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HARMONIC ANALYSIS OF ABELIAN INNER AUTOMORPHISM GROUPS OF VON NEUMANN ALGEBRAS

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1. Introduction.

From work done over the last decade, it is apparent that automorphism groups play a central role in the theory of operators algebras. Therefore, it would be desirable to have a spectral theory for automorphism groups. This paper continues the author's work [9], [10] along these lines.

Let A be a von Neumann algebra, let G be a locally compact abelian group, and let σ be a *representation* (i.e. a homomorphism of G into the group of $*$-automorphisms of A such that the map $t \to \varphi(\sigma_t(x))$ is continuous on G for all x in A and φ in the *predual* A_* of A). The representation σ is said to be *inner* if there is a unitary representation u of G in A (i.e. each u_t is in A) such that u unitarily *implements* σ in the sense that

$$\sigma_t(x) = \operatorname{ad} u_t(x) = u_t x u_t^*$$

for all t in G and x in A. If f is in $L^1(G)$, let $\sigma(f)$ be the element of the algebra $L(A)$ of all σ-weakly continuous linear maps of A into itself given by

$$\sigma(f)x = \int f(t)\sigma_t(x)\,dt$$

and let $L(\sigma)$ be the commutative Banach algebra equal to the closure in $L(A)$ of the set of all $\sigma(f)$ $(f \in L^1(G))$. The *spectrum* $\operatorname{Sp}\sigma$ of σ given by

$$\operatorname{Sp}\sigma = \bigcap \{N(f) \mid f \in L^1(G), \sigma(f)=0\},$$

where $N(f)$ is the set of all γ in the dual group Γ of G at which the Fourier transform $f^\wedge$ of f vanishes, is identified with the carrier space Ω of $L(\sigma)$ under the map $\gamma \to \omega_\gamma$, such that $\sigma(f)^\wedge(\omega_\gamma) = f^\wedge(\gamma)$ for $\gamma \in \operatorname{Sp}\sigma$. Here $\sigma(f)^\wedge$ is the Gelfand transform of $\sigma(f)$ [4; 2.3.7]. This correspondence sets up the elements of a spectral theory of linear maps on von Neumann algebras.

In this paper we show that $L(\sigma)$ is semisimple (i.e. the Gelfand transform is

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one-one) if G is a discrete group. Combining this with our earlier work, we show that $L(\sigma)$ is semisimple whenever its carrier space $\mathrm{Sp}\,\sigma$ is compact. The proof of this can also be used to extend some results of Størmer [16] by showing that the Banach algebra $B \otimes B$ in $L(A)$ generated by maps of the form

$$\pi(x)y = xy, \quad \pi'(x)y = yx,$$

where x lies in an abelian C*-algebra B in A, is semisimple.

Now let B be an abelian von Neumann algebra in A containing the center of A and let L be the commutative Banach algebra in $L(A)$ generated by all $L(\sigma)$, where σ runs through the set of all representations of G on A with compact support unitarily implemented in B. A representation σ has compact support if and only if it is continuous in the norm topology of $L(A)$ [13], and a norm continuous representation of a connected abelian group is inner [12]. We show that the algebra L is equal to the Banach algebra $L(\tau)$ generated by the representation τ on A of the unitary group or the self-adjoint unitary group of B given by $\tau_u = \mathrm{ad}\, u$ if G has an element t not equal to t^{-1} or if $t = t^{-1}$ for every t in G, respectively. We give the exact form of the projections in L and the spectrum of L in terms of those of B. The projections of L generate L in the norm topology. Finally, the form of the spectrum allows us to extend Størmer's notion [16] of positive definite element in L and to characterize an automorphism implemented in B in terms of its spectral properties in L.

2. Semisimplicity.

We show that a representation σ with compact spectrum produces a semisimple algebra $L(\sigma)$. The main part of the proof consists of analyzing the representation of a discrete abelian group.

THEOREM 1. *If σ is a representation with compact spectrum of the locally compact abelian group G on the von Neumann algebra A, then the algebra $L(\sigma)$ generated by $\sigma(f)$ $(f \in L^1(G))$ in $L(A)$ is semisimple.*

PROOF. First assume that G is discrete. The spectrum of any representation σ of G, being a closed subset of the dual Γ of G which is a compact group, is compact. Let A'' be the enveloping von Neumann algebra of A, let C be the center of A'', and let p_0 be the sum of the set $\{p_i\}$ all nonzero minimal projections in C such that $p_i A''$ is a factor of type I. For each σ_t there is a unique θ_t in $\mathrm{Aut}\, A''$ that coincides with σ_t on A. In fact, the algebra A'' is identified with the second dual of A and θ_t is then identified with the second transpose of σ_t. The map $t \to \theta_t$ is a representation θ of G on A''. Since each θ_t maps a minimal projection of C onto a minimal projection of C, the projection

p_0 must be invariant under the action of θ. Thus, the representation θ induces a representation α of G on the von Neumann algebra $A''p_0$.

We show that $L(\sigma)$ is isometric isomorphic to $L(\alpha)$ under the map Φ satisfying $\Phi(\sigma(f))=\alpha(f)$ for all f in $L^1(G)$. In fact, we have that

$$\begin{aligned}\|\alpha(f)(xp_0)\| &= \|\sum f(t)\alpha_t(xp_0)\| \\ &= \|\sum f(t)\sigma_t(x)p_0\| \\ &= \|(\sigma(f)x)p_0\| = \|\sigma(f)(x)\| ,\end{aligned}$$

for all x in A, due to the fact that $x \to xp_0$ is an isometric embedding of A into $A''p_0$. By the Kaplansky density theorem [6, I,3, Theorem 3], we get that

$$\|\alpha(f)\| = \operatorname{lub}\{\|\alpha(f)(x)\| \mid \|x\| \leqq 1,\ x \in A_{p_0}\}$$

and so we get that

$$\|\alpha(f)\| = \|\sigma(f)\| .$$

Since the sets $\{\sigma(f) \mid f \in L^1(G)\}$ and $\{\alpha(f) \mid f \in L^1(G)\}$ are norm dense in $L(\sigma)$ and $L(\alpha)$, respectively, we can conclude that Φ is an isometric isomorphism of $L(\sigma)$ onto $L(\alpha)$.

