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A PROJECTIVE GAUSS MAP ASSOCIATED WITH A HYPERSURFACE AND A HYPERPLANE

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1. Introduction.

In this note we study a map associated with a hypersurface M and a fixed affine n-plane Π in $(n+1)$-Euclidean space. Roughly speaking such a map can be described by associating to each point $m \in M$ the intersection point with Π of the normal line to M at m. The points $m \in M$ such that the normal line is parallel to Π are associated with appropriate points at infinity.

We interpret the critical set of this map in terms of the geometry of the hypersurface, calculate its mod 2 degree and apply it to gain some information about the location of the focal set of M in R^{n+1}.

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2. The projective Gauss map associated with (M, Π).

Let M be a compact, connected, smooth $(=C^\infty)$ hypersurface of R^{n+1}. Through each $m \in M$ there passes a line N_m normal to M at m.

Let Π be an affine n-plane in R^{n+1} such that no normal line to M is contained in Π. We fix $p_0 \in \Pi$ and an orthonormal basis (e_i) in the vector subspace associated with Π. Using p_o and (e_i) we compactify Π to real projective n-space p^n.

DEFINITION 2.1. The projective Gauss map associated with (M, Π) is the map Φ: $M \to p^n$ defined as follows:

if $N_m \cap \Pi = \{p\}$ is such that $p - p_0 = \sum_i \lambda_i e_i$, then $\Phi(m) = [\lambda_1, \ldots, \lambda_n, 1]$;
if $N_m \cap \Pi = \varnothing$ and the direction of N_m is given by $\sum_i \lambda_i e_i$, then $\Phi(m) = [\lambda_1, \ldots, \lambda_n, 0]$.

REMARKS. Φ is a smooth map.

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Sometimes we shall identify $\Phi(m)$ with $N_m \cap \Pi$ if this intersection is non-empty.

3. The results.

Let Δ be the set of elements $m \in M$ such that $N_m \cap \Pi = \varnothing$. Regarding p^{n-1} as a submanifold of p^n, then $\Delta = \Phi^{-1}(p^{n-1})$.

THEOREM 3.1. *If $m \in M \setminus \Delta$, then m is a critical point iff $\Phi(m)$ is a focal point of M with base m.*

If $m \in \Delta$, then m is a critical point iff $K(m)=0$, where $K(m)$ is the Gaussian curvature of M at m.

PROOF: We denote by i the inclusion map $M \to \mathrm{R}^{n+1}$ and assume Π to be determined by $x_{n+1}=0$.

If $m \in M \setminus \Delta$, then there is a chart $\varphi\colon U \to U'$ of M such that $m \in U$ and $i \circ \varphi^{-1}(x) = (x_1,\ldots,x_n, g(x))$, where we write x for $(x_1,\ldots,x_n) \in \mathrm{R}^n$. We obtain a local representative Φ_1 for Φ such that

$$\Phi_1(x) = (x_1 + g(x)\,D_1\,g(x),\ldots,x_n + g(x)\,D_n\,g(x))\,.$$

If $\{q\} = N_m \cap \Pi$ and we consider $L_q(x) = \|i(x) - q\|^2$, where $\|\cdot\|$ is the standard norm in R^{n+1}, we conclude that m is a critical point of Φ iff m is a degenerate critical point of L_q. The theorem now follows (cf.[3]).

If $m \in \Delta$ we may assume without any loss of generality that there is a chart $\varphi\colon U \to U'$ such that

$$m \in U, \quad i \circ \varphi^{-1}(x) = (g(x), x_1,\ldots,x_n), \quad \varphi(m) = (0,\ldots,0,\bar{x}_n)\,,$$

$$D_i g(0,\ldots,0,\bar{x}_n) = 0, \quad i=1,\ldots,n\,.$$

A local representative Φ_1 for Φ is then given by

$$\Phi_1(x) = \left(\frac{x_1 D_n g(x) - x_n D_1 g(x)}{g(x) D_n g(x) + x_n},\ldots,\frac{x_{n-1} D_n g(x) - x_n D_{n-1} g(x)}{g(x) D_n g(x) + x_n}, \frac{D_n g(x)}{g(x) D_n g(x) + x_n}\right)$$

As $K(m)$ is given either by the determinant of the matrix $[D_{ij} g(0,\ldots,0,\bar{x}_n)]$ or by the determinant of $[-D_{ij} g(0,\ldots,0,\bar{x}_n)]$ a straightforward calculation shows that m is a critical point iff $K(m)=0$.

