MATH. SCAND. 50 (1982), 149–166

SOME INFINITE FAMILIES OF U-HYPERSURFACES

ANDREW BAKER and NIGEL RAY

Dedicated to the memory of Adrienne Hand

0. Introduction.

Since Milnor first calculated the complex bordism ring MU_* [6] and thus initiated the study of the bordism functor $MU_*(\cdot)$, its algebraic properties have been exhaustively investigated. However, many geometric questions concerning the ring remain unanswered, and, with current techniques, unanswerable.

We consider here one such question, inspired by work in [4] and [11] connected with finding the least embedding codimension of a representative for a given bordism class. More precisely, we investigate those bordism classes carried by hypersurfaces, i.e. submanifolds of R^{n+1} with codimension 1.

The structure defined by these classes seems interestingly rich, and this is a natural consequence of our observations in section 1 relating the problem to computing $\pi_*(\Omega(S^1 \wedge SO/U))$. Of course, this also tells us that we cannot expect a complete solution!

However, the methods we use, and the small gains we make, are entirely inspired by the geometry of MU_*, and might not otherwise suggest themselves when attacking the homotopy problem by more conventional means.

In section 1 we give a precise formulation of the hypersurface question, and relate it to $\Omega(S^1 \wedge SO/U)$ via a Pontrjagin–Thom construction. In section 2 we discuss in detail the topology of our hypersurfaces, and explain how to describe their U structures.

We develop our calculational machinery in section 3, showing in particular how the problem collapses over the rationals, and proving an interesting result (3.8) on hypersurfaces which are stably a wedge of spheres.

Our main section is section 4, where we investigate hypersurfaces which arise as the boundaries of regular neighbourhoods of embedded 2 cell complexes. It is here that we uncover the infinite families noted in our title. An interesting relationship emerges with the well-known summands $\operatorname{Im} J$ of the stable homotopy groups of spheres.

Received August 22, 1980; in revised form December 1, 1980.

In the concluding section we tie up some loose ends, and describe work currently in progress to improve our understanding of the hypersurface ring.

It is a pleasure to acknowledge the crucial assistance rendered by Sam Gitler, both in the way of suggestions for section 4, and hospitality at the Inst. Politecnico Nac. The referee was also most helpful in preventing us from obscuring our results by clouds of unnecessary elaboration.

1. Preliminary constructions.

Consider, by analogy with [4] and [11], the bordism group $\Omega^{\mathrm{U}}_{n,k}$ of n-manifolds M embedded in S^{n+k} for some fixed k, and with a U structure (i.e. a lift to BU) on their stable normal bundle. The bordisms are required to be of codimension k also, in $\mathrm{S}^{n+k}\times I$.

The images of $\Omega^{\mathrm{U}}_{*,k}$ in MU_* we shall denote by F^k_*. For stability reasons $F^{2n}_{2n}=\mathrm{MU}_{2n}$, so $\{F^k_* : k=0,1,2,\ldots\}$ gives a finite filtration of MU_* by increasing subgroups. Moreover, with respect to the cartesian product structure, $F^k_*\cdot F^l_*\subseteq F^{k+l}_*$.

Note that F^k_* arises from $\Omega^{\mathrm{U}}_{*,k}$ merely by allowing the bordisms to be arbitrary U manifolds.

We are interested here in the groups F^1_*. For a detailed analysis of the higher filtration groups, see [2]. It suffices here to note

(1.1) PROPOSITION. *$\Omega^{\mathrm{U}}_{*,1}$ is a ring, and each $\Omega^{\mathrm{U}}_{*,k}$ is a module over this ring for* $k>1$.

PROOF. Each $x\in\Omega^{\mathrm{U}}_{n,1}$ is represented by some hypersurface $M^n\subset\mathrm{S}^{n+1}$ carrying a U structure (U-hypersurface, for brevity). Similarly, $y\in\Omega^{\mathrm{U}}_{m,k}$ is represented by some $N^m\subset\mathrm{S}^{m+k}$.

So $xy\in\Omega^{\mathrm{U}}_{n+m,k+1}$ is represented by the product U-structure on $M^n\times N^m$. But M^n is oriented by its U structure, so there is a compatible embedding $M^n\times\mathrm{R}\subset\mathrm{S}^{n+1}$, since any hypersurface has trivial normal bundle. So we have

$$M^n\times N^m \subset M^n\times\mathrm{R}^{m+k} = M^n\times\mathrm{R}\times\mathrm{R}^{m+k-1} \subset \mathrm{S}^{n+1}\times\mathrm{R}^{m+k-1}\,.$$

This yields an embedding in S^{n+m+k}, which on suspension is isotopic to the product embedding $M^n\times N^m\subset\mathrm{S}^{n+m+k+1}$. So $xy\in\Omega^{\mathrm{U}}_{n+m,k}$.

Note that the construction also applies to bordisms, and therefore is independent of the representatives chosen for x and y.

Restricting to $k=1$ gives the first part of the result: the second then follows by choosing arbitrary k.

(1.2) COROLLARY. *F^1_* is a graded subring of* MU_*, *and F^k_* is a module over this ring for $k>1$.*

Proof. Apply (1.1) to images in MU_*.

