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AF ALGEBRAS WITH A LATTICE OF PROJECTIONS

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1. Introduction.

AF algebras are abundant in projections. It is therefore natural to ask in what AF algebras the projections form a lattice. We shall say that an AF algebra has the lattice property (l.p.) if its family of projection with the natural order is a lattice. Our main result states that an AF algebra has the l.p. if and only if it has the directed set property (d.s.p.) that is, its collection of finite dimensional *-subalgebras is directed by inclusion. That the d.s.p. implies the l.p. is obvious. We were not able to give a direct proof for the converse and had to use a characterization of the d.s.p. given in [8]. The paper contains also a characterization of the l.p. in terms of the non commutative topology developed in [1], [2], and [6]: the greatest lower bound of each pair of open projections is an open projection in the enveloping von Neumann algebras of a unital AF algebra A if and only if A has the l.p.

The l.p. is not preserved by tensoring with LC (H), the algebra of all compact operators on the Hilbert space H. Also it may happen that the dimension group of an AF algebra $A, K_0(A)$, is lattice ordered but A does not have the l.p. This is the case for the fermion algebra, for instance, which, according to Proposition 2.4, does not have the l.p.

Most of our notation is standard. For a C*-algebra A we denote by $\mathscr{P}(A)$ the set of all projections in A. If $p,q \in \mathscr{P}(A)$ we denote by $p \wedge q$ and $p \vee q$ their greatest lower bound and least upper bound in $\mathscr{P}(A)$, respectively, whenever they exist. The space of a representation π is denoted H_π. A'' denotes the enveloping von Neumann algebra of A and if π is a representation of A we shall use π'' to denote its normal extension to A''. The C*-algebra of all the sequences of 2×2 complex matrices which converge to matrices of the form

$$\begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}$$

will be named $\mathscr{A}$ throughout the paper. Another special notation is the following:

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$$\alpha_n = 2^{-1}\{1+[1-4(n+1)^{-2}]^{\frac{1}{2}}\}\,,$$
$$\beta_n = (n+1)^{-1}\,,$$
$$\gamma_n = 2^{-1}\{1-[1-4(n+1)^{-2}]^{\frac{1}{2}}\}\,,$$

n being a natural number.

We are grateful to Professor L. Brown for suggesting to us to explore the possibility of a connection between the l.p. and the d.s.p. Credit must be given to the referee for the proofs of Lemma 2.1 and Proposition 2.4 and many other improvements to the manuscript.

2. Primitive quotients of AF algebras with the l.p.

In this section we shall describe the images of the unital AF algebras with the l.p. by irreducible representations. First, we shall single out some classes of AF algebras without the l.p.

LEMMA 2.1. *A separable* C*-*algebra A which contains a* C*-*subalgebra* *-*isomorphic to $\mathscr{A}$ does not have the* l.p.

PROOF. To simplify the notation we shall suppose that $\mathscr{A}$ is a C*-subalgebra of A. We assume that A has the l.p. and argue by contradiction.

Put

$$p_n = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad n=1,2,3,\dots\,,$$

$p=\{p_n\}_{n=1}^{\infty} \in \mathscr{A}$. For each set of natural numbers E let $p_E=\{(p_E)_n\}_{n=1}^{\infty} \in \mathscr{A}$ be given by

$$(p_E)_n = \begin{cases} p_n, & n \in E\,, \\ \begin{pmatrix} \alpha_n & \beta_n \\ \beta_n & \gamma_n \end{pmatrix}, & n \notin E\,. \end{cases}$$

Define also $r_n=\{r_{nm}\}_{m=1}^{\infty} \in \mathscr{A}$ by

$$r_{nm} = \begin{cases} 0, & m \neq n\,, \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, & m=n\,. \end{cases}$$

For any projection p, q and r, if r commutes with p and q, then r commutes also with $p \wedge q$ and $r(p \wedge q)=rp \wedge rq$. (Consider the automorphism Ad $(1-2r)$ and use the uniqueness of $p \wedge q$.) Hence, in particular,

$$(p \wedge q_E)r_n = \begin{cases} r_n, & n \in E \\ 0, & n \notin E . \end{cases}$$

Therefore, if E_1, E_2 are sets of natural numbers and $n \in E_1 \setminus E_2$,

$$\|p \wedge p_{E_1} - p \wedge p_{E_2}\| \geqq \|(p \wedge p_{E_1} - p \wedge p_{E_2})r_n\| = \|r_n\| = 1 .$$

The famoly of projections $\{P \wedge P_E\}$ is uncountable, so we obtained a contradiction to the separability of A.