Now it is sufficient to show that $L(\alpha)$ is semisimple. We have that $A''p_0$ is isomorphic to the product $\prod A''p_i$. Each algebra $A''p_i$ is a type I factor. Let φ_i be the canonical trace of $A''p_i$ that sends an abelian projection into 1. Then the map $\varphi=\sum \varphi_i$ is a semifinite, faithful, normal trace of $A''p_0$. We show that φ is invariant under α. Let α_t be given. For each i there is an index $t(i)$ with $\alpha_t(p_i) =p_{t(i)}$. If an element a in $A''p_i^+$ is in the ideal of definition of φ_i, then there is an orthogonal sequence of abelian projections $\{q_i\}$ in $A''p_i$ and a sequence $\{\lambda_i\}$ of positive scalars such that $a=\sum \lambda_i q_i$ and $\sum \lambda_i<\infty$. We have that

$$\alpha_t(a) = \sum \lambda_i \alpha_t(q_i)$$

and $\{\alpha_t(q_i)\}$ are orthogonal abelian projections in $A''p_{t(i)}$. This means that

$$\varphi(a) = \varphi_i(a) = \sum \lambda_i = \varphi_{t(i)}(\alpha_t(a)) = \varphi(\alpha_t(a)) .$$

Because the positive part of the ideal of definition of φ is equal to countable sums $\sum a_i$ with a_i in the positive part of the ideal of definition of φ_i such that $\sum \varphi_i(a_i)<\infty$, we see that α_t leaves φ invariant. By Theorem 2 [10], we conclude that $L(\alpha)$, and consequently, that $L(\sigma)$ are semisimple.

Now let σ be a representation on A with compact spectrum of an arbitrary locally compact abelian group. There is a representation σ_1 with compact spectrum of a locally compact abelian group having a compactly generated dual such that $L(\sigma)=L(\sigma_1)$ [10, Proposition 10]. Then there is a representation

σ_2 of a discrete abelian group such that $L(\sigma_2)=L(\sigma_1)$ [10, Proposition 17]. Thus, the algebra $L(\sigma)$ is semisimple.

REMARK 2. Let $M(G)$ be the algebra of bounded measures on G. If σ is a representation of G on A,

$$\varphi(\upsilon(\sigma)) = \int \varphi(\sigma_t(a))\,d\upsilon(t) \quad (\varphi \in A_*)$$

defines an operator $\upsilon(\sigma)$ in $L(A)$ *for every* υ *in* $M(G)$. Then one can prove that closure $M(\sigma)$ of the set of all $\upsilon(\sigma)$ ($\upsilon \in M(G)$) is also a semisimple Banach algebra if σ has compact spectrum.

3. Spectra.

Let B be an abelian von Neumann subalgebra of the von Neumann algebra A. Suppose B contains the center of A. Let R be the family of all inner representations on A with compact spectrum of a fixed nontrivial locally compact abelian group G such that each representation is unitarily implemented by a unitary representation in B. The commutative Banach algebra L in $L(A)$ generated by all $L(\sigma)$ for σ in R is equal to an algebra of the form $L(\sigma_0)$ for a certain inner representation σ_0 on A of a discrete abelian group and σ_0 is unitarily implemented in B. The algebra L is generated by its *projections* (i.e. elements T with $T^2=T$) of norm 1 [10, Theorem 22]. We now obtain a better description of L. First we need to analyze the projections of L.

LEMMA 3. *Let p and q be orthogonal projections in B. Then there is in L a projection T of the form*

$$Tx = pxq \tag{1}$$

(respectively,

$$Tx = pxq+qxp), \tag{2}$$

if G contains an element $t \neq t^{-1}$ (respectively, if $t=t^{-1}$ for every t in G).

PROOF. Suppose G contains a t with $t^2 \neq 1$; then there is a γ in the dual group Γ of G with $\gamma^2 \neq 1$. The relations

$$u_t = 1-q+\langle\gamma,t\rangle q, \qquad v_t = 1-p+\langle\gamma,t\rangle^- p$$

define unitary representations of G in B such that

$$\operatorname{ad} u(f)x = (1-q)xq, \ \operatorname{ad} v(f)x = px(1-p)$$

for any integrable function f on G with $f^\wedge(\gamma)=1$ and $f^\wedge(\gamma^{-1})=f^\wedge(1)=0$. Then the element $(\operatorname{ad} v)(f)\operatorname{ad} u(f)$ maps x into pxq.

The projections mapping x into

$$px(1-p)+(1-p)xp \quad \text{and} \quad qx(1-q)+(1-q)xq$$

respectively, are in L for a group G in which every element is its own inverse. Thus the operator

$$Tx = pxq+qxp$$

is in L.

Out of two orthogonal projections p_1, p_2 in B of sum 1 together with the projection $p_0=0$, we may form six projections in L, viz.,

(i) $x \to p_1xp_2 = p_1xp_2+p_2xp_0$,
(ii) $x \to p_2xp_1$,
(iii) $x \to p_1xp_2+p_2xp_1$,
(iv) $x \to p_1xp_1+p_2xp_2$,
(v) $x \to x$,
(vi) $x \to 0$.

The complete description of projections in L is an extension of the preceding.

PROPOSITION 4. *Suppose the locally compact abelian group G contains an element not equal to is own inverse. An operator T in $L(A)$ is a projection in B if and only if it can be written in the form*

$$Tx = \sum_j (p_jx \sum \{p_i \mid i \in X_j\}), \tag{3}$$

where $p_0=0, p_1,\ldots,p_n$ are orthogonal projections of sum 1 in B and $X_1, X_2,\ldots,X_n$ are subsets of $\{0,1,2,\ldots,n\}$ such that i is in X_i for one particular i implies i is in X_i for every i.

PROOF. First we obtain formulae to express the combination of elements of the form (3) under certain algebraic operations. The formulae will show that each element of the form (3) is a projection, and that the set of elements of the form (3) is closed under multiplication and orthogonal summation.

Let T be the element in $L(A)$ of the form (3) given in the hypothesis of Proposition 4, and let $\{q_i\}$ be a finite set of orthogonal projections in B of sum 1. Let $\{r_i \mid 1 \leqq i \leqq m\}$ be any enumeration of the nonzero projections p_jq_k and let $r_0=0$. Then we can write T as

$$Tx = \sum_j (r_jx \sum \{r_i \mid i \in I_j\})$$

for such suitable subsets $I_1, I_2, \ldots, I_m$ of $\{0, 1, \ldots, m\}$. This is a simple rearrangement of

$$Tx = \sum_{j,k,l} (p_j q_k x \sum \{p_i q_l \mid i \in X_j\}) .$$

Now we have that

$$\sum r_i = \sum p_j q_k = \sum p_j \sum q_k = 1 .$$

Furthermore, if some i is in I_i, then $T(r_i) = r_i \neq 0$ and thus $T(p_j) \neq 0$ for that p_j with $p_j q_k = r_i$. In fact, we have that $T(r_i) = q_k T(p_j)$. This means that j is in X_j for this particular j, and consequently, that j is in X_j for every j. Thus, the term $p_j q_k x p_j q_k$ appears in the sum for T for every $p_j q_k \neq 0$, that is $i \in I_i$ for every i. This shows that the representation for T in terms of the projections $\{r_i\}$ is still of the form (3).

