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EXTENSIONS OF DERIVATIONS II

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1. Introduction.

Recall that a derivation δ, of an algebra $\mathscr{A}$ of operators on a Hilbert space H into the bounded operators $B(H)$ on this space, is a linear map, which for any two elements a and b in $\mathscr{A}$ satisfies

$$\delta(ab) = a\delta(b)+\delta(a)b .$$

The main result of this article tells that any C*-algebra $\mathscr{A}$ on a Hilbert space H with a cyclic vector has the property, that to any derivation δ of $\mathscr{A}$ into $B(H)$ there exists an operator x in $B(H)$ such that:

$$\forall a \in \mathscr{A} \qquad \delta(a) = [x,a] = xa-ax .$$

In [6] we proved a similar result for properly infinite von Neumann algebras. The present result shows that the general problem is linked to the particular representation rather than to some internal algebraic properties.

In Section 3 we have tried to show why it is interesting to know whether all derivations of a C*-algebra $\mathscr{A}$ into $B(H)$ are implemented by bounded operators on H (such derivations are said to be inner). One of the main reasons is, that if this is the case, then there exists a $k>0$ such that for any operator x in $B(H)$ we have the inequalities (1) below. In order to explain the inequalities we remind the reader that $\mathscr{A}'$ denotes the commutant of $\mathscr{A}$, i.e. all the operators in $B(H)$ which commute with all operators in $\mathscr{A}$. Moreover $d(x,\mathscr{A}')$ is the distance from x to $\mathscr{A}'$, ad (x) denotes the derivation on $B(H)$ implemented by x and $\|\mathrm{ad}\,(x)\,|\,\mathscr{A}\|$ is the norm of the restriction of this map to $\mathscr{A}$. We can now state the inequality which is fulfilled when all derivations of $\mathscr{A}$ into $B(H)$ are implemented by elements in $B(H)$:

$$\tfrac{1}{2}\|\mathrm{ad}\,(x)\,|\,\mathscr{A}\| \leqq d(x,\mathscr{A}') \leqq k\|\mathrm{ad}\,(x)\,|\,\mathscr{A}\| . \tag{1}$$

This means that one can estimate the distance to the commutant by measuring norms of commutators. In the papers [2, 3, 4, 5, 6, 7, 8] it has been proved that a lot of algebras satisfy an inequality as (1) and the inequality has also been used to prove various results related to perturbations of operator algebras.

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The main argument of this paper comes in Section 4 and is based upon Pisiers' non-commutative Grothendieck inequality. Via this inequality we prove that derivations on C*-algebras with values in C*-algebras always satisfy a certain inequality, which in turn shows that derivations between concretely represented C*-algebras are always ultrastrongly continuous.

In Section 5 we prove that in various cases all derivations of a C*-algebra $\mathscr{A}$ on a Hilbert space H into $B(H)$ are inner, Moreover we prove that for any cyclic projection q in $\mathscr{A}'$ and any derivation δ of $\mathscr{A}$ into $B(H)$ the derivation δq of $\mathscr{A}$ into $B(H)$ is always inner.

2. Preliminaries.

The notation and terminology follows Dixmier's book [10], except that we use script capital letters for C*-algebras and capital Roman letters for von Neumann algebras. Hilbert spaces are denoted by the capital Roman letters H, K, whereas vectors in these spaces are denoted by greek letters. Given a Hilbert space H, $B(H)$ denotes the bounded operators on H, $C(H)$ the compact operators, and $B(H)_*$ the predual, or the space of ultraweakly continuous functionals on $B(H)$. The special algebra of all complex $n \times n$ matrices with entries from an algebra $\mathscr{A}$ is denoted $\mathscr{A} \otimes M_n$.

