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THE n-BALL PROPERTIES IN REAL AND COMPLEX BANACH SPACES

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Subspaces of Banach spaces which possess the n-ball property, for some $n \in \mathbf{N}$, have been the subject of considerable attention. Unfortunately, the literature contains a plethora of definitions of the n-ball property. Here we attempt to clarify the relationships that exist between all these properties. Let us begin with the relevant definitions.

Throughout, M will be a closed subspace of a Banach space E. Fix $n \in \mathbf{N}$. We will say that M has the n-ball property in E, if, given n closed balls $B(a_i, r_i)$ such that $M \cap B(a_i, r_i) \neq \emptyset$, for all $i \leqq n$, and $\bigcap_{i=1}^n B(a_i, r_i)$ has non-empty interior, then $M \cap \bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$. If the conditions $M \cap B(a_i, r_i) \neq \emptyset$ for each i, and $\bigcap_{i=1}^n B(a_i, r_i) \neq \emptyset$ imply that $M \cap \bigcap_{i=1}^n B(a_i, r_i + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$, then we say that M has the weak n-ball property in E. It is straightforward to show that the n-ball property implies the weak n-ball property, and that the weak n-ball property is equivalent to the "n-ball property for open balls" considered in [1] and [2]. If we may take $\varepsilon = 0$ in the definition of the weak n-ball property, then M is said to have the strong n-ball property in E. Obviously the strong n-ball property implies the n-ball property.

In [10] we declared M to have the $1\frac{1}{2}$-ball property in E if the conditions $a_1 \in M$, $M \cap B(a_2, r_2) \neq \emptyset$ and $\|a_1 - a_2\| < r_1 + r_2$ implied that $M \cap B(a_1, r_1) \cap B(a_2, r_2) \neq \emptyset$. Similarly we say that M has the weak $1\frac{1}{2}$-ball property in E if the conditions $a_1 \in M$, $M \cap B(a_2, r_2) \neq \emptyset$ and $\|a_1 - a_2\| \leqq r_1 + r_2$ imply that $M \cap B(a_1, r_1 + \varepsilon) \cap B(a_2, r_2 + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. If we can take $\varepsilon = 0$ in the latter definition, M is said to have the strong $1\frac{1}{2}$-ball property in E. After translating and scaling, we see that M has the $1\frac{1}{2}$-ball property in E iff $M \cap B(0,1) \cap B(a,r) \neq \emptyset$ whenever $M \cap B(a,r) \neq \emptyset$ and $\|a\| < r + 1$. Similar remarks apply to the weak and strong $1\frac{1}{2}$-ball properties.

We wish to know which of the trivial implications so far mentioned can be reversed. One easy result is available to us now. Suppose M has the weak n-ball property in E, for some $n \in \mathbf{N} \cup \{1\frac{1}{2}\}$. If M is reflexive, or if E is a dual space and M a weak* closed subspace, an easy compactness argument shows that M actually has the strong n-ball property.

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If there is a projection P from E onto M satisfying $\|x\| = \|Px\| + \|x - Px\|$ (respectively, $\|x\| = \max\{\|Px\|, \|x - Px\|\}$), then M is said to be an L-summand (respectively, an M-summand) in E. It is routine to show that an M-summand has the strong n-ball property, for all $n \in \mathbf{N} \cup \{1\frac{1}{2}\}$. If M°, the polar of M, is an L-summand in E^*, then M is said to be an M-ideal in E. Every M-summand is an M-ideal, but numerous examples show that the converse is false. However, a reflexive M-ideal is easily shown to be an M-summand.

Alfsen and Effros [1, Theorems 5.8 and 5.9] showed that an M-ideal has the n-ball property for every n and, conversely, that a subspace with the weak 3-ball property is already an M-ideal. This paper is devoted to proving that, and related duality results. Some of the results are not new. However, new proofs should be of interest. The $1\frac{1}{2}$-ball property features prominently in our arguments, which use the language of approximation theory rather than the complementary cones considered by other authors [1, 2, 7]. We will also show that the weak n-ball property is equivalent to the n-ball property, for all $n \in \mathbf{N} \cup \{1\frac{1}{2}\}$.

Lastly we show that the strong n-ball property is distinct from the n-ball property, for every value of n. We do this with a single example, an M-ideal which fails the strong $1\frac{1}{2}$-ball property. Throughout, the scalar field $\boldsymbol{K}$ may be either $\mathbf{R}$ or $\mathbf{C}$.

