MATH. SCAND. 50 (1982), 123–134

POWER FACTORIZATION IN BANACH MODULES OVER COMMUTATIVE RADICAL BANACH ALGEBRAS

NIELS GRØNBÆK

1. Introduction.

Let $\mathscr{A}$ be a Banach algebra with a bounded approximate identity and let X be a left Banach $\mathscr{A}$-module. Cohen's factorization theorem states that for each $x \in (\mathscr{A}.X)^-$ there are $a \in \mathscr{A}$ and $y \in X$ such that $x=a.y$. Furthermore we can obtain $\|a\| \leqq d$ and $\|x-y\| \leqq \varepsilon$ where d is the bound of the bounded approximate identity and $\varepsilon > 0$ is a given constant, see [3] and [4]. A number of generalizations have been achieved since Cohen's original result. We want to focus on a paper by G. R. Allan and A. M. Sinclair ([1]), where the authors proved a power factorization result $x=a^n.y_n$, $n \in \mathbf{N}$ with estimates of the norms of the factors. Our main theorem generalizes this result when A is commutative and radical by weakening the hypothesis that $\mathscr{A}$ has a bounded approximate identity.

To have a bounded appriximate identity has been the basic assumption on $\mathscr{A}$ in proofs of Cohen factorization results, i.e. one supposes that there is a constant $d>0$ and there exists a net (e_λ) bounded by d such that for all $a \in \mathscr{A}$ we have $\lim_\lambda e_\lambda a = \lim_\lambda a e_\lambda = a$. In [5] this was weakened by requiring only that the bounded net (e_λ) satisfies $\lim_\lambda e_\lambda e = \lim_\lambda e e_\lambda = e$ for all $e \in \{e_\lambda\}$. However, it has been pertinent that the net (e_λ) in addition to being an approximation of the identity on the Banach module X also satisfied some specified approximation property for elements in the Banach algebra $\mathscr{A}$ even though the factorization was to take place in the module.

When $\mathscr{A}$ is commutative and radical we are able to prove G. R. Allan's and A. M. Sinclair's power factorization theorem for modules with estimates of the norms of the factors without assuming that $\mathscr{A}$ has a bounded approximate identity. Our hypothesis is described in the following definition.

DEFINITION 1.1. Let $\mathscr{A}$ be a Banach algebra and let X be a left Banach $\mathscr{A}$-module. The algebra $\mathscr{A}$ is said to have a bounded approximate identity *for* X

Received September 10, 1980.

bounded by $d>0$ if for each finite set $x_1,\ldots,x_n \in X$ and for each $\varepsilon>0$ there is $e \in \mathscr{A}$, $\|e\| \leqq d$, such that $\|e.x_j - x_j\| \leqq \varepsilon$, $j=1,\ldots,n$.

Hence the usual meaning of "$\mathscr{A}$ has a bounded (left) approximate identity" is in this terminology "$\mathscr{A}$ has a bounded approximate identity for $\mathscr{A}$ when regarded as a (left) module over itself" or just "$\mathscr{A}$ has a bounded (left) approximate identity for itself".

Definition 1.1 is the weakest possible hypothesis if one wants to prove Cohen factorization results, i.e. if one wants to factorize finite sets in a (left) Banach $\mathscr{A}$-module X simultaneously with prescribed estimates of the norms of the factors. One sees readily that if X has Cohen factorization over $\mathscr{A}$, then $\mathscr{A}$ has a bounded approximate identity for X. It follows as a corollary of our main theorem that the converse holds if $\mathscr{A}$ is commutative and radical, thus showing that the connection between Cohen factorization in modules and bounded approximate identities for modules is deeper for commutative radical Banach algebras than for Banach algebras in general. An example will illustrate this (example 5.1). We exhibit a semisimple Banach algebra and a module over the algebra such that the algebra has a bounded approximate identity for the module but such that not every element in the module can be factored, the more impossible to do Cohen factorization.