In the proof of the next lemma we shall use the terminology and the notation introduced in section 2 of [9].

LEMMA 2.2. *If an* AF *algebra has a quotient *-isomorphic to* $\mathscr{A}$ *then it has a* C*-*subalgebra *-isomorphic to* $\mathscr{A}$.

PROOF. Suppose that the AF algebra A has a quotient *-isomorphic to $\mathscr{A}$. It is easily seen that this quotient has a diagram $(D', d'. U')$ as illustrated graphically by Figure 1.

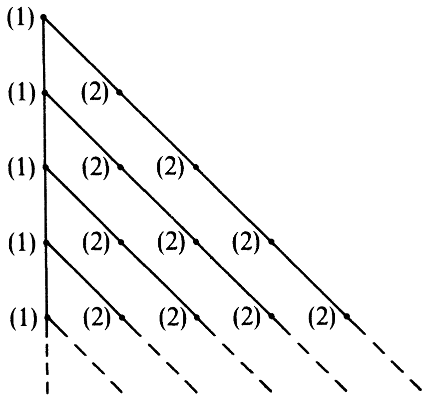

Figure 1.

Let (D, d, U) be a diagram of A. It has an ideal subdiagram $(E, d \mid E, U_E)$ such that $(D \setminus E, d \mid D \setminus E, U_{D \setminus E})$ is equivalent to (D', d', U'). Thus A contains sequences of projections $\{e_n\}_{n=1}^{\infty}$, $\{f_n^1\}_{n=2}^{\infty}$, $\{f_n^2\}_{n=2}^{\infty}$ and a sequence of partial isometries $\{u_n\}_{n=2}^{\infty}$ such that:

$$e_n \geqq f_{n+1}^1,\ e_{n+1} = e_n - f_{n+1}^1,\ u_{n+1}^* u_{n+1} = f_{n+1}^1 ,$$

$$u_{n+1} u_{n+1}^* = f_{n+1}^2,\ f_{n+1}^2\left(e_1 + \sum_{k=2}^{n} f_k^2\right) = 0$$

for every natural number n. Let F_n be the four dimensional C*-subalgebra of A generated by $\{f_n^1, u_n\}$ and put $B_n = \{\lambda e_n : \lambda \in \mathbf{C}\} \oplus F_n$, $n = 2, 3, \ldots$. Clearly

$$B_n \subset B_{n+1} \quad \text{and} \quad B = \overline{\bigcup_{n=2}^{\infty} B_n}$$

is a C*-subalgebra of A *-isomorphic to $\mathscr{A}$.

LEMMA 2.3. *Let A be an* AF *algebra and B a quotient of A which contains a* C*-*subalgebra *-isomorphic to $\mathscr{A}$. Then A does not have the* l.p.

PROOF. Suppose B_1 is a C*-subalgebra of B *-isomorphic to $\mathscr{A}$. By [7, Theorem 2.4] there is an AF subalgebra A_1 of A whose image by the quotient map is B_1. Lemma 2.2 yields a C*-subalgebra of A_1 which is *-isomorphic to $\mathscr{A}$ and the conclusion follows from Lemma 2.1.

PROPOSITION 2.4. *An* AF *algebra with the* l.p. *is postliminal.*

PROOF. Let A be a non-postliminal AF algebra; we shall show that A does not have the l.p. Then

$$A = \overline{\bigcup_{n=1}^{\infty} A_n},$$

where $\{A_n\}_{n=1}^{\infty}$ is an increasing sequence of finite dimensional *-subalgebras. By [7, Theorem 1.1], A contains an AF subalgebra with unit which has a diagram represented by Figure 2.

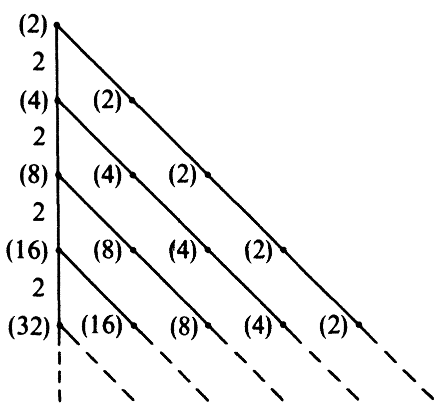

Figure 2.