Since a great deal of this paper is devoted to the study of derivations, we want to introduce some notation which shortens some of the statements. First we remind the reader that a linear map δ of an algebra $\mathscr{A}$ into a bigger algebra $\mathscr{B}$ is called a derivation if for any two operators x and y in $\mathscr{A}$, $\delta(xy) = x\delta(y) + \delta(x)y$. The linear space of all derivations of $\mathscr{A}$ into $\mathscr{B}$ is denoted by $Z^1(\mathscr{A}, \mathscr{B})$. The linear space of all derivations of $\mathscr{A}$ into $\mathscr{B}$ which are implemented by operators in $\mathscr{B}$ i.e. has the form $\delta(a) = [b, a] = ba - ab$, is called $B^1(\mathscr{A}, \mathscr{B})$. The quotient space $Z^1(\mathscr{A}, \mathscr{B})/B^1(\mathscr{A}, \mathscr{B})$ is called the first cohomology group for $\mathscr{A}$ which coefficients in $\mathscr{B}$ and the space is denoted by $H^1(\mathscr{A}, \mathscr{B})$. If $H^1(\mathscr{A}, \mathscr{B}) = 0$ we say that all derivations of $\mathscr{A}$ into $\mathscr{B}$ are inner. For an element b in $\mathscr{B}$ the inner derivation of $\mathscr{A}$ into $\mathscr{B}$ given by $a \rightarrow (ba - ab)$ is called $\text{ad}\,(b) \mid \mathscr{A}$.

Finally we will mention the concept, completely boundedness, which is very important in this context. We say that a linear map φ of a C*-algebra $\mathscr{A}$ into a C*-algebra $\mathscr{B}$ is completely bounded, if there exists a positive real c such that for each n in N $\varphi \otimes \text{id}: \mathscr{A} \otimes M_n \rightarrow \mathscr{B} \otimes M_n$ is bounded and $\|\varphi \otimes \text{id}\| \leq c$.

3. On some properties of C*-algebras.

It has been the aim of much of our work recently to prove that all C*-algebras have the properties listed in Theorem 3.1. We do think this is the case and we find strong evidence of this assumption in the results presented in Section 5 together with the results presented in [5, 6].

3.1. THEOREM. *Let $\mathscr{A}$ be a C*-algebra on a Hilbert space H and let M denote the ultraweak closure of $\mathscr{A}$.*

The following properties are equivalent.

1) *There exists a constant $k>0$ such that for any operator x in $B(H)$:*

$$d(x,\mathscr{A}') \leqq k\|\mathrm{ad}\,(x)\,|\,\mathscr{A}\| .$$

2) $H^1(\mathscr{A},B(H))=0$.
3) $B^1(\mathscr{A},B(H))$ *is closed in* $Z^1(\mathscr{A},B(H))$.
4) *Any derivation of $\mathscr{A}$ into $B(H)$ is completely bounded.*
5) *There exists a constant $c>0$ such that to any trace class operator h on H which is orthogonal to $\mathscr{A}'$ there exist sequences $(a_i)_{i\in\mathrm{N}}$ of elements from $\mathscr{A}$ and $(h_i)_{i\in\mathrm{N}}$ of traceclass operators such that*

$$\sum_i \|a_i\|\,\|h_i\|_1 \leqq c\|h\|_1 \quad \text{and} \quad h = \sum_i [h_i, a_i] .$$

6) *There exists a constant $c>0$ such that to any ultraweakly continuous functional φ, which vanishes on A' there exists a sequence $\omega_{\xi_n\eta_n}$ of vectorfunctionals all vanishing on $\mathscr{A}'$ such that*

$$\sum_n \|\xi_n\|\,\|\eta_n\| \leqq c\|\varphi\| \quad \text{and} \quad \varphi = \sum_n \omega_{\xi_n\eta_n} .$$

PROOF. By [17] every derivation of $\mathscr{A}$ into $B(H)$ has a unique extension to a derivation of M into $B(H)$. Therefore we may always consider derivations of $\mathscr{A}$ into $B(H)$ as being restrictions of derivations of M into $B(H)$.

[1 ⇒ 2]. This result is the main content of [6], more precisely the result follows from [6, Theorem 3.2 and Theorem 4.2].

[2 ⇒ 3]. Obvious.

[3 ⇒ 1]. As in the proof of Theorem 2.4 of [6] we define a complete norm $|||\cdot|||$ on the quotient space $B(H)/\mathscr{A}'$ by

$$x \in B(H) \qquad |||x||| = \|\mathrm{ad}\,(x)\,|\,\mathscr{A}\| .$$

By the closed graph theorem the norm $|||\cdot|||$ is equivalent to the quotient norm, but that implies the existence of a constant $k>0$ such that

$$d(x,\mathscr{A}') \leqq k\|\mathrm{ad}\,(x)\,|\,\mathscr{A}\| .$$

[2 ⇒ 4]. Obvious.