Now let S be the operator of the form (3) given by

$$Sx = \sum q_j x \sum \{q_i \mid e \in Y_j\}) .$$

We have just shown that T and likewise S can be written in the form (3) in terms of the same set $\{r_i\}$ of orthogonal projections of B of sum 1. So there is no loss in generality in the assumption that $p_j = q_i = r_j$. Then we get that

$$STx = \sum_j (r_j x \sum \{r_i \mid i \in X_j \cap Y_j\})$$

and

$$(S+T)(x) = \sum_j \{r_j x \sum \{r_i \mid i \in X_j \cup' Y_j\}\}) .$$

Here $\cup'$ denotes the disjoint union. We see that i in $X_i \cap Y_i$ for one particular index i implies i in $X_i \cap Y_i$ for all indices so that the representation for ST is still in the form (3). We also see that $ST = 0$ implies that $X_j \cap Y_j$ is contained in the set $\{0\}$ for all j. This means that the representation for $S+T$ is still in the form (3) if $ST = 0$. In fact, we have that $X_j \cup' Y_j = X_j \cup Y_j$. In addition, if one particular i is in $X_i \cup Y_i$, then i is in X_i or Y_i and thus i is in $Y_i \cup Y_i$ for every i. Thus, we have that each element of the form (3) is a projection and that set of such elements is closed under multiplication and orthogonal summation.

Now we have that the projection T in $L(A)$ given in (3) is actually in L. Suppose i is not in X_i for every i; then T is in L by Lemma 3. If $\sum p_i = 1$, then the operator

$$Rx = x - \sum_j p_j (x \sum \{p_i \mid i \neq j\}) = \sum p_i x p_i$$

is also in L since the identity operator is in L [10, Theorem 22].

As the first step in showing the converse that every projection in L has the form (3), we show that linear combinations of projection of the form (3) are dense in L. Given a representation σ in the set R, an integrable function f on G, and an $\varepsilon > 0$, then we can find a compact subset G_0 of G and an $\eta > 0$ such that

$$\|\theta(f)-\sigma(f)\| \leqq \int_{G_0} \|\theta_t - \sigma_t\| \, |f(t)| \, dt + 2 \int_{G-G_0} |f(t)| \, dt < \varepsilon ,$$

whenever θ is in R and $\|\theta_t - \sigma_t\| < \eta$ for all t in G_0. There are a finite number of norm continuous unitary representations u_i of G in the center C of A and a corresponding number q_i of orthogonal projections in B of sum 1 such that

$$\theta_t = \operatorname{ad} \sum_i u_{it} q_i$$

is in R and satisfies the relation

$$\|\theta_t - \sigma_t\| < \eta/2$$

for t in G_0. For this we limit ourselves to sketching the partition argument presented in detail in [10]. We can find a unitary representation v of G into B with compact support that implements σ [10, Proposition 20]. Let e be the spectral resolution of v in B and let W be a compact Baire set in the dual group Γ of G. If p is the central support of $e(W)$ in C, there is a continuous function χ of the support of the Gelfand transform of p in the carrier space of C into W. The relation

$$u_t\hat{}(\zeta) = \langle \chi(\zeta), t \rangle p\hat{}(\zeta) + (1-p)\hat{}(\zeta)$$

for ζ in the carrier space of C defines a unitary operator u_t in C and the map $t \to u_t$ defines a norm continuous unitary representation u of G in C. A rearrangement of sums of elements of the form $u_t e(W)$ gives θ_t [10, Proposition 21]. Now working with the spectral resolution of the u_t given by Stone's Theorem, we can find a finite set $\{r_i\}$ of orthogonal projections in C and a corresponding finite subset $\{\mu_i\}$ in the carrier space of C such that the representation α in R given by

$$\alpha_t = \operatorname{ad} \sum u_{it}\hat{}(\mu_j) q_i r_j$$

satisfies the relation

$$\|\alpha_t - \theta_t\| < \eta/2$$

on G_0. Thus, we have that

$$\|\alpha(f) - \sigma(f)\| < \varepsilon .$$

The spectrum of α is finite since it is contained in the subset of Γ given by

$$t \to \langle \gamma_{ijk}, t\rangle^- = \hat{u_{it}}(\mu_j)\hat{u_{kt}}(\mu_j)^-$$

[10, Proposition 21]. We can find a finite set $\{g_n\}$ of integrable functions on G whose Fourier transforms act as the Kronecker delta on the spectrum $\{\gamma_n\}$ of α. Using the fact that $L(\alpha)$ is semisimple (Theorem 1) and the fact that $f^\wedge = \sum f^\wedge(\gamma_n)g_n$ on the carrier space $\operatorname{Sp}\alpha$ of $L(\alpha)$, we get

$$\alpha(f) = \sum f^\wedge(\gamma_n)\alpha(g_n)\,.$$

However, the operators $\alpha(g_n)$ are projections because $(g_n * g_n)^\wedge$ and $g_n^\wedge$ coincide on $\operatorname{Sp}\alpha$ and so $\alpha(g_n)^2 = \alpha(g_n * g_n)$ and $\alpha(g_n)$ coincide. We have that

$$\begin{aligned}\alpha(g_n)x &= \int g_n(t)\,\mathrm{ad}\Big(\sum_{j,k} \hat{u_{jt}}(\mu_k)q_j r_k\Big)(x)\,dt \\ &= \int g_n(t) \sum_j \sum_{i,k} \langle \gamma_{ijk}, t\rangle^- q_i x q_k r_j\,dt \\ &= \sum_j \sum_{i,k} \hat{g_n}(\gamma_{ijk}) q_i x q_k r_j \\ &= \sum \{q_i x q_k r_j \mid (i,j,k) \in I_n\}\,.\end{aligned}$$

Here, I_n will denote the set of all (i,j,k) with $\gamma_{ijk} = \gamma_n$. If γ_n is equal to the identity of Γ, then the set I_n contains the set $\{(i,j,k) \mid i=k\}$; on the other hand, if $\gamma_n \neq 1$, then the set I_n is disjoint from $\{(i,j,k) \mid i=k\}$. In either case $\alpha(g_n)$ is of the form (3). Hence, linear combinations of projections of the form (3) are dense on L.

