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STRONGLY ANNULAR FUNCTIONS WITH GIVEN SINGULAR VALUES

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0. Introduction.

For a function f holomorphic in the unit disk $D=\{z : |z|<1\}$, we consider the closed subset $Z'(f,a)$ of the unit circle $C=\{z : |z|=1\}$ consisting of limit points of the set $\{z : f(z)=a\}$ of a-points of f in D. If f is *strongly annular*, that is, if there is a sequence of positive numbers r_n such that $r_n \uparrow 1$ and

$$\min\{|f(z)| : \ |z|=r_n\} \to \infty$$

as n increases, then the set $Z'(f,a)$ cannot be empty for any finite complex number a, and in fact, as is guaranteed by the Koebe–Gross theorem concerning meromorphic functions omitting three values, $Z'(f,a)$ must coincide with the full circle C for every value a, except possibly for at most countably many a's. Therefore the set $S(f)$ consisting of those exceptional values mentioned above may be viewed as being singular for f. We are interested in the question raised by D. D. Bonar [3]: What cardinalities are possible for $S(f)$?

Partial answers to this question has been given by K. Barth, D. D. Bonar and F. W. Carroll [2], and also the present author [5]. Recently F. W. Carroll [4] constructed a strongly annular function f whose $S(f)$ is an increasing sequence $0=a_0<a_1<\ldots<a_n<\ldots$ on the real line. In connection with this result one is naturally led to ask whether there exists a strongly annular function f whose $S(f)$ is an arbitrary prescribed set, for instance, a countable dense subset of the complex plane. The purpose of this paper is to remark that the procedure involved in the above construction has much wider applicability so that the above question can be positively answered. Namely, we shall prove the following

THEOREM. *Let S be any non-empty subset of the complex plane which is at most countable. Then there exists a strongly annular function f with $S(f)=S$.*

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1. Fundamental lemma.

Let I be an open and proper subarc of C. For any z in D, denote by $G(I,z)$ the Jordan domain in D whose boundary consists of I and two segments connecting z with each of end points of I. When z is a point in $G(I,0)$, the domain $G(I,z)$ will be referred to as a *sectorial neighborhood* of I. Now let $\zeta_1 =1,\ldots,\zeta_{N+1}$ be distinct points on C such that $\arg\zeta_k<\arg\zeta_{k+1}$ and $\arg\zeta_{N+1} <2\pi$ $(k=1,\ldots,N)$. Further denote by I_k the open arc on C having ζ_k and ζ_{k+1} as end points and not containing any other ζ_j. We denote by C_j the circle $\{z : |z|=R_j\}$ $(0<R_1<R_2<1)$ such that C_1 intersects any sectorial $G(I_k,z_k)=G_k$ where z_k is a point in $G(I_k,0)$. Then C_2 also intersects any G_k, and hence we can divide C_2 into non-overlapping closed subarcs A_k, B_k, $B_{k,1}$, and $B_{k,2}$ as follows: First we denote by A_k a subarc whose middle point is the point where C_2 meets the radius $(0,\zeta_k)$ if $2\leqq k\leqq N$ where the argument of the middle point of A_1 is $(\arg\zeta_{N+1}-2\pi)/2$. Here we choose A_k so small that it does not meet the boundary of $G_k\cup G_{k-1}$ $(G_0=G_N)$. Next removing N arcs A_k $(k=1,\ldots,N)$ from C_2, we obtain N remaining arcs $\{J_k\}$ each of which intersects only one of N sectorial neighborhoods $\{G_k\}$. Assume that J_k intersects G_k. Then J_k is also divided into three subarcs B_k, $B_{k,1}$, and $B_{k,2}$. Namely B_k is the intersection $J_k\cap G_k$. Observing here that B_k protrudes outside G_k at each of its end points, we denote by $B_{k,1}$ the subarc protrudent towards A_k, and $B_{k,2}$ the subarc protrudent towards A_{k+1}, where $A_{N+1}=A_1$. Finally we denote by $H(D)$ the set of functions holomorphic in D. The following lemma will play a fundamental role in proving Theorem.

LEMMA. *Assume that for any given N complex numbers w_k $(k=1,\ldots,N)$, there exists a function f in $H(D)$ such that*

(1.1) *$f(z)-w_k$ is bounded away from 0 in G_k $(k=1,\ldots,N-1)$,*
(1.2) *f is bounded in G_k $(k=1,\ldots,N-1)$.*

Then, for each pair of positive numbers M and ϱ, there exists a function g in $H(D)$ such that

(1.1)′ *$g(z)-w_k$ is bounded away from 0 in G_k $(k=1,\ldots,N)$,*
(1.2)′ *g is bounded in G_k $(k=1,\ldots,N)$,*
(1.3) *$|g(z)|>M$ for every z on C_2,*
(1.4) *$|g(z)-f(z)|<\varrho$ for every z in $\{z : |z|\leqq R_1\}$.*

PROOF. Let q be a function defined as follows:

(1.5) $q(z)=f(z)$ on $\{z : |z|\leqq R_1\}$ and also for every z in $\bar G_1,\ldots,\bar G_{N-1}$,
(1.6) $q(z)=f(z_{k,j})$ on $B_{k,j}$ $(k=1,.,N-1;\ j=1,2)$,
(1.6)′ $q(z)=a$ constant, distinct from w_N, on the set $G_N\cup B_{N,1}\cup B_{N,2}$.