We now introduce the required results from elementary approximation theory. Given $a \in E$, let $P(a) = P_M(a)$ be the set of points in M which are as close as possible to a. That is, $P(a) = \{x \in M \colon \|x - a\| = d(a, M)\}$. If $P(a)$ contains exactly (at least/at most) one element, for every $a \in E$, then M is said to be a Chebyshev (proximinal/unicital) subspace of E. (If M is Chebyshev in E, then the closest point map $P \colon E \to M$ is called the metric projection.) We define the metric complement of M by

$$M^\perp = \{x \in E \colon \|x\| = d(x, M)\} = \{x \in E \colon 0 \in P(x)\}.$$

We say that the subspace M has the unique extension property in E if every $f \in M^*$ has a unique norm preserving extension to an element of E^*. Note that, for any $f \in E^*$, we have $g \in P_{M^\circ}(f)$ iff $f - g$ is a norm preserving extension of $f \mid M$. This gives us the following result of Phelps [8, Theorem 1.1]: M has the unique extension property in E iff M° is a Chebyshev subspace of E^*. Dually, if M^0 has the unique extension property in E^*, then M^{00} will be Chebyshev in E^{**}, whence M is unicital in E. Phelps [8, p. 252] showed that the converse of this is false. If $E = c_0$ and M is the one dimensional subspace spanned by the sequence $(1, \frac{1}{2}, \frac{1}{3}, \ldots)$, then M is Chebyshev in E, but M^0 does not have the unique extension property in E^*.

The proof of the Hahn–Banach theorem gives us the following useful result.

Let $f \in M^*$ with $\|f\|=1$ and fix $a \in E$. Then f has a norm preserving extension $g \in E^*$, with $\operatorname{re} g(a)=\lambda$ iff

$$\sup\{\operatorname{re} f(x)-\|x-a\| : x \in M\} \leqq \lambda \leqq \inf\{\operatorname{re} f(x)+\|x-a\| : x \in M\} .$$

Of course, at least one such λ exists.

The following two results are essential for what follows.

LEMMA 1. [7, Theorems 1.1 and 1.2].

(i) *Fix $a_1,\ldots a_n \in E$ and $r_1,\ldots r_n>0$. Then*

$$M \cap \bigcap_{i=1}^{n} B(a_i, r_i+\varepsilon) \neq \varnothing \quad \textit{for all } \varepsilon>0, \textit{ iff}$$

$$\left|\sum_{i=1}^{n} f_i(a_i)\right| \leqq \sum_{i=1}^{n} r_i\|f_i\| \quad \textit{whenever } \sum_{i=1}^{n} f_i \in M^\circ .$$

(ii) *Fix $f_1,\ldots f_n \in E^*$ and $r_1,\ldots r_n>0$. Then*

$$M^\circ \cap \bigcap_{i=1}^{n} B(f_i, r_i) \neq \varnothing, \quad \textit{iff}$$

$$\left|\sum_{i=1}^{n} f_i(a_i)\right| \leqq \sum_{i=1}^{n} r_i\|a_i\| \quad \textit{whenever } \sum_{i=1}^{n} a_i \in M .$$

COROLLARY 2. [7, Corollary 1.3]. *Fix $a_1,\ldots a_n \in E$ and $r_1,\ldots r_n>0$. Then*

$$M \cap \bigcap_{i=1}^{n} B(a_i, r_i+\varepsilon) \neq \varnothing \quad (\textit{in } E) \textit{ for all } \varepsilon>0 ,$$

iff

$$M^{\circ\circ} \cap \bigcap_{i=1}^{n} B(a_i, r_i) \neq \varnothing \quad (\textit{in } E^{**}) .$$

It follows at once that M has the weak n-ball property in E, if $M^{\circ\circ}$ has the (strong) n-ball property in E^{**}.

Now we give some useful characterizations of the $1\frac{1}{2}$-ball property.