G. R. Allan and A. M. Sinclair noted as a corollary of their power factorization theorem that if a radical Banach algebra $\mathscr{R}$ has a bounded approximate identity for itself, then there is arbitrarily slow decrease of powers in $\mathscr{R}$, i.e. for each sequence (α_n) of positive reals tending to zero there is $x \in \mathscr{R}$ such that

$$\lim \frac{\|x^n\|^{1/n}}{\alpha_n} = +\infty .$$

This extended an observation by J. K. Miziotek, T. Müldner, and A. Rek who showed that a radical Banach algebra with bounded approximate identity for itself cannot have $\|x^n\|^{1/n}$ tending uniformly to zero in the unit ball (Proposition 2.4 and Lemma 3.1 of [6]). The corollary of G. R. Allan and A. M. Sinclair can be frased as follows: Let $\mathscr{R}$ be a radical Banach algebra and view $\mathscr{R}$ as an algebra of operators on itself via the left regular representation. If there is a positive sequence (α_n) tending to zero such that for each r in $\mathscr{R}$

$$\liminf \frac{\|r^n\|^{1/n}}{\alpha_n} < +\infty ,$$

then no bounded net in $\mathscr{R}$ tends strongly to the identity operator on $\mathscr{R}$. We strengthen this proving that if $\mathscr{R}$ is a commutative radical Banach algebra of operators on a Banach space X for which there is a positive sequence (α_n) tending to zero such that

$$\liminf \frac{\|r^n\|^{1/n}}{\alpha_n} < +\infty \quad \text{for all } r \in \mathcal{R},$$

then no operator on X with a non-zero eigenvalue is a strong limit of a bounded net in $\mathcal{R}$.

In Section 2 we state the main theorem and discuss the method of proof. In Section 3 we prove the theorem. In Section 4 we give some applications and in Section 5 we give two examples, the one alluded to above showing that the theorem is not necessarily true when the algebra is semisimple and an example of a power factorization in a module over a radical commutative Banach algebra that does not have a bounded approximate identity for itself.

2. Statement of the theorem and method of proof.

THEOREM 2.1. *Let $\mathcal{R}$ be a commutative radical Banach algebra and let X be a left Banach $\mathcal{R}$-module. Assume that $\mathcal{R}$ has a bounded approximate identity for X bounded by $d>0$. Let (α_n) be a sequence diverging to infinity such that $\alpha_n>1$ for all $n \in \mathbb{N}$, let $x \in X$, and let $\delta>0$. Then there exist a sequence (y_n) in X, an element $r \in \mathcal{R}$, and a natural number $N \in \mathbb{N}$ such that*

(i) $x = r^j . y_j$, *for* $j = 1, 2, \ldots$
(ii) $\|r\| \leqq d$
(iii) $y_j \in (\mathcal{R}.x)^-$ *for* $j = 1, 2, \ldots$
(iv) $\|x - y_j\| \leqq \delta$ *for* $j = 1, 2, \ldots, N$
(v) $\|y_j\| \leqq \alpha_j^j \|x\|$ *for* $j = 1, 2, \ldots$

In proving this we shall follow the idea of G. R. Allan's and A. M. Sinclair's proof of the noncommutative version of their theorem. Their slicker proof in the commutative case does not work here even though we do assume that $\mathcal{R}$ is commutative, precisely because we are not supposing that $\mathcal{R}$ has a bounded approximate identity for itself. Let $\mathcal{R}^\#$ denote the algebra $\mathcal{R}$ with a unit adjoined. We shall construct by induction a sequence (b_n) in $\mathrm{Inv}(\mathcal{R}^\#)$ that converges to an element $r \in \mathcal{R}$ and such that $b_n^{-j}x$ is Cauchy in X for each fixed $j \in \mathbb{N}$ with limit y_j. As in [1] the control of growth of the sequence $(\|y_j\|)$ is obtained by considering a subsequence (α_{K_n}) of (α_n) that diverges to infinity fast and doing the construction for the jth power $b_n^{-j}x$ for j belonging to the interval $[K_n, K_{n+1}]$.

One more word about the construction may be in place. In Cohen's original proof a crucial step in the construction of b_{n+1} was to define a certain element

u_{n+1} from b_n and note that if $\|u_{n+1}-b_n\|$ is sufficiently small, then u_{n+1} is invertible and $\|u_{n+1}^{-1}-b_n^{-1}\|$ is also small. What will help us through this step is of course that if $\zeta \in \mathbf{C}\setminus(0)$ then $\zeta+r$ is invertible for all $r \in \mathscr{R}$ since $\mathscr{R}$ is radical.