[4 ⇒ 2]. Suppose δ is a derivation of $\mathscr{A}$ into $B(H)$ then $\delta \otimes \mathrm{id}$ is a bounded derivation of the spatial C* tensorproduct $\mathscr{A} \otimes C(\ell^2(\mathbf{N}))$ into $B(H \otimes \ell^2(\mathbf{N}))$. Let $\hat{\delta}$ denote the unique extension of $\delta \otimes \mathrm{id}$ to a derivation of the properly infinite von Neumann algebra $M \bar{\otimes} B(\ell^2(\mathbf{N}))$ into $B(H \otimes \ell^2(\mathbf{N}))$. By [6, Theorem 3.2] there exists an operator x in $B(H \otimes \ell^2(\mathbf{N}))$ such that $\hat{\delta} = \mathrm{ad}\,(x) \mid M \bar{\otimes} B(\ell^2(\mathbf{N}))$. Now $\hat{\delta}$ is trivial on $\mathbf{C} \otimes B(\ell^2(\mathbf{N}))$, so x belongs to $B(H) \otimes \mathbf{C}$. Consequently there exists an operator y in $B(H)$ such that $x = y \otimes I$ and $\delta = \mathrm{ad}\,(y) \mid \mathscr{A}$.

[1 ⇒ 5]. Let

$$K = \{[a,h] \mid a \in \mathscr{A},\ h \text{ trace class},\ \|a\|\,\|h\|_1 < 1\}\ .$$

We want to consider K as a subset of the predual of $B(H)$, and we prove the theorem by showing that the closed convex hull of K contains $k^{-1}((\mathscr{A}')^{\perp})_1$.

Let now z in $B(H)$ be an element of the polar K° of K,

$$K^{\circ} = \{y \in B(H) \mid \forall k \in K : |\mathrm{tr}\,(yk)| \leqq 1\}.$$

Since z belongs to K° we get for any a in $\mathscr{A}$ with $\|a\| \leqq 1$ and any h with $\|h\|_1 \leqq 1$ that

$$1 \geqq |\mathrm{tr}\,(z(ah - ha))| = |\mathrm{tr}\,((za - az)h)|\ .$$

Therefore $\|\mathrm{ad}\,(z) \mid \mathscr{A}\| \leqq 1$, and then $d(z, \mathscr{A}') \leqq k$. We have then proved that

$$K^{\circ} \subseteqq k(\mathscr{A}' + B(H)_1)\ ,$$

which implies

$$((\mathscr{A})^{\perp})_1 \subseteqq k(K^{\circ\circ}) = k(\text{closed convex hull of } K)\ .$$

A usual approximation argument shows that any $c > k$ will do.

[5 ⇒ 1]. Let x be in $B(H)$ then the distance from x to $\mathscr{A}'$ is

$$\sup\{|\varphi(x)| \mid \varphi \in B(H)_*,\ \|\varphi\| \leqq 1,\ \varphi \in (\mathscr{A}')^{\perp}\}$$

for any such φ there exist sequences $(x_i)_{i \in \mathbf{N}}$, $(h_i)_{i \in \mathbf{N}}$ of operators in $\mathscr{A}$ and traceclass operators respectively such that $\|a_i\| \leqq 1$, $\sum_i \|h_i\|_1 \leqq c\|\varphi\|$ and the density of φ with respect to the trace on $B(H)$ is $\sum_i [a_i, h_i]$. We then get

$$\begin{aligned} |\varphi(x)| &\leqq \sum_i |\mathrm{tr}\,(x(a_i h_i - h_i a_i))| \\ &= \sum_i |\mathrm{tr}\,((x a_i - a_i x) h_i)| \\ &\leqq \sum_i \|\mathrm{ad}\,(x)(a_i)\|\,\|h_i\|_1 \leqq c\|\mathrm{ad}\,(x) \mid \mathscr{A}\|\ . \end{aligned}$$

Consequently $d(x, \mathscr{A}') \leqq c \|\mathrm{ad}\,(x) \mid \mathscr{A}\|$.

[1 ⇒ 6]. As in [1 ⇒ 5] we consider the closed convex hull of a set functionals vanishing on $\mathscr{A}'$.