By inspection one sees that this diagram contains row by row the diagram for $\mathscr{A}$ given in Figure 1. Hence by Lemma 2.1 A does not have the l.p.

As in [8], we can now prove

PROPOSITION 2.5. *If π is an infinite dimensional irreducible representation of a unital* AF *algebra A with the* l.p. *then $\pi(A) = \tilde{\text{LC}}\,(H_\pi)$. Each primitive ideal of A is contained in precisely one maximal two-sided ideal. A maximal two-sided ideal which properly contains a primitive ideal has codimension one.*

PROOF. Proposition 2.4 implies $\pi(A) \supset \tilde{\text{LC}}\,(H_\pi)$. If $\pi(A) \neq \tilde{\text{LC}}\,(H_\pi)$, then there is a projection in $\pi(A)$ not contained in $\tilde{\text{LC}}\,(H_\pi)$. It is readily seen then that $\pi(A)$ must contain a C*-subalgebra *-isomorphic to $\mathscr{A}$, a contradiction to the conclusion of Lemma 2.3. The other assertions of the proposition are immediate consequences of the first.

3. The equivalence of the l.p. and the d.s.p.

Most of the proofs of this section follow closely those of section 3 in [8]. To avoid repetition the reader will be often referred to [8] for details.

LEMMA 3.1. *Let A be a unital* AF *algebra with the* l.p. *The set of all primitive ideals which are contained in a given maximal two-sided ideal of A is closed in* Prim (A). *Those which are properly contained form a discrete set in the relative topology.*

PROOF. Let M be a maximal two-sided ideal of A; if no primitive ideal is properly contained in M, then there is nothing to prove. Suppose $\{P_i\colon i \in \mathscr{I}\}$ is the collection of all primitive ideals properly contained by M. Let $P = \bigcap_{i \in I} P_i$, $B = A/P$, $N = M/P$, and $Q_i = P_i/P$.

Choose an open dense subset U of Prim (N) = Prim $(B) \setminus \{N\}$ which is Hausdorff in tis relative topology (cf. [10, Theorem 6.2.11]). Clearly U is open and dense in Prim (B) and $\{Q_i\colon Q_i \in U\}$ is dense in Prim (B). There is a two-sided ideal Q of B such that Prim (Q) can be identified with U.

We claim that each $Q_i \in U$ is isolated in U. Assume not. Then there is $Q_{i_0} \in U$ and a sequence of distinct points $\{Q_{i_n}\}_{n=1}^{\infty} \subset U$ which converges to Q_{i_0}. Reasoning as in the proof of [8, Lemma 3.1], we can find in $Q/\bigcap_{i=0}^{\infty} (Q_{i_n} \cap Q)$ a C*-subalgebra *-isomorphic to $\mathscr{A}$. Thus B (hence A too) has a quotient which contains a C*-subalgebra *-isomorphic to $\mathscr{A}$. According to Lemma 2.3 this is impossible.

Now let π_i be the irreducible representation of B whose kernel is Q_i and define the representation π of B by

$$\pi(x) = \sum_{Q_i \in U} \oplus \, \pi_i(x), \qquad x \in B .$$

Since $\{Q_i: Q_i \in U\}$ is dense in Prim B, π is a *-isomorphism. Proposition 2.5 and the fact that each $Q_i \in U$ is isolated in U imply that $\pi(N)$ contains the restricted sum $\sum\oplus_{Q_i \in U} \mathrm{LC}(H_{\pi_i})$. If $\pi(N) \neq \sum\oplus_{Q_i \in U} \mathrm{LC}(H_{\pi_i})$, then there is $p \in \mathscr{P}(N)$ such that $\pi_i(p) \neq 0$ for infinitely many $Q_i \in U$. This leads to the existence in $\pi(N)$ of a C*-subalgebra *-isomorphic to $\mathscr{A}$, again a contradiction to the conclusion of Lemma 2.3. Thus $\pi(N) = \sum\oplus_{Q_i \in U} \mathrm{LC}(H_{\pi_i})$ hence B is *-isomorphic to the C*-algebra obtained by adjoining a unit to $\sum\oplus_{Q_i \in U} \mathrm{LC}(H_{\pi_i})$. Since each H_{π_i} is infinite dimensional, this C*-algebra has a unique maximal two-sided ideal and the primitive ideals which are not maximal form a discrete set. Now, B is a quotient of A so the conclusions of the lemma follow immediately.