3. Proof of Theorem 2.1.

In order to prove the theorem we need two approximation lemmas of which the first is standard in Cohen factorization proofs. Throughout $\mathscr{R}$ denotes a commutative radical Banach algebra and X is a Banach $\mathscr{R}$-module. The constant M is the bound of the module action, that is $\|r.x\| \leqq M\|r\|\,\|x\|$ for all $r \in \mathscr{R}$ and all $x \in X$. The number λ is chosen so that $0<\lambda<(d+1)^{-1}$ and $\lambda M \leqq 1$ and the number γ is defined as $\gamma=(1-\lambda-\lambda d)^{-1}$.

LEMMA 3.1. *Let $e \in \mathscr{R}$, $\|e\| \leqq d$, and let $f(e)=((1-\lambda)+\lambda e)^{-1}$. Then*

(i) $\|f(e)\| \leqq \dfrac{1}{1-\lambda-\lambda d} = \gamma$

(ii) *For all $k \in \mathbf{N}$ and for all $x \in X$*

$$\|f(e)^k.x-x\| \leqq \left(\sum_{j=1}^{k} \gamma^j\right)\|e.x-x\| .$$

PROOF.

$$\text{(i):}\quad \|f(e)\| = \left\|(1-\lambda)^{-1}\left(1+\frac{\lambda}{1-\lambda}e\right)^{-1}\right\|$$

$$\leqq (1-\lambda)^{-1}\sum_{k=0}^{\infty}\lambda^k(1-\lambda)^{-k}\|ek\|$$

$$= \frac{1}{1-\lambda-\lambda d}.$$

$$\text{(ii):}\quad \|f(e)^k.x-x\| = \left\|\sum_{j=1}^{k} f(e)^j(1-((1-\lambda)+\lambda e)).x\right\|$$

$$\leqq \lambda M \sum_{j=1}^{k}\gamma^j\|e.x-x\| .$$

LEMMA 3.2. *Let e and $f(e)$ be as above. Let $\mu>0$ and let $r \in \mathscr{R}$. Define $b=\mu-r$ and $E(e)=\mu-f(e)r$. Then there is a function $F\colon \mathbf{N} \to \mathbf{R}_+$ whose construction depends only on b such that*

(i) $F(j)^{1/j} \to 1$ *as* $j \to \infty$.

(ii) $\|(E(e)^{-j}-b^{-j}).x\| \leqq F(j)\|(1-e)b^{-j}.x\|$ *for all* $x \in X$ *and all* $j \in \mathbf{N}$.

PROOF. First we look at

$$\begin{aligned} E(e)^{-1}b &= \frac{1}{\mu}\sum_{k=0}^{\infty}\frac{1}{\mu^k}f(e)^k r^k(\mu-r) \\ &= \sum_{k=0}^{\infty}\frac{1}{\mu^k}f(e)^k r^k - \sum_{k=0}^{\infty}\frac{1}{\mu^{k+1}}f(e)^k r^{k+1} \\ &= 1+\sum_{k=1}^{\infty}\frac{1}{\mu^k}r^k(f(e)^k-f(e)^{k-1}), \end{aligned}$$

so that

$$E(e)^{-1}b = 1+\lambda\sum_{k=1}^{\infty}\left(\frac{1}{\mu}rf(e)\right)^k(1-e). \tag{1}$$

Using (1) and the binomial formula we get

$$\begin{aligned} E(e)^{-j}-b^{-j} &= ((E(e)^{-1}b)^j-1)b^{-j} \\ &= \left[\left(\sum_{i=0}^{j}\binom{j}{i}\lambda^i(1-e)^i\left\{\sum_{k=1}^{\infty}\left(\frac{1}{\mu}f(e)r\right)^k\right\}^i\right)-1\right]b^{-j} \\ &= \sum_{i=1}^{j}\binom{j}{i}\lambda^i(1-e)^i\left\{\sum_{k=1}^{\infty}\left(\frac{1}{\mu}f(e)r\right)^k\right\}^i b^{-j}, \end{aligned}$$

so that

$$E(e)^{-j}-b^{-j} = \lambda\left[\sum_{i=1}^{j}\binom{j}{i}\lambda^{i-1}(1-e)^{i-1}\left\{\sum_{k=1}^{\infty}\left(\frac{1}{\mu}f(e)r\right)^k\right\}^i\right](1-e)b^{-j}. \tag{2}$$