Define

$$K = \{\omega_{\xi\eta} \mid \|\xi\|\,\|\eta\| \leqq 1 \quad \text{and} \quad \omega_{\xi\eta} \mid \mathscr{A}' = 0\}$$

and suppose z in the polar K°. For any non-trivial projection p in M and any two unit vectors ξ in pH and η in $(I-p)H$, $\omega_{\xi\eta}$ belongs to K and therefore $|(z\xi \mid \eta)| \leqq 1$, This shows that for any projection p in M we have $\|(I-p)zp\| \leqq 1$. Hence we get for projections p from M, that

$$\|[z, p]\|^2 = \|pz^*(I-p)zp + (I-p)z^*pz(I-p)\| \leqq 1 .$$

Since the self-adjoint projections are the extremal elements among the positive operators of norm at most one in M, we have proved that for all operators z in K° the norm $\|\mathrm{ad}\,(z) \mid M\|$ is less than or equal to 4.

The assumption 1) then implies that

$$K^\circ \subseteqq 4k(\mathscr{A}' + B(H)_1) ,$$

which by polarisation gives

$$((\mathscr{A}')^\perp)_1 \subseteqq 4k(K^{\circ\circ}) .$$

An approximation argument shows that any $c > 4k$ will do.

[6 ⇒ 1]. The method from [5 ⇒ 1] applies here too.

The theorem has some corollaries, which follow below. Although these results are partly presented in [5, 6, 7] we find it convenient to include them here too.

3.2 Corollary. *Suppose $\mathscr{A}$ is a C*-algebra on a Hilbert space H and that $\mathscr{A}$ for a positive k satisfies* 1) *in Theorem* 3.1.

To any derivation δ of $\mathscr{A}$ into $B(H)$ there exists an operator x in $B(H)$ such that

$$\delta = \mathrm{ad}\,(x) \mid \mathscr{A} \quad \textit{and} \quad \|x\| \leqq k\|\delta\| .$$

Proof. Since 1) is fulfilled, $H^1(\mathscr{A}, B(H)) = 0$ and there exists y in $B(H)$ such that $\delta = \mathrm{ad}\,(y) \mid \mathscr{A}$. By 1) and the weak compactness of the unitball in $\mathscr{A}'$ there exists a z in $\mathscr{A}'$ such that $\|y - z\| \leqq k\|\delta\|$. Define $x = y - z$ and x has the desired properties.

3.3. COROLLARY. *Let M be a von Neumann algebra on a Hilbert space H and let δ be a derivation of M into $B(H)$.*

1) *If M is injective then, there exists an operator x in $B(H)$ implementing δ such that $\|x\| \leqq \|\delta\|$.*
2) *If M is properly infinite, then x can be chosen such that $\|x\| \leqq \frac{3}{2}\|\delta\|$.*
3) *If M is finite of type II_1 and M is isomorphic to $M \bar{\otimes} R$, where R is the hyperfinite II_1 factor, then x can be found such that $\|x\| \leqq \frac{5}{2}\|\delta\|$.*

PROOF. By [9] and [11], M has property P of Schwartz, when M is injective and 1) follows from [5, Theorem 2.3].

If M is properly infinite we use [5, Theorem 2.4] to obtain the statement. The last statement follows from [5, Corollary 2.9].

4. An inequality for derivations.

In our first work on perturbations [3], the key lemma [3, Lemma 2.5] says that if Ψ is a linear map of a C*-algebra $\mathscr{A}$ into another C*-algebra $\mathscr{B}$, then for any finite family of pairwise orthogonal projections $(p_1, p_2, \ldots, p_n)$ in $\mathscr{A}$

$$\|\psi(p_1)^*\psi(p_1) + \Psi(p_2)^*\Psi(p_2) + \ldots + \Psi(p_n)^*\Psi(p_n)\| \leqq \|\Psi\|^2 .$$

It has been clear to us since 1975 that a more general inequality of this type, as presented in Theorem 4.1 below, is valid for all ultrastrongly continuous maps [6, pp. 239–240] and moreover, that such an inequality has to be useful in the study of linear maps on operatoralgebras.

In 1975 in [18] Ringrose discussed the problem whether ultraweakly continuous linear maps on operatoralgebras are automatically ultrastrong-star continuous, and he proved that it is so if and only if there exists a constant $H > 0$ such that any linear map φ of a C*-algebra $\mathscr{A}$ into another $\mathscr{B}$ satisfies

(*) $\forall a_1, \ldots, a_n \in \mathscr{A}$:

$$\left\| \sum_{i=1}^{n} [\varphi(a_i)^*\varphi(a_i) + \varphi(a_i)\varphi(a_i)^*] \right\| \leqq H^2\|\varphi\|^2 \left\| \sum_{i=1}^{n} [a_i^*a_i + a_ia_i^*] \right\| .$$

In 1976 Pisier proved in [16, Corollary 2.3], that (*) is valid and that $H^2 \leqq 6$.