THEOREM 3. *The following are equivalent*:

(i) *Suppose $M \cap B(a,r) \neq \varnothing$, $f \in M^*$ and $\operatorname{re} f(x) \geqq c$ for all $x \in M \cap B(a,r)$. Then f has a norm preserving extension $g \in E^*$ such that $\operatorname{re} g(x) \geqq c$ for all $x \in B(a,r)$.*

(ii) *M has the $1\frac{1}{2}$-ball property in E.*

(iii) *M has the weak $1\frac{1}{2}$-ball property in E.*
(iv) *M^0 has the (strong) $1\frac{1}{2}$-ball property in E^*.*

Moreover, any subspace satisfying these conditions is proximinal.

PROOF. (i) $\Rightarrow$ (ii). Suppose $M \cap B(a,r) \neq \emptyset$ but that $M \cap B(0,1) \cap B(a,r) = \emptyset$. We will show that $\|a\| \geqq r+1$. By the Hahn–Banach theorem, there is $f \in M^*$ with $\operatorname{re} f(x) \geqq 1$ for all $x \in M \cap B(a,r)$ and $\operatorname{re} f(x) < 1$ for all $x \in M$ with $\|x\| < 1$. By hypothesis, f has an extension $g \in E^*$ with $\|g\| \leqq 1$ and $\operatorname{re} g(x) \geqq 1$ for all $x \in B(a,r)$. It follows that $B(0,1) \cap B(a,r)$ has no interior points.

(ii) $\Rightarrow$ (i). Assume without loss of generality that $K = \mathbf{R}$ and $\|f\| = 1$. We claim that $f(x) \geqq c + r - \|x-a\|$ for all $x \in M$. First suppose $\|x-a\| \leqq r$. Then

$$y \in M,\ \|y\| \leqq 1 \Rightarrow x - (r - \|x-a\|)y \in M \cap B(a,r)$$
$$\Rightarrow f(x) \geqq c + (r - \|x-a\|)f(y) .$$

Hence

$$f(x) \geqq c + (r - \|x-a\|)\|f\| .$$

Now suppose $\|x-a\| > r$. Then, for all $\varepsilon > 0$, the $1\frac{1}{2}$-ball property gives us some $y \in M \cap B(a,r) \cap B(x, \|x-a\| - r + \varepsilon)$.
Hence

$$f(x) = f(y) + f(x-y) \geqq c - \|x-y\|$$
$$\geqq c + r - \|x-a\| - \varepsilon .$$

Letting $\varepsilon \to 0$ establishes the claim.

Now the Hahn–Banach theorem gives us a norm preserving extension $g \in E^*$ with $g(a) \geqq r + c$. Then, for any $x \in B(a,r)$, $g(x) \geqq g(a) - \|x-a\| \geqq c$.

(ii) $\Leftrightarrow$ (iii). Obviously the $1\frac{1}{2}$-ball property implies the weak $1\frac{1}{2}$-ball property. Conversely, assume M has the weak $1\frac{1}{2}$-ball property in E. Given $a \in E$, $r > 0$ with $d(a,M) \leqq 1 < \|a\| < r+1$, we will show that $M \cap B(0,r) \cap B(a,1) \neq \emptyset$. (If $\|a\| \leqq 1$, $0 \in M \cap B(0,r) \cap B(a,1)$.)

Let $\varepsilon = \frac{1}{3}(r + 1 - \|a\|)$. Then $\varepsilon > 0$, $M \cap B(a, 1 + \frac{1}{2}\varepsilon) \neq \emptyset$ and $\|a\| \leqq 1 + (r - 3\varepsilon)$. The weak $1\frac{1}{2}$-ball property then gives us some $x_0 \in M \cap B(0, r - 2\varepsilon) \cap B(a, 1 + \varepsilon)$. By induction, we will construct a sequence $(x_n) \subset M$ satisfying

(1) $$\|x_n - x_{n+1}\| \leqq 2^{-n}\varepsilon ,$$

(2) $$\|x_n - a\| \leqq 1 + 2^{-n}\varepsilon .$$

Given x_n satisfying (2), the weak $1\frac{1}{2}$-ball property gives us a suitable

$$x_{n+1} \in M \cap B(x_n, \tfrac{3}{4}2^{-n}\varepsilon + \tfrac{1}{4}2^{-n}\varepsilon) \cap B(a, 1 + \tfrac{1}{4}2^{-n}\varepsilon + \tfrac{1}{4}2^{-n}\varepsilon)\,.$$

Now (x_n) is a Cauchy sequence, whose limit $x \in M$ satisfies $\|x - x_0\| \leqq 2\varepsilon$ and $\|x - a\| \leqq 1$. Then $\|x\| \leqq \|x_0\| + 2\varepsilon$, so $x \in M \cap B(0, r) \cap B(a, 1)$.