In order to finish we must estimate the expression in the bracket [...]. Using that $\|\lambda(1-e)\|<1$ and $\|f(e)\| \leqq \gamma$ we get

$$\begin{aligned} &\left\|\sum_{i=1}^{j}\binom{j}{i}\lambda^{i-1}(1-e)^{i-1}\left\{\sum_{k=1}^{\infty}\left(\frac{1}{\mu}f(e)r\right)^k\right\}^i\right\| \\ &\qquad \leqq \sum_{i=1}^{j}\binom{j}{i}\left\|\left(\sum_{k=1}^{\infty}\left(\frac{1}{\mu}f(e)r\right)^k\right)^i\right\| \\ &\qquad \leqq \sum_{i=0}^{j}\binom{j}{i}\sum_{k_1=1}^{\infty}\cdots\sum_{k_i=1}^{\infty}\left(\frac{\gamma}{\mu}\right)^{k_1+\ldots+k_i}\|r^{k_1+\ldots+k_i}\|. \end{aligned}$$

We estimate this expression by means of a weighted powerseries algebra in an indeterminate z. Let

$$\mathscr{P} = \left\{ \sum_{n=0}^{\infty} \lambda_n z^n \;\middle|\; \sum_{n=0}^{\infty} |\lambda_n| \, \|r^n\| \equiv \left\| \sum_{n=0}^{\infty} \lambda_n z^n \right\| < \infty \right\}.$$

Since r is quasinilpotent the algebra $\mathscr{P}$ is, with the usual operations on powerseries, a commutative radical Banach algebra with unit adjoined. Consider the element

$$p = \sum_{k=0}^{\infty} \left(\frac{\gamma}{\mu}\right)^k z^k .$$

We have

$$\begin{aligned} \|p^j\| &= \left\| \left(1 + \sum_{k=1}^{\infty} \left(\frac{\gamma}{\mu}\right)^k z^k \right)^j \right\| \\ &= \left\| \sum_{i=0}^{j} \binom{j}{i} \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} \left(\frac{\gamma}{\mu}\right)^{k_1+\ldots+k_i} z^{k_1+\ldots+k_i} \right\| \\ &= \sum_{i=0}^{j} \binom{j}{i} \sum_{k_1=1}^{\infty} \cdots \sum_{k_i=1}^{\infty} \left(\frac{\gamma}{\mu}\right)^{k_1+\ldots+k_i} \|r^{k_1+\ldots+k_i}\| \end{aligned}$$

where we have used that since the coefficients of p are positive we can interchange summation and taking norms freely (Lebesgue's monotone convergence theorem). If we put $F(j) = \|p^j\|$ and note that the spectral radius of p is 1 we are done, since then

$$\begin{aligned} \|(E(e)^{-j} - b^{-j}).x\| &\leqq \lambda M F(j) \|(1-e)b^{-j}.x\| \\ &\leqq F(j)\|(1-e)b^{-j}.x\| \end{aligned}$$

for all $x \in X$ and all $j \in \mathbf{N}$.

PROOF OF THEOREM 2.1. From here on we shall follow G. R. Allan's and A. M. Sinclair's construction. Without loss of generality we may assume that $\|x\| = 1$ and $\delta \leqq \min\{1, \alpha_n^n - 1 \mid n \in \mathbf{N}\}$. Choose a constant $C > 0$ so that $C^j \geqq M\gamma^j(2^j+1)$ for all $j \in \mathbf{N}$, choose $K_0 \geqq N$ so that $\alpha_j \geqq C+1$ for $j \geqq K_0$, and choose an increasing sequence $K_1, K_2, \ldots$ with $K_1 > K_0$ so that for all $n \in \mathbf{N}$, $\alpha_j \geqq C^n + 1$ for $j \geqq K_n$. These choices are possible since $\alpha_n > 1$ for all $n \in \mathbf{N}$ and $\alpha_n \to \infty$. Then we inductively choose a sequence $e_1, e_2, \ldots$ in $\mathscr{R}$ bounded by d such that if we define $e_0 = 0$ and

$$b_n = (1-\lambda)^n + \sum_{i=0}^{n} \lambda(1-\lambda)^{i-1} e_i$$

$$E(e_{n+1}) = (1-\lambda)^n + f(e_{n+1}) \sum_{i=0}^{n} \lambda(1-\lambda)^{i-1} e_i$$

we have

(I) $\|b_{n+1}^{-j}.x-b_n^{-j}x\| \leqq 2^{-(n+1)}\delta$ for all $n \in \mathbf{N}$ and $j \leqq K_{n+1}$,

(II) $\|E(e_{n+1})^{-j}.x-b_n^{-j}.x\| \leqq 2^j\|b_n^{-j}.x\|$ for all $n \in \mathbf{N}$ and all $j \in \mathbf{N}$.