We will now study how (*) yields the desired inequality for derivations on operator algebras.

4.1 THEOREM. *Let $\mathscr{B}$ be a C*-algebra, $\mathscr{A}$ a C*-subalgebra and δ a derivation of $\mathscr{A}$ into $\mathscr{B}$.*

For any finite set $a_1, \ldots, a_n$ in $\mathscr{A}$

$$\left\|\sum_{i=1}^{n} \delta(a_i)^*\delta(a_i)\right\| \leqq 14\|\delta\|^2\left\|\sum_{i=1}^{n} a_i^*a_i\right\| .$$

PROOF. Suppose first that δ is hermitian and also that $\mathscr{B}$ is represented faithfully as operators on a Hilbert space. Then it is possible [17] to extend δ to a derivation $\bar{\delta}$ of the ultraweak closed $\bar{\mathscr{A}}$ of $\mathscr{A}$ into the ultraweak closure $\bar{\mathscr{B}}$ of $\mathscr{B}$. Inside $\bar{\mathscr{A}}$ the elements a_i all have polardecompositions $a_i = v_i h_i$, with $h_i \geqq 0$, $h_i^2 = a_i^* a_i$ and v_i partial isometries. Hence $\delta(a_i) = v_i\bar{\delta}(h_i) + \bar{\delta}(v_i)h_i$, and

$$\begin{aligned}\sum_{i=1}^{n} \delta(a_i)^*\delta(a_i) &= \sum_{i=1}^{n} [\bar{\delta}(h_i)v_i^* + h_i\bar{\delta}(v_i)^*][v_i\bar{\delta}(h_i) + \bar{\delta}(v_i)h_i] \\ &= \sum_{i=1}^{n} [\bar{\delta}(h_i)v_i^*v_i\bar{\delta}(h_i) + h_i\bar{\delta}(v_i)^*\bar{\delta}(v_i)h_i \\ &\quad + \bar{\delta}(h_i)v_i^*\bar{\delta}(v_i)h_i + h_i\bar{\delta}(v_i)^*v_i\bar{\delta}(h_i)] .\end{aligned}$$

Since for arbitrary operators x and y

$$(x-y)^*(x-y) \geqq 0 .$$

We also have

$$x^*y + y^*x \leqq x^*x + y^*y ,$$

and then

$$\begin{aligned}(1)\qquad \sum_{i=1}^{n} \delta(a_i)^*\delta(a_i) &\leqq 2\sum_{i=1}^{n} [\bar{\delta}(h_i)v_i^*v_i\bar{\delta}(h_i) + h_i\bar{\delta}(v_i)^*\bar{\delta}(v_i)h_i] \\ &\leqq 2\sum_{i=1}^{n} [\bar{\delta}(h_i)\bar{\delta}(h_i) + \|\delta\|^2 h_i^2] .\end{aligned}$$

Since $h_i^2 = a_i^*a_i = h_i^*h_i = h_ih_i^*$ and $\bar{\delta}$ is hermitian we get from (1) and next from (*)

$$\begin{aligned}\left\|\sum_{i=1}^{n} \delta(a_i)^*\delta(a_i)\right\| &\leqq \left\|\sum_{i=1}^{n} [\bar{\delta}(h_i)^*\bar{\delta}(h_i) + \bar{\delta}(h_i)\bar{\delta}(h_i)^*]\right\| + 2\|\delta\|^2\left\|\sum_{i=1}^{n} a_i^*a_i\right\| \\ &\leqq 6\|\delta\|^2\left\|\sum_{i=1}^{n} 2a_i^*a_i\right\| + 2\|\delta\|^2\left\|\sum_{i=1}^{n} a_i^*a_i\right\| \\ &= 14\|\delta\|^2\left\|\sum_{i=1}^{n} a_i^*a_i\right\| .\end{aligned}$$