Now we show that an arbitrary projection T in L has the form (3). Let $\{T_i\}$ be a finite set of projections of the form (3) and let $\{\lambda_i\}$ be scalars such that

$$\|T - \sum \lambda_i T_i\| < 1/3\,.$$

We may assume that $T_i T_j = 0$ for $i \neq j$. We show that T is equal to the projection

$$S = \sum \{T_i \mid |1-\lambda_i| < 1/3\}\,.$$

Since L is semisimple, it is sufficient to show that the support of their Gelfand transforms on the carrier space Ω of L are equal. Let $|\lambda_i - 1| < 1/3$ and let ω be a point in Ω with $\hat{T_i}(\omega) = 1$. We have that

$$\begin{aligned}|(1-T)^\wedge(\omega)| &= |(T_i - T)^\wedge(\omega)| \\ &= |(\sum \lambda_j T_j - T)^\wedge(\omega)| + |1-\lambda_i| \\ &\leqq \|\sum \lambda_j T_j - T\| + |1-\lambda_i| < 1\,.\end{aligned}$$

Thus, we get that $T^\wedge(\omega) = 1$. Conversely, let $T^\wedge(\omega) = 1$. Since we have that

$$\|(\sum \lambda_j T_j - T)\hat{}(\omega)\| \leqq \|\sum \lambda_j T_j - T\| < 1/3 ,$$

there is precisely one T_i with $T_i\hat{}(\omega) = 1$. We have that

$$|1 - \lambda_i| = |(\sum \lambda_j T_j - T)\hat{}(\omega)| < 1/3 .$$

This shows that $S\hat{}$ and $T\hat{}$ have the same support and thus are equal. So we get $S = T$.

The two classes of projections described in Proposition 4 can also be described more conveniently as

(5) $\{T \mid i \notin X_i \text{ in form } (3)\} = \{T \mid T\hat{}(1) = 0\} = \{T \mid T(1) = 0\}$

and

(6) $\{T \mid i \in X_i \text{ in form } (3)\} = \{T \mid T\hat{}(1) = 1\} = \{T \mid T(1) = 1\}$.

Here 1 denotes the identity of dual of the discrete abelian group.

Using of proof similar to that of Proposition 4, we can also prove the following.

PROPOSITION 5. *Suppose every element of the locally compact abelian group G is equal to its own inverse. An operator T in L(A) is a projection in L if and only if it can be written in the following form.*

(4) $$Tx = \sum (p_i x \sum \{p_j \mid j \in X_i\}) ,$$

where $p_0 = 0, p_1, \ldots, p_n$ *are orthogonal projections in B and* $X_1, X_2, \ldots, X_n$ *are subsets of* $\{0, 1, \ldots, n\}$ *such that i is in* X_i *for one i implies i is in* X_i *for every i and* $\sum p_i = 1$, *and i is in* X_j *implies j is in* X_i.

Now it is possible to identify the algebra L with $L(\tau)$ for a concrete representation τ.

THEOREM 6. *Let A be a von Neumann algebra and let G be a locally compact abelian group. Let L be the commutative Banach algebra generated by all the algebras* $L(\sigma)$ *for representations* σ *of G on A with compact spectrum that are unitarily implemented by unitary representations of G into the abelian von Neumann subalgebra B of A containing the center of A. Then the algebra L is equal to the algebra* $L(\tau)$ *for the representation* τ *on A of the discrete group of unitary (respectively, self-adjoint unitary) operators of B given by* $\tau_u = \mathrm{ad}\, u$ *provided G contains an element not equal to its own inverse (respectively every element in G is its own inverse).*

PROOF. Let G contain an element that is not equal to its own inverse. Let u be a unitary operator in B. Given a positive number ε, there are orthogonal projections $p_1,\ldots,p_n$ of sum 1 in B and complex numbers $\lambda_1,\ldots,\lambda_n$ of modulus 1 such that

$$\|\sum \lambda_i p_i - u\| < \varepsilon .$$

The operator

$$Tx = \sum \lambda_i \bar{\lambda_j} p_i x p_j$$

is in L (Proposition 4) and

$$\|T - \operatorname{ad} u\| < 2\varepsilon .$$

This proves that $L(\tau)$ is contained in L.

Conversely, let σ be a representation in R. Let Γ_0 be a compactly generated subgroup of the dual Γ of G that contains $\operatorname{Sp}\sigma$. The representation σ induces a representation σ' of G modulo the annihilator G_0 of Γ_0 by $tG_0 \to \sigma_t$ [4; 2.3.9]. Since the unitary representation v of G in B implementing σ can be chosen so that its spectrum lies in $\operatorname{Sp}\sigma$ [10, Proposition 21], the relation $tG_0 \to v_t$ induces a unitary representation of G/G_0 in B that implements σ'. There is a discrete subgroup D of G/G_0 such that the Banach algebra $L(\sigma'')$ generated by the restriction σ'' to D coincides with the original algebra $L(\sigma)$ [10, Proof of Theorem 22]. This means that $L(\sigma)$ is contained in $L(\tau)$. Thus, we have that L is equal to $L(\tau)$.

If every element is its own inverse, then a suitable modification of the preceding argument based on Lemma 3 shows that L is generated by the representation of the discrete group of self-adjoint unitary operators of B given by $u \to \operatorname{ad} u$.

The preceding theorem show that some of the harmonic analysis inherent in the analysis of inner representations with compact spectrum of G on A disappear in favor of the harmonic analysis of the representations induced by a canonical discrete abelian group of unitarires.

We now compute the spectrum of L. We have already computed it in terms of the implementing unitary representation [9]. Here we compute it in terms of the carrier space of B. We first consider a more general subalgebra $B \otimes B$ in $L(A)$ generated by left and right multiplications by elements of B on A. This was studied by E. Størmer [14], when B is a closed *-subalgebra of the algebra A of all bounded linear operators on Hilbert space. The next proposition extends Størmer's results by reducing the present case to the case considered by Størmer.

THEOREM 7. *Let A be a von Neumann algebra with center C, let B be an abelian C^*-algebra in A containing C, let Z be the carrier space of B, and let $B \otimes B$ be the Banach algebra of operators on A generated by the left and right multiplications $\pi(x)y = xy$ and $\pi'(x)y = yx$ ($y \in A$, $x \in B$) of elements of B on A. Then $B \otimes B$ is a semisimple Banach algebra with carrier space*

$$\{(\zeta, \xi) \in Z \times Z \mid \zeta \cap C = \xi \cap C\},$$

and the carrier space acts on $B \otimes B$ according to the relation

$$(\pi(x)\pi'(y))\hat{\ }(\zeta, \xi) = x\hat{\ }(\zeta) y\hat{\ }(\xi).$$

PROOF. Let M be the carrier space of C, let μ be in M, and let Z_μ be the set

$$Z_\mu = \{\zeta \in Z \mid \zeta \cap C = \mu\}.$$

There is an irreducible representation $\varrho = \varrho_\mu$ of A on a Hilbert space $H = H_\mu$ with kernel equal to the ideal μ in A generated by μ [8]. We note that the carrier space of the C^*-algebra $\varrho(B)$ can be identified with Z_μ by the relation