Note that if we put $\mu=(1-\lambda)^n$ and $r=-\lambda\sum_{i=0}^n(1-\lambda)^{i-1}e_i$, Lemma 3.2 is applicable with $b=b_n$. The choice of e_1 is just Lemma 3.1. We have that $b_1^{-1} = f(e_1)$ and $E(e_1)=b_0=1$. Now suppose we have chosen $e_0,\ldots,e_n$. Let $\|e_{n+1}\| \leqq d$. Then

$$b_{n+1} = ((1-\lambda)+\lambda e_{n+1})E(e_{n+1}) ,$$

hence

$$\begin{aligned}\|b_{n+1}^{-j}.x-b_n^{-j}.x\| &= \|f(e_{n+1})^jE(e_{n+1})^{-j}.x-b_n^{-j}.x\|\\ &\leqq \|f(e_{n+1})^j(E(e_{n+1})^{-j}-b_n^{-j}).x\|\\ &\quad+\|(f(e_{n+1})^j-1)b_n^{-j}.x\| .\end{aligned}$$

Hence using Lemma 3.2 on the first term and Lemma 3.1 (ii) on the second term we see that if we choose e_{n+1} so that $\|e_{n+1}.x-x\|$ is sufficiently small we can get (I) satisfied.

By Lemma 3.2 there is $m \in \mathbf{N}$ depending only on b_n, so that for $j \geqq m$ we have

$$\begin{aligned}\|(E(e_{n+1})^{-j}-b_n^{-j}).x\| &\leqq F(j)\|(e_{n+1}-1)b_n^{-j}.x\|\\ &\leqq M(d+1)F(j)\|b_n^{-j}.x\|\\ &\leqq 2^j\|b_n^{-j}.x\| ,\end{aligned}$$

since $F(j)^{1/j} \to 1$ as $j \to \infty$. Choose $\|e_{n+1}.x-x\|$ so small that (I) is satisfied and such that also for $j \leqq m$ we have

$$\|(E(e_{n+1})^{-j}-b_n^{-j}).x\| \leqq 2^j\|b_n^{-j}.x\| .$$

This finishes the induction.

We now want to estimate $\|b_n^{-j}.x\|$. We have for all $n \in \mathbf{N}$ and all $j \in \mathbf{N}$ that

$$\begin{aligned}\|b_{n+1}^{-j}.x\| &= \|f(e_{n+1})^jE(e_{n+1})^{-j}.x\|\\ &\leqq \|f(e_{n+1})^j(E(e_{n+1})^{-j}-b_n^{-j}).x\|+\|f(e_{n+1})^jb_n^{-j}.x\|\\ &\leqq (M\gamma^j2^j+M\gamma^j)\|b_n^{-j}.x\| ,\end{aligned}$$

so that with the choice of $C>0$ we have

$$\|b_{n+1}^{-j}.x\| \leqq C^j\|b_n^{-j}.x\|$$

for all $j,n \in \mathbf{N}$. Using $b_0=1$ we get

(III) $\|b_n^{-j}.x\| \leqq C^{nj}$ for all $j,n \in \mathbf{N}$.

Clearly the sequence b_n is convergent. Denote the limit by r. Then

$$\|r\| = \lambda\left\|\sum_{i=1}^{\infty} (1-\lambda)^{i-1}e_i\right\|$$
$$\leqq \lambda d \frac{1}{1-(1-\lambda)} = d\,.$$

By (I) the sequence $(b_n^{-j}.x)$ is convergent for each $j \in \mathbf{N}$. Denote the limit by y_j. Since $x \in (\mathscr{R}.x)^-$, we get $y_j \in (\mathscr{R}.x)^-$ for all $j \in \mathbf{N}$. Let $j \leqq K_0$. Then

$$\|x-y_j\| = \left\|\sum_{n=0}^{\infty} b_{n+1}^{-j}.x - b_n^{-j}.x\right\|$$
$$\leqq \sum_{n=0}^{\infty} 2^{-(n+1)}\delta = \delta$$

and by the choice of δ

$$\|y_j\| \leqq \delta+1 \leqq \alpha_j^j\,.$$

If $K_n < j \leqq K_{n+1}$, we get

$$\|y_j\| = \left\|b_n^{-j}.x + \sum_{k=n}^{\infty} b_{k+1}^{-j}.x - b_k^{-j}.x\right\|$$
$$\leqq \|b_n^{-j}.x\| + \sum_{k=n}^{\infty} \|b_{k+1}^{-j}.x - b_k^{-j}.x\|$$
$$\leqq C^{nj}+\delta$$
$$\leqq (C^n+1)^j \leqq \alpha_j^j$$

by the choice of the sequence (K_n). Finally since $x = (b_n^j).(b_n^{-j}.x)$ for all $j, n \in \mathbf{N}$, we see that the power factorization holds.

