Väisälä, *Definitions for quasiregular mappings*, Ann. Acad. Sci. Fenn. Ser. A I 448 (1969), 1–40.
8. N. G. Meyers, *An L^p-estimate for the gradient of solutions of second order elliptic divergence equations*, Ann. Scuola Norm. Sup. Pisa 17 (1963), 189–206.
9. N. G. Meyers and A. Elcrat, *Some results on regularity for solutions of non-linear elliptic systems and quasi-regular functions*, Duke Math. J. 42 (1975), 121–136.
10. Ju. G. Rešetnjak, *Space mappings with bounded distortion*, Siberian Math. J. 8 (1967), 466–487.
11. Ju. G. Rešetnjak, *Mappings with bounded deformation as extremals of Dirichlet type integrals*, Siberian Math. J. 9 (1968), 487–498.
12. E. W. Stredulinsky, *Higher integrability from reverse Hölder inequalities*, Indiana Univ. Math. J. 29 (1980), 408–413.

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