Here $\bar{G}_k$ denotes the closure of G_k with respect to D and $\{z_{k,1}, z_{k,2}\}$ the end points of B_k with $\arg z_{k,1} < \arg z_{k,2}$. Then, by virtue of the Arakelian approximation theorem [1], we can find, for any $\varepsilon > 0$, a function h in $H(D)$ such that

$$|h(z) - q(z)| < \varepsilon . \tag{1.7}$$

If we choose a positive number ε properly, then h is easily seen to possess required properties of the lemma except for (1.3). In order to make h satisfy (1.3) without losing the other requirements of the lemma, we first use the Runge approximation theorem. Recall here that the union of B_k, $B_{k,1}$, and $B_{k,2}$ form a single closed arc J_k which is off the boundary of the annular sector

$$S_k = \{z : R_1 < |z| \leqq 1, \arg \zeta_k < \arg z < \arg \zeta_{k+1}\} ,$$

using $(\arg \zeta_{N+1} - 2\pi)/2$ in place of $\arg \zeta_1$ for S_1 and $(\arg \zeta_{N+1} + 2\pi)/2$ in place of $\arg \zeta_{N+1}$ for S_N, and again consider a function q_k defined as follows:

(1.8) $q_k(z) = a_k$, a positive number, on the arc J_k,
(1.9) $q_k(z) = 0$ on the set $\{z : |z| \leqq 1\} - S_k$.

For this function q_k and a positive number ε_k, Runge's theorem assures the existence of a polynomial p_k such that

$$|p_k(z) - q_k(z)| < \varepsilon_k . \tag{1.10}$$

Using h and $\{p_k\}$, we define a function in $H(D)$ by

$$\begin{aligned} H(z) = {} & (h(z) - w_N) \exp(p_N(z) + \dots + p_1(z)) \\ & + (w_N - w_{N-1}) \exp(p_{N-1}(z) + \dots \\ & + p_1(z)) + \dots + (w_2 - w_1) \exp(p_1(z)) + w_1 . \end{aligned} \tag{1.11}$$

To see that H satisfies all the requirements (1.1)′–(1.4) in the lemma, except for that H is large on A_k $(k = 1, \dots, N)$ in modulus, we have only to note, as well as (1.5)–(1.11), the equality

$$\begin{aligned} H(z) - w_k = {} & \{u_k(z) + (h(z) - w_k) \exp(p_{k-1}(z) + \dots \\ & + p_1(z))\} \exp(p_k(z)) + v_k(z) , \end{aligned} \tag{1.12}$$

where

$$\begin{aligned} u_k(z) = {} & \{(h(z) - w_N)[\exp(p_N(z) + \dots + p_{k+1}(z)) - 1] + \dots \\ & + (w_{k+2} - w_{k+1})[\exp(p_{k+1}(z)) - 1]\} \exp(p_{k-1}(z) + \dots + p_1(z)) \\ & \qquad (1 \leqq k \leqq N-2) \end{aligned}$$

$$u_{N-1}(z) = (h(z) - w_N)(\exp(p_N(z)) - 1) \exp(p_{N-2}(z) + \dots + p_1(z)) ,$$

and

$$u_N(z) = 0 \quad \text{for all } z$$

$$v_k(z) = (w_k - w_{k-1})[\exp(p_{k-1}(z) + \ldots + p_1(z)) - 1] + \ldots$$
$$+ (w_2 - w_1)[\exp(p_1(z)) - 1] .$$

In fact, by virtue of (1.1), (1.5), (1.6), and (1.7), we can find a positive number K, independent of ε and N numbers ε_k $(k=1,\ldots,N)$, such that

$$\inf\{|h(z) - w_k| \ : \ z \in G_k \cup B_{k,1} \cup B_{k,2}\} > 4K \text{ for any } k . \tag{1.13}$$

Therefore, if we make each of ε_k's sufficiently small, it follows from (1.9) and (1.10) together with (1.2), (1.5), and (1.7) that

$$\inf\{|(h(z) - w_k)\exp(p_{k-1}(z) + \ldots + p_1(z))| \ : \ z \in G_k \cup B_{k,1} \cup B_{k,2}\} > 2K$$