4. Applications.

The first two corollaries are strengthenings of the corresponding observations in [1].

COROLLARY 4.1. *Let $\mathscr{R}$ be a commutative radical Banach algebra and suppose that $\mathscr{R}$ has a bounded approximate identity for some Banach $\mathscr{R}$-module X. Then there is arbitrary slow decrease of powers in $\mathscr{R}$*, i.e. *for each positive sequence (β_n) tending to zero there is $r \in \mathscr{R}$ such that $\|r^n\|^{1/n} \geqq \beta_n$ for all $n \in \mathbf{N}$.*

PROOF. Let $\alpha_n = \beta_n^{-1} \vee \frac{3}{2}$, let $\|x\| = 1$, and let $x = r^n . y_n$ be the power factorization from Theorem 2.1. Then for n sufficiently large we have

$$\|r^n\|^{1/n} \geqq \left(\frac{\|x\|}{\|y_n\|}\right)^{1/n} \geqq \beta_n .$$

By multiplying r with a suitable constant, we get $\|r^n\|^{1/n} \geqq \beta_n$ for all $n \in \mathbf{N}$.

COROLLARY 4.2. *Let $\mathscr{R}$ be a commutative Banach algebra of quasinilpotent operators on a Banach space X. Suppose there is a positive sequence (β_n) tending to zero such that*

$$\liminf \frac{\|r^n\|^{1/n}}{\beta_n} < +\infty \quad \textit{for all } r \in \mathscr{R} .$$

Let $T \in B(X)$ be the strong operator topology limit of a bounded net (r_λ) in $\mathscr{R}$. Then 0 is the only possible eigenvalue for T.

PROOF. Suppose $\varkappa$ is a nonzero eigenvalue for T and let $x \in X \setminus (0)$ be an eigenvector. Then $(1/\varkappa) r_\lambda . x \to x$, and so $\mathscr{R}$ has a bounded approximate identity for the Banach $\mathscr{R}$-module $Y = (\mathscr{R} . x)^-$. This contradicts Corollary 4.1.

EXAMPLE 4.3. Let $\mathscr{R} = C_*[0,1]$, the continuous functions on $[0,1]$ with the uniform norm and algebra product given by

$$(f * g)(t) = \int_0^t f(t-x) g(x)\, dx .$$

Let $e(t) = 1$ for all $t \in [0,1]$ and let $\|f\| \leqq 1$. Then

$$\|f^{*n}\| \leqq \|e^{*n}\| \leqq \frac{1}{(n-1)!} \quad \text{for all } n \in \mathbf{N} .$$

So $\|f^n\|^{1/n}$ tends uniformly to zero in the unit ball. Hence if X is a Banach $C_*[0,1]$-module we have for all bounded nets (f_λ) in $C_*[0,1]$ and all $x \in X \setminus (0)$ that $\liminf \|f_\lambda . x - x\| > 0$.

We shall now discuss the connection between Cohen factorization and bounded approximate identities for modules. First let us make the notion of Cohen factorization precise.

DEFINITION 4.4. Let $\mathscr{A}$ be a Banach algebra and let X be a left Banach $\mathscr{A}$-module. We shall say that X has Cohen factorization over $\mathscr{A}$ bounded by $d > 0$ if for each finite set $x_1, \ldots, x_n$ and for each $\varepsilon > 0$ there exists $a \in \mathscr{A}$ and $y_1, \ldots, y_n \in X$ such that

(i) $x_i = a.y_i$, $\quad i = 1, \dots, n$
(ii) $\|a\| \leqq d$,
(iii) $\|x_i - y_i\| \leqq \varepsilon$, $\quad i = 1, \dots, n$.