If δ is non hermitian, then the derivation Δ of $\mathscr{A}\otimes C_{C^2}$ into $\mathscr{B}\otimes M_2$ given by

$$\Delta\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 0 & \delta(a^*)^* \\ \delta(a) & 0 \end{pmatrix}$$

has the same norm as δ and is hermitian,

$$\left\|\begin{pmatrix} 0 & y \\ x & 0 \end{pmatrix}\right\|^2 = \left\|\begin{pmatrix} x^*x & 0 \\ 0 & y^*y \end{pmatrix}\right\| .$$

From above we then get for any set $(a_1,\ldots,a_n)$ in $\mathscr{A}$ that

$$\left\|\sum_{i=1}^n \delta(a_i)^*\delta(a_i)\right\| \leqq \left\|\sum_{i=1}^n \Delta\left(\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix}\right)^* \Delta\left(\begin{pmatrix} a_i & 0 \\ 0 & a_i \end{pmatrix}\right)\right\| \leqq 14\|\delta\|^2 \left\|\sum_{i=1}^n a_i^* a_i\right\| .$$

The theorem follows.

A linear ultraweakly continuous map between concrete C*-algebras $\mathscr{A}$ and $\mathscr{B}$ is ultrastrongly continuous if and only if, it satisfies an inequality similar to the one presented in Theorem 4.1. This was proved in [6], but we did not state the result in the form needed here. We therefore reformulate some arguments from [6].

4.2. PROPOSITION. *Let ϱ be an ultraweakly continuous linear map of a von Neumann algebra M into a von Neumann algebra N.*

Suppose that for any finite set $(m_1,\ldots,m_n)$

$$\left\|\sum_{i=1}^n \varrho(m_i)^*\varrho(m_i)\right\| \leqq \left\|\sum_{i=1}^n m_i^* m_i\right\| ,$$

then to any normal state φ on N there exists a normal state ψ on M such that for any m in N

$$\varphi(\varrho(m)^*\varrho(m)) \leqq \psi(m^*m) .$$

PROOF. As mentioned the proof follows the version given in [6, p. 240] of arguments from [18]. Let

$$S_1 = \{m \in M \mid \varphi(\varrho(m)^*\varrho(m)) = 1\}$$

and let S_2 be the convex hull of the set $\{m^*m \mid m \in S_1\}$, then for $x=\sum \lambda_i m_i^* m_i$ an element of S_2

$$\|x\| = \|\sum (\lambda_i^{\frac{1}{2}} m_i)^*(\lambda_i^{\frac{1}{2}} m_i)\| \geqq \|\sum \lambda_i \varrho(m_i)^*\varrho(m_i)\| \geqq 1 .$$

Since S_2 does not meet the open unit ball in M, there exists a hermitian functional ω with $\|\omega\| \leqq 1$ such that $\omega(x) \geqq 1$ for all x in S_2.

Since all elements in S_2 are positive, the positive part of ω will have the same properties so we may assume that ω is positive and satisfies

$$\forall m \in M: \varphi(\varrho(m)^*\varrho(m)) \leqq \omega(m^*m).$$

In [6, p. 240] it is shown that since φ is normal, the normal part [19] ω_n of ω also satisfies

$$\forall m \in M: \varphi(\varrho(m)^*\varrho(m)) \leqq \omega_n(m^*m).$$

Since $\|\omega_n\| \leqq 1$ one finds that the normal state $\psi = \|\omega_n\|^{-1}\omega_n$ has the property that $\forall m \in M$: $\varphi(\varrho(m)\varrho^*(m)) \leqq \psi(m^*m)$.

4.3 COROLLARY. *Let $\mathscr{A}$ be a C*-algebra on a Hilbert space H and δ be a derivation of $\mathscr{A}$ into $B(H)$. To any normal state φ on $B(H)$ there exists a normal state ψ on $B(H)$ such that*

$$\forall a \in \mathscr{A} \quad \varphi(\delta(a)^*\delta(a)) \leqq 14\|\delta\|^2\psi(a^*a).$$

PROOF. By [17] δ can be extended to a derivation $\bar{\delta}$ of $\bar{\mathscr{A}}$, the ultraweak closure of $\mathscr{A}$, into $B(H)$ with the same norm. The result then follows by combination of 4.1 and 4.2.

5. Sone consequences of the automatic ultrastrong continuity of derivations.

Even though we can not prove in general, that derivations on C*-algebras are completely bounded, we can through the ultrastrong continuity show this in a fair amount of the possible cases.