(iii) $\Leftrightarrow$ (iv). First suppose M° has the strong $1\frac{1}{2}$-ball property in E^*. Given $M \cap B(a, r) \neq \emptyset$ and $\|a\| \leqq r + 1$, we must show that $M \cap B(0, 1 + \varepsilon) \cap B(a, r + \varepsilon) \neq \emptyset$ for all $\varepsilon > 0$. By lemma 1 (i), we need only show that $|f_2(a)| \leqq \|f_1\| + r\|f_2\|$, whenever $f_1, f_2 \in E^*$, $f_1 + f_2 \in M^\circ$. If $\|f_2\| \leqq \|f_1\|$, then $|f_2(a)| \leqq (r+1)\|f_2\| \leqq \|f_1\| + r\|f_2\|$ as required. If $\|f_1\| < \|f_2\|$ then, since

$$f_1 + f_2 \in M^\circ \cap B(f_2, \|f_1\|)\,,$$

the $1\frac{1}{2}$-ball property gives us $f \in M^\circ \cap B(0, \|f_2\| - \|f_1\|) \cap B(f_2, \|f_1\|)$. Now $M \cap B(a, r) \neq \emptyset$, so $|f(a)| \leqq r\|f\|$. Thus

$$\begin{aligned} |f_2(a)| &= |f(a) - (f - f_2)(a)| \\ &\leqq r(\|f_2\| - \|f_1\|) + (r+1)\|f_1\| \\ &= \|f_1\| + r\|f_2\|\,. \end{aligned}$$

The converse follows from a similar argument, using Lemma 1 (ii).

Finally, suppose M satisfies these conditions and choose $a \in E$ with $d(a, M) = 1$. The proof of (ii) $\Leftrightarrow$ (iii) shows that $M \cap B(a, 1) \neq \emptyset$. Thus M is proximinal.

Next, we give a characterization of subspaces with the 2-ball property.

THEOREM 4. *The following are equivalent*:
(i) *M has the weak 2-ball property in E,*
(ii) *M has the $1\frac{1}{2}$-ball property and the unique extension property in E,*
(iii) *M has the 2-ball property in E.*

PROOF. (i) $\Rightarrow$ (ii). Obviously the weak 2-ball property implies the $1\frac{1}{2}$-ball property. Fix $f \in M^*$ with $\|f\| = 1$. To show that M has the unique extension property, we must prove that, for all $a \in E$,

$$\sup\{\operatorname{re} f(x) - \|a - x\| : x \in M\} \geqq \inf\{\operatorname{re} f(x) + \|a - x\| : x \in M\}\,.$$

We may assume that $\|a\| = 1$. Now fix $\varepsilon > 0$, and choose $x \in M$ with $\|x\| = 1$, $f(x) > 1 - \varepsilon$. Then $a \in B(a + x, 1) \cap B(a - x, 1)$ and $\pm x \in M \cap B(a \pm x, 1)$. Hence we can find $y \in M \cap B(a + x, 1 + \varepsilon) \cap B(a - x, 1 + \varepsilon)$. Then $y + x$, $y - x \in M$ and

$$\begin{aligned}\{\operatorname{re} f(y+x)-\|a-(y+x)\|\}-\{\operatorname{re} f(y-x)+\|a-(y-x)\|\}\\ = 2f(x)-\|a-x-y\|-\|a+x-y\|\\ > 2(1-\varepsilon)-2(1+\varepsilon)\,.\end{aligned}$$

This establishes the inequality.

(ii) $\Rightarrow$ (iii). Suppose $M\cap B(a_i,r_i)\neq\emptyset$ for $i=1,2$ but that $M\cap B(a_1,r_1)\cap B(a_2,r_2)=\emptyset$. We must show that $B(a_1,r_1)\cap B(a_2,r_2)$ has no interior points. Suppose that $B(y,\delta)\subset B(a_1,r_1)\cap B(a_2,r_2)$. We will show that $\delta<\varepsilon$, for any given $\varepsilon>0$.