PROPOSITION 4.5. *If X has Cohen factorization over $\mathscr{A}$ bounded by d, then $\mathscr{A}$ has a bounded approximate identity for X bounded by d.*

PROOF. Let $x_1, \dots, x_n \in X$ and let $\delta > 0$. Choose $y_1, \dots, y_n \in X$ and $a \in \mathscr{A}$ such that (i), (ii), and (iii) in Definition 4.4 hold with $\varepsilon = \delta/dM$, where M is the boud of the module action. Then

$$\begin{aligned} \|x_i - a.x_i\| &= \|a.y_i - a.x_i\| \\ &\leqq M\|a\|\,\|y_i - x_i\| \\ &\leqq \delta \end{aligned}$$

for $i = 1, \dots, n$.

It is surprising that the converse holds for commutative radical Banach algebras. This follows readily from Theorem 2.1.

COROLLARY 4.6. *Let $\mathscr{R}$ be a commutative radical Banach algebra and let X be a (left) Banach $\mathscr{R}$-module. Then $\mathscr{R}$ has a bounded approximate identity for X bounded by d if and only if X has Cohen factorization over $\mathscr{R}$ bounded by d.*

PROOF. Let $x_1, \dots, x_n \in X$ and let $\varepsilon > 0$. Define $Y = X \times \dots \times X$ (n copies). Make Y a (left) Banach $\mathscr{R}$-module by defining

$$\|(\xi_1, \dots, \xi_n)\| = \max\{\|\xi_1\|, \dots, \|\xi_n\|\}$$

and $r.(\xi_1, \dots, \xi_n) = (r.\xi_1, \dots, r.\xi_n)$. The statement now follows from (i), (ii), and (iv) of Theorem 2.1.

It is clear that Corollary 4.6 is also true for a Banach algebra with a bounded left approximate identity for itself, since if $\mathscr{A}$ has a bounded approximate identity for a left Banach $\mathscr{A}$-module X, then $(\mathscr{A}.X)^- = X$ and a bounded left approximate identity for $\mathscr{A}$ will also be a bounded approximate identity for X so that the usual proof of Cohen's factorization theorem works. Example 5.1 below shows that Corollary 4.6 does not hold for all Banach algebras. It would be interesting to know exactly for which class of Banach algebras Corollary 4.6 is true.

5. Examples.

EXAMPLE 5.1. Let $\mathscr{A}=l^1(\mathsf{R}_+)$ and let $X=L^1(\mathsf{R}_+)$. Let δ_t, $t \in \mathsf{R}_+$, denote the characteristic function of the set $\{t\}$. If $f \in \mathscr{A}$ we can write

$$f = \sum_{t>0} \lambda_t\delta_t, \quad \sum_{t>0} |\lambda_t| = \|f\| .$$

We define a product on $\mathscr{A}$ by

$$\left(\sum_{t>0} \lambda_t\delta_t\right) * \left(\sum_{t>0} \mu_t\delta_t\right) = \sum_{t>0}\left(\sum_{s+u=t} \lambda_s\mu_u\right)\delta_t .$$

It is easily seen that with $*$ as product $\mathscr{A}$ becomes a commutative Banach algebra. If we view δ_t as the Dirac measure at $\{t\}$, this product is nothing but convolution of measures, and the norm on $l^1(\mathsf{R}_+)$ is just absolute value of measures. We make X a Banach $\mathscr{A}$-module by defining the module action to be convolution. Then $\mathscr{A}$ has a bounded approximate identity for X since for all $x \in X$ we have $\lim_{t\to 0}\|\delta_t * x - x\| = 0$.

Let now $x \in X$ be a function with $\alpha(x) \equiv \inf \operatorname{supp} x = 0$ and suppose we have a factorization $x = f * y$, $f \in l^1(\mathsf{R}_+)$, $y \in L^1(\mathsf{R}_+)$. Then clearly $\alpha(f)=0$ so that $f = \sum_{t>0}\lambda_t\delta_t$ and for each $\delta>0$ there is $t<\delta$ such that $\lambda_t \neq 0$. If we apply the Laplace transform we get $\mathscr{L}(x)=\mathscr{L}(f)\mathscr{L}(y)$. Now $\mathscr{L}(f)(\zeta)=\sum\lambda_t e^{-t\zeta}$, $\operatorname{Re}\zeta>0$ so $\mathscr{L}(f)(\zeta)$ is an almost periodic analytic function. By a result of H. Bohr (Satz 27 of [2]) $\mathscr{L}(f)$ has zeros in any halfplane $\{\operatorname{Re}\zeta>\eta\}$ forcing $\mathscr{L}(x)$ to have the same property. But clearly there exist functions $x \in L^1(\mathsf{R}_+)$ with $\alpha(x)=0$ and $\mathscr{L}(x)$ zero free. Take for instance $x(t)=e^{-t}$. Hence factorization is not always possible.