5.1 DEFINITION. Let δ be a derivation of a C*-algebra $\mathscr{A}$ on a Hilbert space H into $B(H)$. For any x in $\mathscr{A}'$, δx denotes the derivation of $\mathscr{A}$ into $B(H)$ given by $\delta x(a) = \delta(a)x$.

5.2. PROPOSITION. *Let $\mathscr{A}$ be a C*-algebra on a Hilbert space H and let δ be a derivation of $\mathscr{A}$ into $B(H)$.*

For any cyclic projection q in $\mathscr{A}'$ — that is $\exists\, \xi \in qH$ *such that* $qH = \overline{\mathscr{A}\xi}$ — *the derivation δq of $\mathscr{A}$ into $B(H)$ is completely bounded. Moreover for any natural number n, $(\delta q \otimes \mathrm{id})$: $\mathscr{A} \otimes M_n \to B(H) \otimes M_n$ has norm less than or equal to* $8\|\delta\|$.

PROOF. As usual we suppose that δ is defined on $\bar{\mathscr{A}}$ also. Let $\xi \in H$, $\underline{(a_{ij})} \in \mathscr{A} \otimes M_n$ and let $\Gamma = (b_1\xi, b_2\xi, \ldots, b_n\xi)$ be in $qH \oplus qH \oplus \ldots \oplus qH$ $(qH = \overline{A\xi})$ such that $b_i \in \mathscr{A}$ and $\sum \|b_i\xi\|^2 \leqq 1$. It is now enough to show that $\|(\delta(a_{ij}))\Gamma\| \leqq 8\|\delta\|$.

Define c as the positive squareroot of $\sum_{i=1}^n b_i^* b_i$, then there exist operators $(d_1, \ldots, d_n)$ in $\bar{\mathscr{A}}$ such that $b_i = d_i c$ and $\sum_{i=1}^n d_i^* d_i \leqq I$. One should remark that $\|c\xi\| \leqq 1$.

We will now estimate the norm $\|(\delta(a_{ij}))\Gamma\|$, by estimating $(\delta(a_{ij})\Gamma \mid \Omega)$ for vectors Ω of the form $(\omega_1,\ldots,\omega_n)$ with $\sum \|\omega_i\|^2 \leqq 1$. The derivation property yields

$$\sum_{ij} (\delta(a_{ij})d_j c\xi \mid \omega_i) = \sum_{ij} ((\delta(a_{ij}d_j) - a_{ij}\delta(d_j))c\xi \mid \omega_i) .$$

Now, we get when looking at $n \times n$ matrices

$$\left\|\sum_i \left(\sum_j d_j^* a_{ij}^*\right)\left(\sum_k a_{ik}d_k\right)\right\| =$$

$$\left\|\begin{bmatrix} d_1^* & \ldots & d_n^* \\ & & \\ & 0 & \end{bmatrix}(a_{ij})^*(a_{ij})\begin{bmatrix} d_1 & \\ \vdots & 0 \\ d_n & \end{bmatrix}\right\| \leqq \|(a_{ij})\|^2$$

so by Cauchy–Schwarz inequalities and Theorem 4.1 we obtain

$$\left|\sum_{ij} (\delta(a_{ij}d_j)c\xi \mid \omega_i)\right| \leqq \left(\sum_i \left\|\delta\left(\sum_j a_{ij}d_j\right)c\xi\right\|^2\right)^{\frac{1}{2}}\left(\sum_i \|\omega_i\|^2\right)^{\frac{1}{2}}$$

$$\leqq \sqrt{14}\|\delta\|\left\|\sum_i \left[\left(\sum_j d_j^* a_{ij}^*\right)\left(\sum_k a_{ik}d_k\right)\right]\right\|^{\frac{1}{2}}$$

$$\leqq \sqrt{14}\|\delta\|\;\|(a_{ij})\| .$$

The second term is measured in a similar way

$$\left|\sum_{ij} (a_{ij}\delta(d_j)c\xi \mid \omega_i)\right| = |((a_{ij})\Delta \mid \Omega)| ,$$

where

$$\Delta = (\delta(d_1)c\xi,\ldots,\delta(d_n)c\xi) .$$

By Theorem 4.1 we have

$$\|\Delta\|^2 = \sum \|\delta(d_i)c\xi\|^2 \leqq 14\|\delta\|^2 \|\sum d_i^* d_i\|\;\|c\xi\|^2 \leqq 14\|\delta\|^2 .$$

Hence

$$\left|\sum_{ij} (a_{ij}\delta(d_j)c\xi \mid \omega_i)\right| \leqq \sqrt{14}\|\delta\|\;\|(a_{ij})\| .$$

Finally we get since $2\sqrt{14} < 8$ that

$$\|(\delta q \otimes \mathrm{id})(a_{ij})\| \leqq 8\|\delta\|\;\|(a_{ij})\| ,$$

and the proposition follows.