Let A be the closure of A_1-A_2, where $A_i=M\cap B(a_i,r_i)$. Now A_1 and A_2 are closed, bounded, convex subsets of M, and $0\notin A_1-A_2$. It follows from [6, Corollary 22.5] that 0 is not an interior point of A. Thus we can find $x_0\in M$ with $\|x_0\|\leqq\varepsilon$ and $x_0\notin A$. The Hahn–Banach theorem then gives us $f\in M^*$ with $\|f\|=1$ and $\operatorname{re} f(x)<\operatorname{re} f(x_0)$ for all $x\in A$. Putting $c=\sup\operatorname{re} f(A_1)$, we have $\operatorname{re} f(x)\leqq c$ for all $x\in A_1$ and $\operatorname{re} f(x)\geqq c-\varepsilon$ for all $x\in A_2$. Since M has the $1\frac{1}{2}$-ball property, f has norm preserving extensions $f_1,f_2\in E^*$ with $\operatorname{re} f_1(x)\leqq c$ for all $x\in B(a_1,r_1)$ and $\operatorname{re} f_2(x)\geqq c-\varepsilon$ for all $x\in B(a_2,r_2)$. The unique extension property forces $f_1=f_2$. Now

$$\sup\{\operatorname{re} f_1(x_1-x_2):\ x_1,x_2\in B(y,\delta)\}\ =\ 2\delta\,.$$

However $x_i\in B(y,\delta)\ \Rightarrow\ x_i\in B(a_i,r_i)\ \Rightarrow\ \operatorname{re} f_1(x_1-x_2)\leqq\varepsilon$. Thus $2\delta\leqq\varepsilon$.

(iii) $\Rightarrow$ (i). This is trivial.

In [11] we showed that, for certain Banach spaces E, $c_0(E)$ is a subspace of $l_\infty(E)$ which has the n-ball property for every n, but not the strong 2-ball property. It is an easy exercise to show that $c_0(E)$ does have the strong $1\frac{1}{2}$-ball property in $l_\infty(E)$. Thus, despite Theorem 4, a subspace with the strong $1\frac{1}{2}$-ball property and the unique extension property need not have the strong 2-ball property.

Let us say M is a semi-L-summand in E if M is proximinal, and $\|x-y\|=\|x\|+\|y\|$, whenever $x\in M$ and $y\in M^\perp$. This is equivalent to the definition made by Lima [7, section 5].

Theorem 5. *The following are equivalent.*
(i) *M is a semi-L-summand in E,*
(ii) *M is Chebyshev, and has the strong $1\frac{1}{2}$-ball property in E,*
(iii) *M is unicital, and has the $1\frac{1}{2}$-ball property in E.*

Proof. (i) $\Rightarrow$ (ii). Suppose $M\cap B(a,r)\neq\emptyset$ and $\|a\|\leqq r+1$. If $x\in P(a)$, then

$\|a-x\|=d(a,M)\leqq r$ and, since $a-x\in M^{\perp}$, $\|x\|=\|a\|-\|a-x\|\leqq 1+(r-\|x-a\|)$. Choose $\lambda\in[0,1]$, so that $\lambda\|x\|\leqq 1$ and $(1-\lambda)\|x\|\leqq r-\|x-a\|$. Then

$$\|a-\lambda x\| \;=\; \|x-\lambda x\|+\|x-a\| \;\leqq\; r\,,$$

so $\lambda x\in M\cap B(0,1)\cap B(a,r)$. To show M is Chebyshev, suppose $a\in E$, $x,y\in P(a)$. Then $x-y\in M$ and $a-y\in M^{\perp}$, so

$$d(a,M) \;=\; \|x-a\| \;=\; \|x-y\|+\|a-y\| \;=\; \|x-y\|+d(a,M)\,.$$

Thus $x=y$.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). By Theorem 3, M is proximinal. Now let $x\in M$, $y\in M^{\perp}$ and fix $\varepsilon>0$. Since M is Chebyshev, $M\cap B(y,\|y\|)=\{0\}$. Since M has the $1\frac{1}{2}$-ball property,

$$M\,\cap\, B(x,\|x-y\|-\|y\|+\varepsilon)\,\cap\, B(y,\|y\|)\;\neq\;\varnothing\,.$$

Thus $0\in B(x,\|x-y\|-\|y\|+\varepsilon)$, and so $\|x\|\leqq\|x-y\|-\|y\|+\varepsilon$. Letting $\varepsilon\to 0$ finishes the proof.