In the same way we see that if we have a power factorization $x = f^{*n} * y_n$, then $x \equiv 0$. If $\alpha(f)=B>0$, then $\alpha(x) \geqq nB$ for all $n \in \mathsf{N}$, so that $x=0$ and if $\alpha(f)=0$, then $\mathscr{L}(x)$ has a zero of infinite order and consequently is identically zero.

EXAMPLE 5.2. We now modify Example 5.1. Let $\omega(t)=e^{-t^2}$, $t>0$. Then $\omega(t+s) \leqq \omega(t)\omega(s)$ for all $s,t>0$. Define

$$\mathscr{R} = l^1(\mathsf{R}_+,\omega) \equiv \left\{(\lambda_t)_{t>0} \;\middle|\; \sum |\lambda_t|\omega(t) \equiv \|(\lambda_t)_{t>0}\| < \infty\right\}$$

and

$$X = L^1(\mathsf{R}_+,\omega) \equiv \left\{f \;\middle|\; f \text{ is Lebesgue measurable and } \int_{\mathsf{R}_+} |f|\omega \equiv \|f\| < \infty\right\}.$$

One checks easily that we can define convolution as before so that X is a Banach $\mathscr{R}$-module and $\mathscr{R}$ has a bounded approximate identity for X. Since

$$\|\delta_t^{*n}\|^{1/n} = \|\delta_{nt}\|^{1/n} = (e^{-(nt)^2})^{1/n} \to 0$$

as $n \to \infty$ for all $t>0$, $\mathscr{R}$ is a commutative radical Banach algebra. But $\mathscr{R}$ does not have a bounded approximate identity for itself. To see this suppose that (e_i) is a net in $\mathscr{R}$ bounded by $d>0$ such that for all $r \in \mathscr{R}$ we have $\lim_i \|e_i * r - r\| = 0$. By passing to a subnet if necessary we may assume that the net (e_i) is pointwise convergent to a function $e\colon \mathbf{R}_+ \to \mathbf{C}$. We claim that $e \in \mathscr{R}$. Let $F \subseteqq \mathbf{R}_+$ be a finite set. Then

$$\sum_{t \in F} |e(t)|\omega(t) = \lim_i \sum_{t \in F} |e_i(t)|\omega(t) \leqq d .$$

Since F was arbitrary it follows that $e \in \mathscr{R}$ and $\|e\| \leqq d$. Now for all $t>0$ the products $e_i * \delta_t$ converge to δ_t in norm and pointwise to $e * \delta_t$, so that e is an identity for $\mathscr{R}$. This is clearly impossible.

Using Theorem 2.1 we see that each $x \in L^1(\mathbf{R}_+, \omega)$ can be power factorized over $l^1(\mathbf{R}_+, \omega)$; $x = f^n * y_n$, $n \in \mathbf{N}$, with estimates of norms of the factors and from corollary 4.1 it follows that there is arbitrarily slow decrease to zero of $\|r^n\|^{1/n}$ in $\mathscr{R}$ even though $\mathscr{R}$ does not have a bounded approximate identity for itself.

REFERENCES

1. G. R. Allan and A. M. Sinclair, *Power factorization in Banach algebras with a bounded approximate identity*, Studia Math. 56 (1976), 31–38.
2. H. Bohr, *Zur Theorie der fastperiodischen Funktionen* III, Acta Math. 47 (1926), 237–281.
3. F. F. Bonsall and J. Duncan, *Complete normed algebras* (Ergebnisse der Mathematik und ihrer Grenzgebiete 80), Springer-Verlag, Berlin - Heidelberg - New York, 1973.
4. P. J. Cohen, *Factorization in group algebras*, Duke Math. J. 26 (1959), 199–205.
5. P. C. Curtis, Jr. and Figa-Talamanca, *Factorization theorems for Banach algebras* in *Function algebras* ed. F. T. Birtel, pp. 169–185, Scott Foresman and Co., Chicago, Illinois, 1966.
6. J. K. Miziotek, T. Müldner, and A. Rek, *On topologically nilpotent algebras*, Studia Math. 43 (1972), 41–50.

MATEMATISK INSTITUT
H. C. ØRSTED INSTITUTTET
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø
DENMARK