$$\varphi_\zeta(\varrho(x)) = x\hat{\ }(\zeta).$$

Let $\varrho(B) \otimes \varrho(B)$ be the Banach algebra in the algebra $L(L(H))$ of σ-weakly continuous linear operators on the algebra $L(H)$ of bounded linear operators on H generated by left and right multiplications by the C^*-algebra $\varrho(B)$. If T is in $B \otimes B$; then there is a unique operator $\varrho(T)$ in $\varrho(B) \otimes \varrho(B)$

$$\varrho(T)\varrho(X) = \varrho(Tx)$$

for x in A. In fact, we can pass from the generating set of $B \otimes B$ to the full algebra because of the relation

$$\text{lub}\{\textstyle\sum \pi(\varrho_\mu(x_i))\pi'(\varrho_\mu(y_i))\| \mid \mu \in M\} = \|\textstyle\sum \pi(x_i)\pi'(y_i)\|,$$

which follows from the Kaplansky Density Theorem. If T has spectrum equal to $\{0\}$ in $B \otimes B$, then $\varrho_\mu(T)$ has spectrum equal to 0 in $\varrho(B) \otimes \varrho(B)$. Since $\varrho(B) \otimes \varrho(B)$ is semisimple [14, Proposition 4.2], we have that $\varrho(T) = 0$ for all ϱ, and thus, that $T = 0$. This shows that $B \otimes B$ is semisimple.

The center C is isometrically isomorphically embedded in $B \otimes B$ by the map $x \to \pi(x) = \pi'(x)$. This means that any nonzero multiplicative linear functional φ on $B \otimes B$ induces a multiplicative linear functional on the center C. So there is a unique μ in M with

$$\varphi(\pi(x)) = x\hat{\ }(\mu)$$

for every x in C. Setting $\varrho = \varrho_\mu$, we show that there is a unique multiplicative linear functional ψ on $\varrho(B) \otimes \varrho(B)$ such that

$$\varphi(T) = \psi(\varrho(T))$$

for every T in $B \otimes B$ by showing that

$$|\varphi(T)| \leqq \|\varrho(T)\|$$

for every T of the form

$$T = \sum \{\pi(x_i)\pi'(y_i) \mid 1 \leqq i \leqq n\}$$

with x_i, y_i in B. To prove this we use the fact that

$$\eta \to \|\varrho_\eta(x)\|$$

is continuous on M for every fixed x in A [7, Lemma 9]. Since the relation

$$\varphi(T\pi(p)) = \varphi(T)\varphi(\pi(p)) = \varphi(T)\hat{p}(\mu) = \varphi(T)$$

holds for every p in the set of projections P_μ in the complement of μ in C, we see that

$$|\varphi(T)| \leqq \mathrm{glb}\{\|T\pi(p)\| \mid p \in P_\mu\} = \lambda .$$

We show that $\|\varrho(T)\|$ coincides with λ. Because $\varrho(p)=1$ for every p in P_μ, it is clear that $|\varrho(T)| \leqq \lambda$. There is no loss of generality in the assumption that $\lambda > 0$. Let $0 < \varepsilon < \lambda/2$. If p is a projection in C with $\|T\pi(p)\| > \lambda - \varepsilon$, there is an element x in A with

$$\|T\pi(p)x\| > (\lambda-\varepsilon)\|\pi(p)x\| .$$

This means that there is a nonzero projection q in C_p such that for every nonzero projection r in C_q, the relation

$$\|T\pi(r)x\| > (\lambda-\varepsilon)\|\pi(r)x\|$$

holds; otherwise, a maximal set $\{r_i\}$ of nonzero orthogonal projection in C_p with

$$\|T\pi(r_i)x\| \leqq (\lambda-\varepsilon)\|\pi(r_i)x\|$$

must have sum p and this would mean that

$$\begin{aligned}\|T\pi(\varrho)x\| = \mathrm{lub}_i \|T\pi(r_i)x\| &\leqq (\lambda-\varepsilon)\,\mathrm{lub}\, \|r_i x\| \\ &\leqq (\lambda-\varepsilon)\|\pi(p)x\| .\end{aligned}$$

So there is a nonzero projection q in C_p with

$$\|T\pi(r)x\| > (\lambda-\varepsilon)\|rx\|$$

for every nonzero projection r in C_q. Since the norm of qx is given by

$$\|qx\| = \text{lub}\{\|\varrho_\nu(v)\| \mid \nu \in \text{supp}\, q\hat{}\}$$

(cf. [8, p. 210]) and since the map $\nu \to \|\varrho_\nu(x)\|$ is continuous on M, we may replace q by perhaps a smaller projection and multiply x by an element of C_q given by $c\hat{}(\nu) = \|\varrho_\nu(x)\|^{-1}$ to obtain a nonzero projection q in C_p and a vector x in A_q such that $\|\varrho_\nu(x)\| = 1$ for every ν in $\text{supp}\, q\hat{}$, and such that

$$\|T\pi(r)x\| \geqq (\lambda - 2\varepsilon)\|rx\| = \lambda - 2\varepsilon$$

for nonzero every projection r in C_q. Now we are in a position to show $\|\varrho(T)\| \geqq \lambda - 2\varepsilon$ by a maximality argument. Let $\{q_i\}$ be a maximal orthogonal set of nonzero projections in C such that there is a corresponding set $\{x_i\}$ of elements of A such that $q_i x_i = x_i$, $\|\varrho_\nu(x_i)\| = 1$ for ν in $\text{supp}\, \hat{q_i}$, and

$$\|T\pi(r)x_i\| \geqq \lambda - 2\varepsilon$$

for every nonzero projection r in C_{q_i}. Setting $x = \sum x_i$, we have that

$$\|\varrho_\mu(x)\| = \|\varrho(x)\| = 1$$

since μ is in closure of the union of the supports of the $\hat{q_i}$ due to the maximality of the set $\{q_i\}$ and the previous construction for q. We also have that

$$\begin{aligned} \|\varrho_\nu(Tx)\| &= \text{glb}\{\|\pi(r)Tx\| \mid r \in P_\nu\} \\ &\geqq (\lambda - 2\varepsilon) \end{aligned}$$

for every ν in $\text{supp}\, \hat{q_i}$ so that

$$\|\varrho(T)\| \geqq \|\varrho(T)\varrho(x)\| \geqq \lambda - 2\varepsilon .$$

This shows that

$$\|\varrho(T)\| \geqq \lambda .$$

Thus, we have that

$$|\varphi(T)| \leqq \lambda = \|\varrho(T)\| .$$

There exists a multiplicative linear functional ψ on $\varrho(B) \otimes \varrho(B)$ such that

$$\varphi(T) = \psi(\varrho(T))$$

for all T in $B \otimes B$. Using [14, § 5], we can find a point (ζ, ξ) in the carrier space $Z_\mu \times Z_\mu$ of $\varrho(B) \otimes \varrho(B)$ such that

$$\begin{aligned} \varphi(\pi(y)\pi'(z)) &= \psi(\pi(\varrho(y))\pi'(\varrho(z))) \\ &= \varrho(y)\hat{}(\zeta)\varrho(z)\hat{}(\xi) \\ &= y\hat{}(\zeta)z\hat{}(\xi) . \end{aligned}$$

Conversely, if (ζ, ξ) are in $Z_\mu \times Z_\mu$, then the map

$$T \to \varrho_\mu(T)\hat{}(\zeta, \xi)$$

defines a nonzero multiplicative linear functional on $B \otimes B$.