THEOREM 3.2. $\mathrm{Deg}_2 \Phi \equiv e(M)$ (mod 2), *where $e(M)$ is the Euler number of M.*

PROOF. Choose a regular value $q \in \Pi$. By Sard's theorem we know that such regular value exists. Then $\Phi^{-1}(q) = \{p_1, \ldots, p_k\}$, where p_i, $i = 1, \ldots, k$, are the critical points of $L_q\colon M \to \mathsf{R}$ and all of them are non-degenerate. As

$$\sum_{i=0}^{n} (-1)^i N_i = e(M)\,,$$

where N_i denotes the number of critical points of L_q of index i, we conclude that $k = \alpha + e(M)$ with α even.

The following well known result is usually proved using the Gauss map [5]. We give a proof which basically follows the same pattern but now using Φ.

THEOREM 3.3. *Let M be a hypersurface such that, for every $m \in M$, $K(m) \neq 0$. Then M is diffeomorphic to S^n.*

PROOF. If the Gaussian curvature is never zero, then the focal set $F(M)$ of M is bounded. Choose an n-plane Π sufficiently far away from $F(M)$. Define Φ associated with (M, Π). Then Φ has no critical points and therefore it is a covering map. The order of the covering is greater than one and consequently M is diffeomorphic to S^n.

The projective map can be used to give us some information about the focal set $F(M)$ in R^{n+1}. For results concerning the relation between the topological structure of M and the location of $F(M)$, see [1].

THEOREM 3.4. *Let $M \setminus \Delta = M'$. If there is no focal point of M in Π, then all the components of M' are diffeomorphic to R^n.*

PROOF. Let us consider in p^n the set U of elements of the form $[\lambda_1, \ldots, \lambda_n, 1]$ $\lambda_i \in \mathsf{R}$, $i = 1, \ldots, n$. It is an open set diffeomorphic to R^n. We also have $M' = \Phi^{-1}(U)$ and consequently $\Phi \mid M'$ can be looked at as a map $\Phi'\colon M' \to \mathsf{R}^n$. This map is a surjection. In fact, for $q \in \mathsf{R}^{n+1}$, the map $L_q\colon M \to \mathsf{R}$ has always critical points and therefore there are normal lines to M passing through q [3]. Moreover if we assume that no focal point lies in Π, then Φ' is a covering map. This is a consequence of the fact that if y is a regular value of Φ, then there is an open neighbourhood V of y such that $\Phi^{-1}(V) = V_1 \cup \ldots \cup V_k$, where the V_i's are open, $V_i \cap V_j = \emptyset$ for $i \neq j$, and $\Phi\colon V_i \to V$, $i = 1, \ldots, k$, is a diffeomorphism. The theorem now follows from standard results in the theory of covering spaces [4].

If Π and Π' are parallel hyperplanes in R^{n+1} containing no normal line to M, then the associated subsets Δ and Δ' are identical, and the following corollary can be deduced from theorem 3.4.

COROLLARY 3.1. *If the focal set $F(M)$ is bounded then the components of $M \setminus \Delta$ are diffeomorphic to R^n for all Π containing no normal line to M.*

Note, however, that if $F(M)$ is not bounded, it may happen that some component of $M \setminus \Delta$ is not diffeomorphic to R^n. The standard "anchor-ring" torus in R^3 illustrates this phenomenon.

THEOREM 3.5. *Let M be a hypersurface of R^{n+1}, $n \geqq 2$, such that, for every $m \in \Delta$, $K(m) \neq 0$. If Δ is not connected then there is a focal point of M in Π.*

PROOF. If there is no focal point of M in Π, then Φ is a covering map. The inclusion i: $p^{n-1} \to p^n$ induces an epimorphism i_*: $\Pi_1(p^{n-1}) \to \Pi_1(p^n)$. Therefore $\Delta = \Phi^{-1}(\mathrm{p}^{n-1})$ is connected [2].

Theorem 3.5 is false, if we let $n = 1$. To obtain a counterexample take a round 1-sphere and any straight line not through the centre of the sphere.

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