LEMMA 3.2. *Let π be an irreducible representation of a unital* AF *algebra with the* l.p. *If* $\dim H_\pi > 1$, *then π is isolated in $\hat{A}$.*

PROOF. Suppose $\dim H_\pi > 1$ but π is not isolated in $\hat{A}$. By using the previous lemma one can construct as in the proof of [8, Lemma 3.3] a sequence $\{\pi_n\}_{n=1}^\infty$ such that π is a limit of it and each $\{\pi_m\}$ is open in $\overline{\{\pi_n\}_{n=1}^\infty}$. There is a quotient B of A whose spectrum can be identified with $\overline{\{\pi_n\}_{n=1}^\infty}$. Let $\varrho_n, \varrho \in \hat{B}$ satisfy $\varphi \circ \varrho_n = \pi_n$, $\varphi \circ \varrho = \pi$, where φ is the quotient map of A onto B. There is $p \in \mathscr{P}(B)$ such that $\varrho(p) \neq 0 \neq \varrho(e-p)$, e being the unit of B. By [10, Proposition 4.4.4] we have eventually $\varrho_n(p) \neq 0 \neq \varrho_n(e-p)$. In order to keep the notation simple we shall suppose that this is true for every natural number n. The open and discrete set $\{\varrho_n\}_{n=1}^\infty$ is the spectrum of some ideal I of B. Clearly there are $p_n, q_n, u_n \in I$ such that $\varrho_n(p_n)$, $\varrho_n(q_n)$ are one dimensional projections $\varrho_n(p_n) \leqq \varrho_n(p)$, $\varrho_n(q_n) \leqq \varrho_n(e-p)$, $\varrho_n(u_n)\varrho_n(u_n^*) = \varrho_n(p_n)$, $\varrho_n(u_n^*)\varrho_n(u_n) = \varrho_n(q_n)$ and $\varrho_m(p_n) = \varrho_m(q_n) = \varrho_m(u_n) = 0$ for every $m \neq n$. It is then readily seen that the C*-subalgebra of B generated by p and $\{p_n, q_n, u_n\}_{n=1}^\infty$ is *-isomorphic to $\mathscr{A}$, a contradiction.

LEMMA 3.3. *Let A be a unital* AF *algebra with the* l.p. *Then* $\{\pi \in \hat{A}: \dim H_\pi < \infty\}$ *is Hausdorff in its relative topology. If for $\pi \in \hat{A}$ one defines $\varphi(\pi)$ to be that irreducible representation of A whose kernel is the unique maximal two-sided ideal containing $\pi^{-1}(0)$, then φ is a continuous map of $\hat{A}$ onto* $\{\pi \in \hat{A}: \dim H_\pi < \infty\}$.

PROOF. See the proofs of Lemmas 3.4, 3.5, and 3.6 in [8].

THEOREM 3.4. *An* AF *algebra has the* l.p. *if and only if it has the* d.s.p.

PROOF. Suppose the AF algebra A has the l.p. and let A_1, A_2 be two finite dimensional *-subalgebras of A with units p_1, p_2, respectively. Put $p=p_1 \vee p_2$. Then pAp is a unital AF subalgebra of A (cf. [5, Theorem 3.1]) which, obviously, has the l.p. Clearly $A_1 \subset pAp$, $A_2 \subset pAp$. It follows from Proposition 2.5, Lemma 3.2, Lemma 3.3, and [8, Theorem 3.7] that pAp has the d.s.p. Thus, there is a finite dimensional *-subalgebra of pAp which contains A_1 and A_2. This proves that A has the d.s.p. The converse is obvious.

COROLLARY 3.5. *An* AF *algebra A has the* l.p. *if and only if $\tilde{A}$ has the* l.p. *If an* AF *algebra has the* l.p. *then all its quotients and* AF *subalgebras have the* l.p.

PROOF. If A has the l.p. then $\tilde{A}$ has the l.p. by [8, Proposition 2.1] and the above theorem. The other assertions are immediate for AF algebras with the d.s.p.

4. Non-commutative topology and the l.p.

For the next two lemmas A will be an AF algebra with the l.p. having e as unit. We let $E=\{\pi \in \hat{A}\colon \dim H_\pi < \infty\}$.