Now we show that the map Ψ of the compact subset $\{Z_\mu \times Z_\mu \mid \mu \in M\}$ of $Z \times Z$ onto the carrier space of $B \otimes B$ given by

$$T\hat{}(\Psi(\zeta, \xi)) = \varrho_\mu(T)\hat{}(\zeta, \xi)$$

for (ζ, ξ) in $Z_\mu \times Z_\mu$ is a homeomorphism onto the carrier space of $B \otimes B$. It is sufficient to verify that Ψ is continuous. We must show that, for every T in $B \otimes B$, the net $\{T\hat{}(\Psi(\zeta_n, \xi_n))\}$ converges to $T\hat{}(\Psi(\zeta, \xi))$ whenever $\{(\zeta_n, \xi_n)\}$ converges to (ζ, ξ). For this it is sufficient to assume that T is of the form $T = \pi(y)\pi'(z)$ for $y, z \in B$. But then we have that

$$\begin{aligned} \lim T\hat{}(\Psi(\zeta_n, \xi_n)) &= \lim y\hat{}(\zeta_n) z\hat{}(\xi_n) \\ &= y\hat{}(\zeta) z\hat{}(\xi) = T\hat{}(\Psi(\zeta, \xi)) . \end{aligned}$$

So the map Ψ is continuous and thus a homeomorphism.

We now compute the carrier space of L.

THEOREM 8. *Let A be a von Neumann algebra with center C, let B be an abelian von Neumann subalgebra of A containing C, let Z be the carrier space of B, let τ be the representation of the unitary group U of B on A given by $\tau_u = \mathrm{ad}\, u$, and let $L = L(\tau)$ be the Banach algebra of operators on A generated by the operators of the form $\tau(f)$ (f in $L^1(U)$). Then the spectrum of L is the one point compactification of the subset of $Z \times Z$ given by*

$$\Omega_0 = \{(\zeta, \xi) \in Z \times Z \mid \zeta \cap C = \xi \cap C, \zeta \neq \xi\}$$

and the action of the carrier space on L is determined by

$$\hat{\tau_u}(\zeta, \xi) = u\hat{}(\zeta) u\hat{}(\xi)^-$$

and

$$\hat{\tau_u}(\infty) = 1$$

where ∞ is the point at infinity.

PROOF. We preserve the notation of Theorem 7. We have that the transpose Φ of the identity map of L into $B \otimes B$ is a continuous map of the carrier space of $B \otimes B$ onto a compact subset of the carrier space Ω of L. Since Ω has a base of open and closed sets each one of which corresponds to the support of the Gelfand transform of a projection in L [10, Corollary 24] and since the algebra

$B \otimes B$ is semisimple (Theorem 7), the map Φ is a surjection. Now we can show by a simple exhaustion argument, which is based on the fact that the projections of B separate the points of Z, that $\Psi(\zeta_1, \xi_1) = \Psi(\zeta_2, \xi_2)$ on L if and only if $\zeta_1 = \xi_1$ and $\zeta_2 = \xi_2$ or $\zeta_1 = \zeta_2$ and $\xi_1 = \xi_2$. It is worthwhile noting that the entire unitary group and not just the self-adjoint group is necessary to obtain this separation. From this we see that Φ is one-one on Ω_0; and that Φ maps the elements (ζ, ζ) into the identity element in the dual group of U, which is the spectrum of τ, or equivalently, the carrier space of L (cf. [10, Theorem 11]). Finally, we observe that $\Phi(\Omega_0)$ does not contain this identity element. Thus, the map Φ induces a homeomorphism of the one point compactification of Ω_0 onto the carrier of L with the stated properties. Indeed, we set the value at infinity of the map induced by Φ equal to the identity of the dual group. The only point that now needs verification is the continuity at infinity. Given a net $\{(\zeta_n, \xi_n)\}$ in Ω_0 converging to ∞ and given a T in L, we must show that the net $\{T\hat{}(\zeta_n, \xi_n)\}$ contains a subnet converging to $T\hat{}$(identity). For this we show that every subnet of $\{T\hat{}(\zeta_n, \xi_n)\}$ contains a subnet converging to $T\hat{}$(identity). Then there is no loss of generality in the assumption that $\{(\zeta_n, \xi_n)\}$ converges to (ζ_0, ξ_0) in $Z \times Z$ with $\zeta_0 \cap C = \xi_0 \cap C$. But if p is any projection in B different from one and zero, the set

$$\{(\zeta, \xi) \mid p\hat{}(\zeta)(1-p)\hat{}(\xi) = 0\} \cup \{\infty\}$$

is a neighborhood of ∞ in the one-point compactification of Ω_0. The net $\{(\zeta_n, \xi_n)\}$ is eventually in this neighborhood. This means that $\zeta_0 = \xi_0$. Thus, we get

$$\lim T\hat{}(\zeta_n, \xi_n) = T\hat{}(\Phi(\zeta_0, \zeta_0)) = T\hat{}(\text{identity}) .$$

If A is a factor, we note that the carrier of $B \otimes B$ is $Z \times Z$ and the carrier of L is the one-point compactification of the complement in $Z \times Z$ of the diagonal.

Finally, we determine the spectrum of the identity representation of U on the Hilbert space H.

PROPOSITION 9. *Let B be an abelian von Neumann algebra on the Hilbert space H and let Z be the carrier space of B. Then the spectrum of the identity representation ι of the unitary group U of B on H is homeomorphic to Z under the map $\zeta \to \gamma_\zeta$ where γ_ζ is the character of U given by*

$$\langle u, \gamma_\zeta \rangle = u\hat{}(\zeta)^- .$$

PROOF. It is clear that the relation $\langle u, \gamma_\xi \rangle = u\hat{}(\zeta)^-$ defines a character γ_ζ of the group U. We show that the spectrum Sp ι of the unitary representation $\iota(u) = u$ of U on the Hilbert space H given by

$$\mathrm{Sp}\,\iota = \bigcap \{N(f) \mid f \in L^1(U), \iota(f) = \sum f(u)u = 0\}$$

is precisely the set $\{\gamma_\zeta \mid \zeta \in Z\}$ and that map $\zeta \to \gamma_\zeta$ is a homeomorphism of Z. First let $\iota(f)=0$; then we have that