The following theorem is now at hand.

5.3 THEOREM. *Let δ be a derivation of a C*-algebra $\mathscr{A}$ on a Hilbert space H into $B(H)$ and let q be a cyclic projection in $\mathscr{A}'$, then there exists an operator x in $B(H)$ such that*

$$\delta q = \mathrm{ad}\,(x)\,|\,\mathscr{A} \qquad \text{and} \qquad \|x\| \leqq 12\|\delta\| .$$

PROOF. Combine the proof of [4 ⇒ 2] in Theorem 3.1 with Corollary 3.3.

The following corollaries are also immediate.

5.4 COROLLARY. *Let $\mathscr{A}$ be a C*-algebra on a Hilbert space H. Suppose $\mathscr{A}$ has a cyclic vector, then:*

a) *For any operator x in $B(H)$, $d(x, \mathscr{A}') \leqq 12\|\mathrm{ad}\,(x)\,|\,\mathscr{A}\|$.*
b) *Any derivation δ of $\mathscr{A}$ into $B(H)$ is implemented by an operator x such that $\|x\| \leqq 12\|\delta\|$.*

5.5 COROLLARY. *Let M be a finite von Neumann algebra on a Hilbert space H. If M' is finite and the coupling function* [10, § 6,1 Definition 1] *is essentially bounded, say by the natural number n, then:*

a) *For every operator x in $B(H)$, $d(x, M') \leqq 25n\|\mathrm{ad}\,(x)\,|\,M\|$.*
b) *Any derivation δ of M into $B(H)$ is implemented by an operator x such that $\|x\| \leqq 25n\|\delta\|$.*

PROOF. Let x be in $B(H)$ and let Z denote the center of M. Since Z is injective there exists by Corollary 3.3 an operator y in $B(H)$ such that

$$\|y\| \leqq \|\mathrm{ad}\,(x)\,|\,Z\|, \qquad (x-y) \in Z', \qquad \text{and}$$
$$\|\mathrm{ad}\,(y)\,|\,M\| \leqq \|\mathrm{ad}\,(x)\,|\,M\| .$$

(See [5, Proof of Theorem 2.4].)
Let $z = x - y$, then $\|\mathrm{ad}\,(z)\,|\,M\| \leqq 2\|\mathrm{ad}\,(x)\,|\,M\|$ and $\mathrm{ad}\,(z)$ is trivial on Z.

Let $(E_\alpha)_{\alpha \in A}$ be a family of pairwise orthogonal central and σ-finite projections with sum I. By [10, III § 6,3 Proposition 5 and Proposition 6] it turns out that for any α in A, E_α is the sum of n cyclic projections from M'.

The Theorem 5.3 now yields that to any α there exists an operator v_α in $B(E_\alpha H)$ such that:

$$\mathrm{ad}\,(v_\alpha)\,|\,ME_\alpha = \mathrm{ad}\,(z)\,|\,ME_\alpha \qquad \text{and} \qquad \|v_\alpha\| \leqq 24n\|\mathrm{ad}\,x\,|\,M\| .$$

Let us then define $v = (\sum_A v_\alpha) + y$, then:

$$\mathrm{ad}\,(v)\,|\,M = \mathrm{ad}\,(x)\,|\,M \qquad \text{and} \qquad \|v\| \leqq 25n\|\mathrm{ad}\,x\,|\,M\| .$$

The corollary follows.

5.6 REMARKS. When combining Corollary 5.5 with Corollary 3.3 we find that the question whether $H^1(\mathscr{A}, B(H))$ vanishes or not is by now not linked to algebraic properties of $\mathscr{A}$ but rather to the particular representation of $\mathscr{A}$. To be more precise we can state that if M is any von Neumann algebra in standard position on a Hilbert space H, then $H^1(M, B(H)) = 0$.

It should also be remarked that a general positive answer to the question $H^1(M, B(H)) = 0$? will be very useful in the study of perturbations of operator algebras, and it will imply positive completions to the works [1] and [12].

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