LEMMA 4.1. *Let $p \in \mathscr{P}(A)$. Then*

(i) *$\{\pi \in E\colon \pi(p) \neq 0\}$ is open and closed in E;*

(ii) *if $\pi(p) \neq 0$ for $\pi \in \hat{A}$ with $\dim H_\pi = 1$, then there is a neighbourhood U of π in $\hat{A}$ such that $\varrho(p)=\varrho(e)$ for all $\varrho \in U$ except possible finitely many elements of U whose kernels are contained in the kernel of π;*

(iii) *The set of all $\pi \in \hat{A}$ with $\dim H_\pi = 1$, $\pi(p)=0$ for which there is some $\varrho \in \hat{A}$ satisfying $\varrho^{-1}(0) \subset \pi^{-1}(0)$ and $\varrho(p) \neq 0$ is finite.*

PROOF. The first statement follows from [10, Proposition 4.4.4] and [8, Theorem 3.7]. Suppose now $\pi \in \hat{A}$, $\dim H_\pi = 1$ and $\pi(p) \neq 0$. We have then $\pi(e-p)=0$, so by [8, Lemma 3.4] and [4, 3.9.4] there is a neighbourhood U' of π in

$$\hat{A} \setminus \{\varrho \in \hat{A}\colon \ \varrho^{-1}(0) \subset \pi^{-1}(0), \quad \varrho \neq \pi\}$$

such that $\sigma(e-p)=0$ for every $\sigma \in U'$. Lemma 3.2 and the proof of Lemma 3.1 show that

$$U = U' \cup \{\varrho \in \hat{A}\colon \ \varrho^{-1}(0) \subset \pi^{-1}(0)\}$$

has the required properties so (ii) is established.

The set $E'=\{\pi \in \hat{A}\colon \dim H_\pi = 1, \ \pi(p)=0\}$ is closed by [10, Proposition

4.4.10], Lemma 3.2 and (i). As in the proof of (ii) we can find for each $\pi \in E'$ a neighbourhood U_π such that $\varrho(p)=0$ for every $\varrho \in U_\pi$ except possibly finitely many irreducible representations whose kernels are contained in the kernel of π. Then open covering $\{U_\pi: \pi \in E'\}$ has a finite sub-covering $\{U_{\pi_i}\}_{i=1}^n$. Suppose $\pi \in E'$, $\pi \notin \{\pi_i\}_{i=1}^n$. We claim that if $\varrho \in \hat{A}$, $\varrho^{-1}(0) \subset \pi^{-1}(0)$, then $\varrho(p)=0$. Indeed $\pi \in U_{\pi_i}$ for some i hence $\varrho \in U_{\pi_i}$. Since the kernel of π_i does not include the kernel of ϱ we must have $\varrho(p)=0$. It follows that if $\pi \in E'$, and there is $\varrho \in \hat{A}$ with $\varrho^{-1}(0) \subset \pi^{-1}(0)$ and $\varrho(p) \neq 0$, then $\pi \in \{\pi_i\}_{i=1}^n$.

The notions of Borel operator and open projection which we shall use below are defined in [10, 3.11.10, and 4.5.6], respectively.

LEMMA 4.2. *Let $p \in \mathscr{P}(A'')$ be a Borel operator. Then p is open if and only if the following are satisfied:*

(i) $\{\pi \in E: \pi''(p) \neq 0\}$ *is open in* E;

(ii) *for $\pi \in \hat{A}$ with* $\dim H_\pi = 1$, $\pi''(p) \neq 0$, *there is a neighbourhood U_π of π in $\hat{A}$ such that $\varrho''(p)=\varrho(e)$ for each $\varrho \in U_\pi$ except possibly finitely many irreducible representations all whose kernels are contained in the kernel of π; the image of p by such an exceptional representation is a projection of finite codimension*;

(iii) *The set of all $\pi \in \hat{A}$ with* $\dim H_\pi = 1$, $\pi''(p)=0$ *for which there is $\varrho \in \hat{A}$ with $\varrho^{-1}(0) \subset \pi^{-1}(0)$, $\varrho''(p) \neq 0$ is countable.*

PROOF. Let p be an open projection in A''. Then the hereditary C*-subalgebra $(pA''p) \cap A$ of A has an increasing approximate unit $\{p_n\}_{n=1}^\infty$ consisting of projections by [5, Theorem 3.1]. The sequence $\{p_n\}_{n=1}^\infty$ strongly converges to p by the proof of [10, Proposition 3.11.9].