and

$$\sup\{|u_k(z)| \ : \ z \in G_k \cup B_{k,1} \cup B_{k,2}\} < K$$

for any k. Consequently we obtain

$$|H(z) - w_k| > K|\exp(p_k(z))| - |v_k(z)| \tag{1.14}$$

for every z in $G_k \cup B_{k,1} \cup B_{k,2}$ and for each k $(k=1,\ldots,N)$. On the other hand, since v_k does not involve p_j $(j \geqq k)$, we can make v_k arbitrarily and uniformly small in G_j $(j \geqq k)$ if we choose $k-1$ positive numbers $\varepsilon_{k-1},\ldots,\varepsilon_1$ suitably. Therefore, first, using N as a value of k in each of the conditions (1.14), (1.10), (1.9), and (1.8), we can make H possess the required properties in the lemma, except for that $H(z) - w_k$ is bounded away from 0 in G_k $(k=1,\ldots,N-1)$ and that H is large on A_k $(k=1,\ldots,N)$ in modulus. Here we put emphasis on the fact that $H(z) - w_N$ is really bounded away from 0 in G_N. Subsequently letting ε_N be fixed and keeping in mind that v_{N-1} does not involve p_N and p_{N-1}, we shall next use $N-1$ as a value of k in each of (1.14), (1.10), (1.9), and (1.8). We continue this procedure until we use 1 as a value of k in each of the four conditions just mentioned above. Then, as a consequence of this process, we can conclude that H satisfies the requirements of the lemma, except for that $|H(z)|$ is large for every z on A_k $(k=1,\ldots,N)$. Moreover, adding a small vector if necessary, we may assume that H does not vanish on C_2. Thus the remaining task is to make $|H(z)|$ large on A_k $(k=1,\ldots,N)$. To this end, we need some more geometric definitions. Namely, let $A_{k,1}$ and $A_{k,2}$ denote small arcs encroaching on $B_{k-1,2}$ and $B_{k,1}$ at the end points of A_k, and further T_k a small "triangular" region including A_k, $A_{k,1}$, and $A_{k,2}$ in its interior, and pointing at ζ_k if $2 \leqq k \leqq N$ where T_1 is pointing at the point $\exp\{(\arg\zeta_{N+1} - 2\pi)/2\}i$ and $B_{0,2} = B_{N,2}$. Now let H_k be a function "closing the gaps" [4, Lemma A], that is, a function in $H(D)$ such that

$$|H_k(z)| > a \quad \text{for every } z \text{ on } A_k\,, \tag{1.15}$$

$$\operatorname{Re} H_k(z) > -\sigma \quad \text{for every } z \text{ on } A_{k,1},\ A_{k,2}\,, \tag{1.16}$$

$$|H_k(z)| < 2\sigma \quad \text{for every } z \text{ in } D - T_k\,. \tag{1.17}$$

Using H defined by (1.11) and $\{H_k\}$, we consider

$$g(z) = H(z)\{1+H_1(z)\} \dots \{1+H_N(z)\}\,.$$

Then choosing a pair of positive numbers a and σ suitably, and making use of (1.15), (1.16), and (1.17), we can easily show that g satisfies all the requirements of the lemma. Thus the proof is complete.

2. Proof of Theorem.

Let $\{r_n\}$ and $\{\sigma_n\}$ be two sequences of positive numbers with $r_n \uparrow 1$ and $\sum \sigma_n < \infty$. Let $\{G_k\}_1^\infty$ be sectorial neighborhoods defined as in Section 1 such that the circle $C_n = \{z : |z| = r_n\}$ intersects all sectorials $\{G_k\}_1^{n+1}$, $n = 1, 2, \dots$ To prove the theorem, we have only to construct inductively a sequence $\{f_n\}$ in $H(D)$ such that, for any n,

(2.1) $f_n(z) - w_k$ is bounded away from 0 in G_k $(k = 1, \dots, n)$,
(2.2) f_n is bounded in G_k $(k = 1, \dots, n)$,
(2.3) $|f_n(z)| > j$ for every z on C_j $(j = 1, \dots, n)$,
(2.4) $|f_n(z) - f_{n-1}(z)| < \sigma_{n-1}$ for every z on $\{z : |z| \leqq r_{n-1}\}$.

In fact, suppose that we have constructed $\{f_n\}$ satisfying (2.1)–(2.4). Then, by virtue of (2.4), $\{f_n\}$ converges to a function f in $H(D)$. Further, it follows from (2.3) that f is strongly annular. Moreover, the Hurwitz theorem together with (2.1) assures that $f(z) \neq w_k$ for every z in G_k, $k = 1, 2, \dots$ To prove the existence of the sequence $\{f_n\}$, let f_1 be a constant, distinct from each of the w_k's, whose modulus is greater than 1, and suppose that we have obtained $\{f_1, \dots, f_{n-1}\}$ satisfying (2.1)–(2.4). To construct f_n, we use the lemma, taking $r_{n-1} = R_1$, $r_n = R_2$, $f_{n-1} = f$, and $n + 1 = M$. Consequently, we get a function g in $H(D)$ which is denoted by f_n. Here, in order that the inequality

$$|f_n(z)| > j$$

holds for every z on C_j $(j = 1, 2, \dots n-1)$, a sufficiently small number σ_n' $(< \sigma_n)$ must be used as a value of ϱ. Thus, the proof of the theorem is complete.

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