$$f\hat{\ }(\gamma_\zeta) = \sum f(u)\langle u, \gamma_\zeta\rangle^- = (\sum f(u)u)\hat{\ }(\zeta) = \iota(f)\hat{\ }(\zeta) = 0\,,$$

and consequently, that γ_ζ is in $\mathrm{Sp}\,\iota$. Conversely, let γ be in $\mathrm{Sp}\,\iota$.For any absolutely summable sequence $\{\lambda_n\}$ of complex numbers and any sequence $\{u_n\}$ in U, we have that $\sum \lambda_n\langle u_n, \gamma\rangle^-$ vanishes whenever $\sum \lambda_n \iota(u_n) = \sum \lambda_n u_n$ vanishes. In particular, the relation

$$\{(\sum \lambda_i u_i, \sum \lambda_i \langle u_i, \gamma\rangle^-) \mid \lambda_1, \ldots, \lambda_n \text{ complex numbers,}$$
$$u_1, \ldots u_n \text{ in } U, \quad n = 1, 2, \ldots\}$$

defines a linear functional φ on the set of linear combination of U, viz. B due to [6; I,1, Proposition 3]. Since φ satisfies

$$\varphi(uv) = \varphi(u)\varphi(v)$$

for u, v in U, the functional φ is a multiplicative linear functional on B, i.e. there is a ζ in Z with $\varphi(u) = u\hat{\ }(\zeta)$ for all u in U. Thus, we get that $\gamma = \gamma_\zeta$. So the spectrum of ι is $\{\gamma_\zeta \mid \zeta \in Z\}$.

Since the set of linear combinations of U equals B, the map $\zeta \to \gamma_\zeta$ is one-one. The definition of the topology of Z shows that the map is bicontinuous. Thus, the function $\zeta \to \gamma_\zeta$ is a homeomorphism of Z onto $\mathrm{Sp}\,\iota$.

REMARK 10. We see from the preceding two results that the map

$$(\zeta, \xi) \to \gamma_\zeta \gamma_\xi^{-1}$$

for (ζ, ξ) in Ω_0, and

$$\infty \to \text{identity}$$

defines a homeomorphism of the one point compactification of Ω_0 onto the spectrum of τ with

$$\hat{\tau_u}(\gamma_\zeta \gamma_\xi^{-1}) = u\hat{\ }(\zeta)^- u\hat{\ }(\xi)\,.$$

This also makes precise the relationship between Proposition 5.7 and Theorem 5.1 of [14].

REMARK 11. The set $\{\gamma_\zeta \mid \zeta \in Z\}$ generates the dual group of U.

4. Application to harmonic analysis.

Let A be a von Neumann algebra with center C, let B be an abelian C*-algebra in A containing C, let Z be the carrier space of B, and let M be the carrier space of C. An operator T in the algebra of operators $B \otimes B$ on A is said to be *positive definite* if, for every μ in M and every finite subset $\{\zeta_i\}$ of the set Z_μ of all ζ in Z with $\zeta \cap C = \mu$, the scalar matrix $(T\hat{\ }(\Psi(\zeta_i, \zeta_j))) = (T\hat{\ }(\zeta_i, \zeta_j))$ is positive. Here Ψ is the map of $\bigcup Z_\mu \times Z_\mu$ onto the carrier space of $B \otimes B$ described in Theorem 7. The operator T is said to be *positive* (respectively *completely positive*) if T maps positive elements in A into positive elements of A (respectively, if for every $n = 1, 2, \ldots$, the map induced by T on the tensor product of A with the $n \times n$ scalar matrices by the formula $(x_{ij}) \to (T(x_{ij}))$ maps positive elements into positive elements). Then the following theorem extends the results of Størmer [14].

THEOREM 12. *Let A be a von Neumann algebra, with center C and let B be an abelian* C**-subalgebra of A containing the center C of A. Let T be an operator of $B \otimes B$. Then the following are equivalent*:

1. *T is positive definite*;
2. *T is positive*; *and*
3. *T is completely positive.*

PROOF. Let μ be in the carrier space M of C and let $\varrho = \varrho_\mu$ be an irreducible representation of A on the Hilbert space H with kernel equal to the ideal generated by μ (cf. proof, Theorem 7). The operator $\varrho(T)$ in the algebra $\varrho(B) \otimes \varrho(B)$ acting on the algebra of all bounded operators on H is defined by the formula

$$\varrho(T)\varrho(x) = \varrho(Tx) \qquad (x \in A) .$$

It is positive definite if T is positive definite. This follows from Theorem 7 since the spectrum of $\varrho(B) \otimes \varrho(B)$ is $Z_\mu \times Z_\mu$, where Z_μ is the set of all ζ in Z with $\zeta \cap C = \mu$. Also, if T is positive (respectively, completely positive), the same is true about $\varrho(T)$ since the set of positive elements of $\varrho(A)$ (respectively $\varrho(A)$ tensor the $n \times n$ matrices) is strongly dense in the set of positive bounded linear operators on H (respectively the bounded linear operators tensor the $n \times n$ matrices). Thus, if T satisfies any of the three properties of Theorem 12, T satisfies all three [14, Corollary 5.3]. Because μ is arbitrary, it follows that the three properties are equivalent. In fact, an element x in A is positive if and only if $\varrho_\mu(x)$ is positive for all μ in M due to the continuity of the map $\mu \to \|\varrho_\mu(x)\|$. A corresponding statement holds for completely positive operators.

Let τ be the representation on the von Neumann algebra A of the unitary group U of the abelian von Neumann subalgebra B of A given by $\tau_u = \mathrm{ad}\, u$; then an operator T in $L(\tau)$ is positive definite if all the matrices $(T\hat{}(\gamma_i \gamma_j^{-1})) = (T\hat{}(\zeta_j, \zeta_i))$ are positive whenever $\gamma_i = \gamma_{\zeta_i}$ and $\zeta_1, \ldots, \zeta_n$ elements in the carrier space of B having the same intersection with the center of A. In particular, if A is a factor, the operator T is positive definite if every matrix $(T\hat{}(\gamma_i \gamma_j^{-1}))$ is positive for $\gamma_i = \gamma_{\zeta_i}$ with $\zeta_1, \ldots, \zeta_n$ in Z.

Now we characterize the operators τ_u in L in terms of their spectral properties.