Clearly (i) follows immediately from

$$\{\pi \in E: \pi''(p) \neq 0\} = \bigcup_{n=1}^{\infty} \{\pi \in E: \pi(p_n) \neq 0\}$$

and Lemma 4.1 (i).

Suppose $\pi \in \hat{A}$, $\dim H_\pi = 1$, $\pi''(p) \neq 0$. Then $\pi(p_n) \neq 0$ for some n. The needed neighbourhood U_π is the neighbourhood given by Lemma 4.1 (ii) for p_n. From $\varrho(p) \geqq \varrho(p_n)$ for every $\varrho \in \hat{A}$ one obtains (ii).

Next, if $\pi \in \hat{A}$, $\dim H_\pi = 1$, $\pi''(p)=0$ and there is $\varrho \in \hat{A}$ with $\varrho^{-1}(0) \subset \pi^{-1}(0)$, $\varrho''(p) \neq 0$, then $\varrho(p_n) \neq 0$ for some n. Thus (iii) is a consequence of Lemma 4.1 (iii).

Suppose now that p is a Borel projection in A'' which fulfills the conditions (i)–(iii) of the lemma. Let $\{\pi_1, \pi_2, \ldots, \pi_k, \ldots\}$ be the set described in (iii). Let also

$$\{\varrho \in \hat{A}: \varrho^{-1}(0) \subset \pi_k^{-1}(0), \varrho \neq \pi_k\} = \{\varrho_{k,1}, \varrho_{k,2}, \ldots, \varrho_{k,i}, \ldots\}.$$

For each i and k choose an increasing sequence $\{r_{k,i,m}\}_{m=1}^{\infty}$ of finite dimensional projections in the space of the representation $\varrho_{k,i}$ which strongly converges to $\varrho_{k,i}''(p)$.

The topology of E has a basis of closed and open subsets by [3, Theorem 4.1], Lemma 3.2, and Lemma 3.3. Thus

$$\{\pi \in E:\ \pi''(p) \neq 0\} = \bigcup_{n=1}^{\infty} E_n ,$$

where $\{E_n\}_{n=1}^{\infty}$ is an increasing sequence of relatively open and closed subsets of E. For each natural number n we shall define f_n: $\hat{A} \to \bigcup\{\varrho(A): \varrho \in \hat{A}\}$ as described below. Let $\varrho \in \hat{A}$. If there is $\pi \in E_n$ with $\varrho^{-1}(0) \subset \pi^{-1}(0)$, then $f_n(\varrho) = \varrho''(p)$. If there is $\pi \in E \setminus E_n$, $\pi \notin \{\pi_1, \pi_2, \ldots\}$ with $\varrho^{-1}(0) \subset \pi^{-1}(0)$, then $f_n(\varrho) = 0$. Put $f_n(\pi_k) = 0$ for every k and $f_n(\varrho_{k,i}) = r_{k,i,n}$ if $k, i \leqq n$, $f_n(\varrho_{k,i}) = 0$ otherwise.

It is now tedious but easy to verify that Theorem 4.15 of [8] yields a projection $p_n \in A$ such that $f_n(\varrho) = \varrho(p_n)$ for each $\varrho \in \hat{A}$. The increasing sequence of projections $\{p_n\}_{n=1}^{\infty}$ converges strongly to some Borel projection in A''. Since $\{\varrho(p_n)\}_{n=1}^{\infty}$ converges strongly to $\varrho(p)$ for every $\varrho \in \hat{A}$ and the atomic representation of A is faithful on the enveloping Borel *-algebra of A (cf. [10, Corollary 4.5.13]), the strong limit of $\{p_n\}_{n=1}^{\infty}$ is p. Thus p is an open projection.

THEOREM 4.3. *Let A be an* AF *algebra with unit e. A has the* l.p. *if and only if for every two open projections $p, q \in A''$ their greatest lower bound, $p \wedge q$, is open too.*

PROOF. Suppose that A has the l.p. and $p, q \in A''$ are open projections. Remark first that $p \wedge q$ is a Borel projection in A'' by [10, Proposition 4.5.7] and

$$\pi''(p \wedge q) = \pi''(p) \wedge \pi''(q), \qquad \pi \in \hat{A} , \tag{1}$$

by functional calculus.