We can now give easy proofs of the duality results concerning the 2-ball property.

THEOREM 6. [7, Theorem 6.10]. *M has the 2-ball property in E iff M° is a semi-L-summand in E^*.*

PROOF.

M has the 2-ball property in E,

$\Leftrightarrow$ M has the $1\frac{1}{2}$-ball property and the unique extension property ,

$\Leftrightarrow$ M° has the $1\frac{1}{2}$-ball property, and is Chebyshev in E^* ,

$\Leftrightarrow$ M° is a semi-L-summand in E^* .

When M° is a semi-L-summand in E^*, M is said to be a semi-M-ideal in E [7, section 6].

THEOREM 7. [7, Theorem 6.14]. *M is a semi-L-summand in E iff M° has the 2-ball property in E^*.*

PROOF. ($\Leftarrow$). Combine Theorems 4, 3, and 5.

($\Rightarrow$). Suppose $M^{\circ}\cap B(f_i,r_i)\neq\varnothing$ for $i=1,2$ and $\|f_1-f_2\|\leqq r_1+r_2$. It follows that, for any $a_1,a_2\in E$,

$$a_1, a_2 \in M \Rightarrow |f_1(a_1)+f_2(a_2)| \leqq r_1\|a_1\|+r_2\|a_2\|$$

and

$$a_1+a_2 = 0 \Rightarrow |f_1(a_1)+f_2(a_2)| \leqq r_1\|a_1\|+r_2\|a_2\| .$$

Now suppose $a_1+a_2 \in M$. We may write $a_i = b_i + c_i$, where $b_i \in M$ and $c_i \in M^\perp$. Then $b_1+c_1 = (a_1+a_2-b_2)-c_2$ and $a_1+a_2-b_2 \in M$. Since M is Chebyshev in E, this forces $c_1 = -c_2$. Hence

$$\begin{aligned} |f_1(a_1)+f_2(a_2)| &\leqq |f_1(b_1)+f_2(b_2)|+|f_1(c_1)+f_2(c_2)| \\ &\leqq r_1\|b_1\|+r_2\|b_2\|+r_1\|c_1\|+r_2\|c_2\| \\ &= r_1\|a_1\|+r_2\|a_2\| . \end{aligned}$$

By Lemma 1(ii), $M^\circ \cap B(f_1, r_1) \cap B(f_2, r_2) \neq \emptyset$.

This result gives us a weak converse to [8, Theorem 1.3]. If M is a unicital subspace of E, with the $1\frac{1}{2}$-ball property, then M° has the unique extension property in E^*.

We need two elementary results before presenting the duality results for the 3-ball property.

LEMMA 8. [2, Proposition 2.19]. *For $n \in \mathbb{N}$, the weak $(n+1)$-ball property implies the n-ball property.*

LEMMA 9. *Necessary and sufficient conditions for M to be an L-summand in E are that M has the $1\frac{1}{2}$-ball property, with $M^\perp$ convex.*

PROOF. If M has the $1\frac{1}{2}$-ball property, it is proximinal. If $M^\perp$ is a subspace, we may write $E = M \oplus M^\perp$. This implies that M is Chebyshev, with linear metric projection. By Theorem 5, M is a semi-L-summand in E. Thus the metric projection is an L-projection.

The converse is easy.

THEOREM 10. [7, Theorem 6.16]. *The following are equivalent.*

(i) *M is an L-summand in E.*
(ii) *M° has the (strong) n-ball property, for all n.*
(iii) *M° has the (strong) 3-ball property in E^*.*

PROOF. (i) $\Rightarrow$ (ii). M° will be an M-summand in E^*.
(ii) $\Rightarrow$ (iii). This is trivial.
(iii) $\Rightarrow$ (i). By Theorem 3, M has the $1\frac{1}{2}$-ball property in E. Thus it suffices to show that $M^\perp$ is convex.