Proposition 13. *Let A be a von Neumann algebra, let B be a maximal abelian *-subalgebra of A, and let τ be a representation of the unitary group U of B on A given by $\tau_u = \mathrm{ad}\, u$. Then an operator T in $L(\tau) = L$ is of the form $T = \tau_u$ for same u in U if and only if T is positive definite and the spectrum of T in L is contained in the unit circle.*

Proof. Suppose the spectrum of T in L is contained in the unit circle and that T is positive definite. The operator T^{-1} exists and is positive definite since the matrices for T and T^{-1} are related by

$$(T^{-1}\hat{}(\zeta_i, \zeta_j)) = (T\hat{}(\zeta_i, \zeta_j)^{-}),$$

where the notation is the same as Theorem 8. Hence, both T and T^{-1} are completely positive (Theorem 12). If S is a completely positive operator, we recall that

$$(Sx)^*(Sx) \leqq S(x^*x)$$

for all x in A [15, Theorem 3.1]. Hence, we have that

$$\begin{aligned} x^*x = (T^{-1}Tx)^*(T^{-1}Tx) &\leqq T^{-1}((Tx)^*(Tx)) \\ &\leqq T^{-1}T(x^*x) = x^*x \end{aligned}$$

for every x in X. Using the polarization identity, we see that T is a *-automorphism of A.

We can find orthogonal projections $S_1, \ldots, S_m$ in L of norm 1 in L such that

$$\|T - \sum \lambda_i S_i\| < 1/4$$

[11, Theorem 22]. Since $T(1) = 1$, there is a unique projection $S = S_i$ with $S(1) = 1$ (cf. relations (5), (6)). The projection S has the form

$$Sx = \sum_j \sum \{p_j x p_k \mid k \in X_j\}$$

for a set $p_1,\ldots,p_n$ of orthogonal projections in B of sum 1 with $p_0=0$ and X_j a subset of $\{1,\ldots,n\}$. (Proposition 4). We know that $j \in X_j$ for all $j=1,2,\ldots,n$ (Proposition 4). Thus, we get

$$\begin{aligned}\|(T-1)(p_jxp_j)\| &= \|(T-S)p_jxp_j\| \\ &\leqq \|T-\sum \lambda_k S_k\| \, \|S\| + |1-\lambda_i| \, \|S\| \\ &\leqq 1/4 + \left\|\left(T-\sum_j \lambda_j S_j\right)(1)\right\| \\ &\leqq 1/2\end{aligned}$$

for every unit vector x in A. This proves that T restricted to each subalgebra $p_j Ap_j$ is inner [6; III, § 9, Theorem 6]. Therefore, the map T is an inner automorphism of A [14, 8.9.1]. Let u be a unitary operator in A with $Tx=uxu^*$ for x in A. For x in B we have that

$$uxu^* = Tx = T\hat{}(1)x = x .$$

Because B is a maximal commutative *-subalgebra of A, we have that u is in B.

Conversely, we see that the $m \times m$ matrix

$$(\hat{\tau_u}(\zeta_i,\zeta_j)) = (u\hat{}(\zeta_i)u\hat{}(\zeta_j)^-)$$

is positive for $\zeta_1,\ldots,\zeta_n$ in the subset of the carrier space of B whose intersection with the center of A is fixed.

As a final application of harmonic analyses, we find the positive projections of $L(\tau)$.

PROPOSITION 14. *Let A be a von Neumann algebra, let B be an abelian von Neumann subalgebra of A, and let τ be the representation of the unitary group U of B on A given by $\tau_u = \operatorname{ad} u$. Let T be a positive projection in $L(\tau)$. Then there are orthogonal projections $p_1,\ldots,p_n$ of sum 1 in B such that*

$$Tx = \sum p_i x p_i$$

for every x in A.

PROOF. There are orthogonal projections $p_0,p_1,\ldots,p_n$ in A of sum 1 with $p_0 = 0$ satisfying

$$Tx = \sum_i \{p_i x \sum \{p_j \mid j \in X_i\}\}$$

where X_i is a subset of $\{0,1,\ldots,n\}$ (Proposition 4). There is a finite set $\{q_j\}$ of orthogonal projections in the center C of A of sum 1 such that each projection $p_i q_j$ is 0 or has central support q_j. It is sufficient to show that T restricted to Aq_j has the desired forms. So we may assume each p_i $(1 \leqq i \leqq n)$ has central support 1.

For each μ in the carrier space M of C, the set Z_μ of elements ζ in the carrier space Z of B with $\zeta \cap C = \mu$ is nonvoid. Furthermore, given μ and $p = p_i$ $(1 \leqq i \leqq n)$ there is a $\zeta = \zeta_i$ in $Z\mu$ with $\hat{p}(\zeta) = 1$. Indeed, if $\hat{p}(\zeta)$ vanished for every ζ in Z_μ, then we would have that p is in $\bigcap \{\zeta \in Z \mid \zeta \supset \mu\}$ which is the ideal in B generated by μ. But the central support q of p is given by

$$\hat{q}(\nu) = \|\varrho_\nu(p)\|$$

for ν in M. Recall that ϱ_ν denotes an irreducible representation of A with kernel equal to the ideal of A generated by ν (cf. proof, Theorem 7). This would contradict the assumption that the central support of p is 1. So such a point ζ in Z_μ exists. Therefore, we get that

$$\begin{aligned} 1 = \hat{T}(\zeta_1, \zeta_1) &= \sum \hat{p_i}(\zeta_1) \sum \{\hat{p_j}(\zeta_1) \mid j \in X_i\} \\ &= \hat{p_1}(\zeta_1) \sum \{\hat{p_j}(\zeta_1) \mid j \in X_1\}. \end{aligned}$$

This shows that 1 is in X_1 and so i is in X_i for every i (Proposition 4). Moreover, if i is in X_j, then j is in X_i. In fact, if i is in X_j, then the 2×2 matrix $(\hat{T}(\zeta_k, \zeta_l))$ $(k, l = i, j)$ is positive and has the rows $(1, 1)$, $(\lambda, 1)$. This means that $\lambda = \hat{T}(\zeta_j, \zeta_i)$ is 1 or that i is in X_j.

Now we can show that X_i and X_j are either disjoint or coincide. If k is in $X_i \cap X_j$, then i, j are in X_k. This means that i is in X_j; equivalently, j is in X_i; otherwise, we would ge the 3×3 matrix $(\hat{T}(\zeta_l, \zeta_m))$ $(l, m = i, j, k)$ with rows $(1, 0, 1)$, $(0, 1, 1)$, $(1, 1, 1)$ which is not positive. So if k is in $X_i \cap X_j$, then i, j are in $X_i \cap X_j$. Now completing this argument, we see that if m is in X_i, then i is in X_m as well as X_j; and thus, m is in $X_m \cap X_j$. This proves X_i is contained in X_j. Likewise, we get that X_j is contained in X_i. This demonstrates that $X_i = X_j$ once $X_i \cap X_j$ is nonvoid. So we get that

$$Tx = \sum p_i x \sum \{p_j \mid j \in X_i\} = \sum \sum \{p_j x p_k \mid j, k \in X_i\} = \sum q_i x q_i$$

where $\sum \{p_j \mid j \in X_i\} = q_i$ and the last sum is extended over some subset of $X_1, \ldots, X_n$ which forms a partition of the index set.

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