Let $\pi \in \hat{A}$, $\dim H_\pi = 1$. If $\pi''(p \wedge q) \neq 0$, then $\pi''(p) = \pi''(q) = \pi(e)$. By lemma 4.2 (ii), there is a neighbourhood U_π of π in $\hat{A}$ such that $\varrho''(p) = \varrho''(q) = \varrho(e)$ for every $\varrho \in U_\pi$ except possibly finitely many irreducible representations σ with $\sigma^{-1}(0) \subset \pi^{-1}(0)$; for these representations $\sigma''(p)$ and $\varrho''(q)$ are projections of finite codimension in H_σ. This together with (1) and Lemma 3.2 imply that $\{\pi \in E: \pi''(p \wedge q) \neq 0\}$ is relatively open in E. We also see that condition (ii) of Lemma 4.2 is satisfied by $p \wedge q$.

If $\pi \in \hat{A}$ with $\dim H_\pi = 1$ satisfies $\pi''(p)\pi''(q) = \pi''(p \wedge q) = 0$ and there is $\varrho \in \hat{A}$ with $\varrho^{-1}(0) \subset \pi^{-1}(0)$ and $\varrho''(p \wedge q) \neq 0$, then $\pi''(p) = 0$ or $\pi''(q) = 0$. Clearly $\varrho''(p) \neq 0 \neq \varrho''(q)$, whenever $\varrho''(p \wedge q) \neq 0$, so the condition (iii) of the previous lemma is fulfilled by $p \wedge q$. We may conclude that $p \wedge q$ is an open projection.

Suppose now that $p \wedge q$ is an open projection whenever $p, q \in A''$ are open projections. Let $p, q \in \mathscr{P}(A)$. Their greatest lower bound in A'', $p \wedge q$, is open by our assumption and closed by [1, Proposition II.5]. Thus $p \wedge q \in A$ by [1, Proposition II.18]. Since A is unital, it follows that $\mathscr{P}(A)$ is a lattice.

REMARK. The conclusion of the above theorem is valid for non unital AF algebras as well. Indeed, by passing to $\tilde{A}$ it is not hard to see that the "only if" implication holds for a non unital A. In the other direction, the proof of the theorem shows that $p \wedge q \in A$, whenever $p, q \in \mathscr{P}(A)$. The proof of Lemma 2.1 is valid with this property in place of the l.p. For a unital algebra the l.p. itself is not used anywhere else in the proof of Theorem 3.4, — only the consequence of it given in Lemma 2.1. It remains to observe that if A does not contain a C*-subalgebra *-isomorphic to $\mathscr{A}$, then neither does $\tilde{A}$. Indeed, if $B \subset \tilde{A}$ is *-isomorphic to $\mathscr{A}$, then $\tilde{B} \subset \tilde{A}$, $\tilde{B} \cap A$ has codimension one in $\tilde{B}$, hence $\tilde{B} \cap A$ is *-isomorphic to $\mathscr{A}$.

REFERENCES

1. C. A. Akemann, *The general Stone–Weierstrass problem*, J. Funct. Anal. 4 (1969), 277–294.
2. C. A. Akemann, *Left ideal structure of* C*-*algebras*, J. Funct. Anal. 6 (1969), 305–318.
3. O. Bratteli, *Structure spaces of approximately finite dimensional* C*-*algebras*, J. Funct. Anal. 16 (1974), 192–204.
4. J. Dixmier, *Les* C*-*algèbres et leur représentations* (Cahiers Scientifiques 24), Gauthier–Villars, Paris, 1969.
5. G. A. Elliott, *Automorphisms determined by multipliers on ideals of a* C*-*algebra*, J. Funct. Anal. 23 (1976), 1–10.
6. R. Gilles and H. Kummer, *A non-commutative generalization of topology*, Indiana Univ. Math. J. 21 (1971), 91–102.
7. A. J. Lazar, *On some elementary properties of* AF *algebras*, to appear in Indiana Univ. Math. J.
8. A. J. Lazar, AF *algebras with directed sets of finite dimensional *-subalgebras*, to appear.
9. A. J. Lazar and D. C. Taylor, *Approximately finite dimensional* C*-*algebras and Bratteli diagrams*, Trans. Amer. Math. Soc. 259 (1980), 599–619.
10. G. K. Pedersen, C*-*algebras and their automorphism groups* (London Mathematical Society Monographs 14), Academic Press, London, New York, San Francisco, 1979.

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