So let $x_1, x_2 \in M^\perp$. Since M is proximinal, we may write $x_1+x_2=y-x_3$, where $y \in M$, $x_3 \in M^\perp$. Now the Hahn–Banach theorem gives us $f_i \in M^\circ$ and $g \in E^*$ with

$$\|f_i\| = \|g\| = 1, \quad f_i(x_i) = d(x_i, M) = \|x_i\|, \quad \text{and } g(y) = \|y\| .$$

Note that

$$g \in \bigcap_{i=1}^{3} B(g+f_i, 1) \quad \text{and } f_i \in M^\circ \cap B(g+f_i, 1) .$$

Hence we can find

$$h \in M^\circ \cap \bigcap_{i=1}^{3} B(g+f_i, 1) .$$

Then

$$\begin{aligned} \|y\| &= g(y) = (g-h)(y) \\ &= (g-h)\left(\sum_{i=1}^{3} x_i\right) \\ &= \sum_{i=1}^{3} (g+f_i-h)(x_i) - \sum_{i=1}^{3} \|x_i\| \leqq 0 , \end{aligned}$$

so $y=0$. Thus $x_1+x_2=-x_3 \in M^\perp$, so $M^\perp$ is convex.

Defining L-ideals in a manner analogous to the definition of M-ideals does not introduce a new concept. For, if M° is an M-summand in E^*, then by Theorem 10, M is an L-summand in E. This was first proved by Cunningham, Effros, and Roy [4], who showed that every M-summand in a dual space is weak* closed.

The final, and most useful, theorem of this paper was first proved by Alfsen and Effros [1, Theorems 5.8 and 5.9] for real Banach spaces. Lima [7, Theorem 6.9] gave a simpler proof, valid for either scalar field, but he only worked with the weak n-ball property. Behrends [2, chapter 2] has given another account, including the useful Lemma 8.

THEOREM 11. *The following are equivalent.*

(i) *M is an M-ideal in E.*
(ii) *M has the n-ball property, for every n.*
(iii) *M has the weak 3-ball property in E.*

PROOF. (i) $\Rightarrow$ (ii). If M° is an L-summand in E^*, then $M^{\circ\circ}$ will be an M-

summand in E^{**}. Hence $M^{\circ\circ}$ has the strong n-ball property in E^{**}, so M has the weak n-ball property in E. Since this is true for every n, Lemma 8 ensures that M has the n-ball property.

(ii) $\Rightarrow$ (iii). This is trivial.

(iii) $\Rightarrow$ (i). This follows, mutatis mutandis, from the corresponding part of the proof of Theorem 10.

At last, we see that the weak n-ball property is equivalent to the n-ball property, for $n \geqq 3$. Several authors [3, 5] have shown, under suitable hypotheses, that M-ideals have an intersection property for an infinite number of balls.

It is worth noting that the $1\frac{1}{2}$-ball property does not imply the 2-ball property. Any non-trivial L-summand is a suitable counter-example; further examples may be found in [10]. Alfsen and Effros [1, Theorem 5.9] showed that the 2-ball property does not imply the 3-ball property.

Finally, we show that the strong n-ball property is strictly stronger than the n-ball property, for all $n \in \mathbf{N} \cup \{1\frac{1}{2}\}$.

LEMMA 12. *Suppose M has the strong $1\frac{1}{2}$-ball property in E, and that x is an extreme point of E_1. Then $x \in M \cup M^{\perp}$.*

PROOF. If $\delta = d(x, M)$, then the strong $1\frac{1}{2}$-ball property ensures that

$$M \cap B(0, 1-\delta) \cap B(x, \delta) \neq \emptyset .$$

However $B(0, 1-\delta) \cap B(x, \delta) = \{(1-\delta)x\}$, so $(1-\delta)x \in M$. Thus either $x \in M$ or $\delta = 1$.

EXAMPLE 13. There is an M-ideal which does not have the strong $1\frac{1}{2}$-ball property.

PROOF. Let E be the disc algebra (i.e. the sup normed space of functions continuous on Δ, the closed unit disc in $\mathbf{C}$, and analytic on the interior of Δ). Let $M = \{x \in E \colon x(1) = 0\}$. According to [7, Theorem 7.6], M is an M-ideal in E. Given δ with $0 < \delta < 1$, let x be a conformal mapping of Δ onto $\{z \in \Delta\colon \operatorname{re} z \geqq \delta\}$, with $x(1) = \delta$. Then the unit circle $\mathbf{T}$ contains an arc J such that $x(J) \subset \mathbf{T}$. It follows from [9, Theorem 11.22] that x is an extreme point of E_1. However $d(x, M) = |x(1)| = \delta$, so $x \notin M \cup M^{\perp}$.

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