Now consider F^0_* awhile. The only closed codimension zero submanifold of S^n is S^n itself, so $\Omega^U_{n,0} \cong \pi_n(SO/U)$ and F^0_n is represented by bordism classes containing U-structures on S^n. This is precisely the group $\operatorname{Im} J_n \subset MU_n$ determined in [8], being the image of $J\colon \pi_*(SO/U) \to MU_*$. The result is

(1.3) Proposition.

$$\begin{aligned} \operatorname{Im} J_{2n} &\cong Z \quad && \textit{if } n \textit{ odd} \\ &= 0 && \textit{if } n \textit{ even} . \end{aligned}$$

In addition, if $n=1$ *or* $n \equiv 3(4)$, *the group is a direct summand, whereas if* $n>1$ *and* $n \equiv 1(4)$ *it is divisible by* 2.

Also in MU_n, as explained in [8], there is the subring J^S_n consisting of the image of the stable J homomorphism $J^S\colon \pi^S_*(SO/U) \to MU_*$. This contains elements representable on frameable manifolds.

(1.4) Notation. Let Hyp_* denote the subring $F^1_* \subset MU_*$, and let $R\operatorname{Im} J_*$ denote the polynomial subring generated by $\operatorname{Im} J_*$.

(1.5) Proposition. *There are inclusions of subrings*

$$R\operatorname{Im} J_* \subset \mathrm{Hyp}_* \subset J^S_* .$$

Proof. The first inclusion holds since a product of spheres is a hypersurface, and the second since a hypersurface is always frameable.

One of our main interests is the extent to which the above inclusions are strict.

Before proceeding, it is important to record the homotopy interpretation of the above geometry. We shall write $\Omega^\infty MU(\infty)$ for the space $\lim \Omega^{2n} MU(n)$, and QX for the space $\lim \Omega^n(S^n \wedge X)$.

Now recall (e.g. from [8]) the commutative triangle

$$\begin{tikzcd}
SO/U \arrow[d] \arrow[dr, "j"] & \\
Q(SO/U_+) \arrow[r, "j^S"'] & \Omega^\infty MU(\infty)
\end{tikzcd}$$

where j^S is the infinite loop extension of j. Then on homotopy groups $j_* = J$, whilst $j^S_* = J^S$.

We may restrict j^S to a map j' defined on $\Omega(S^1 \wedge (SO/U)_+)$.

(1.6) PROPOSITION. *There is an isomorphism*

$$\varphi\colon \Omega^{\mathrm{U}}_{n,1} \to \pi_n(\Omega(\mathrm{S}^1 \wedge \mathrm{SO/U}_+)),$$

and the map $\Omega^{\mathrm{U}}_{n,1} \to \mathrm{MU}_n$ *is induced by* j' *(and so may be written as* $J' = j'_*$*).*

PROOF. To define φ, suppose given a U-hypersurface $M \subset \mathrm{S}^{n+1}$. It has normal bundle trivialised by the orientation, so the U-structure corresponds to a map $\tilde{\nu}\colon M \to \mathrm{SO/U}$. Performing a Pontrjagin–Thom collapse c onto a tubular neighbourhood of M yields the composition

$$\mathrm{S}^{n+1} \xrightarrow[c]{} \mathrm{S}^1 \wedge M_+ \xrightarrow[S\tilde{\nu}]{} \mathrm{S}^1 \wedge \mathrm{SO/U}_+ .$$

This process sends bordisms to homotopies, so, after adjointing, φ is well defined.

To construct φ^{-1}, start by adjointing a map $\mathrm{S}^n \to \Omega(\mathrm{S}^1 \wedge \mathrm{SO/U}_+)$ to a map $\mathrm{S}^{n+1} \to \mathrm{S}^1 \wedge \mathrm{SO/U}_+$. Making this transverse to SO/U produces (up to bordism) an oriented hypersurface $M \times \mathrm{R} \subset \mathrm{S}^{n+1}$. M is equipped with a map into SO/U, applying which to the normal line yields a compatible U-structure. Such a φ^{-1} sends homotopies to bordisms, and so is well defined.

Clearly φ and φ^{-1} are mutually inverse, whilst j' forgets the restrictive nature of the bordism.

The other filtration groups F^k_* have a more complicated homotopy interpretation, fully documented in [2].

Once (1.6) is established, we should note that the product structure in $\Omega^{\mathrm{U}}_{*,1}$ is induced by smashing two homotopy classes together via the map

$$\mu\colon \Omega(\mathrm{S}^1 \wedge \mathrm{SO/U}_+) \wedge \Omega(\mathrm{S}^1 \wedge \mathrm{SO/U}_+) \to \Omega(\mathrm{S}^1 \wedge \mathrm{SO/U}_+) .$$

μ is defined on the pair (l_1, l_2) to be the composite

$$\mathrm{S}^1 \xrightarrow[l_1]{} \mathrm{S}^1 \wedge \mathrm{SO/U}_+ \xrightarrow[l_2 \wedge 1]{} \mathrm{S}^1 \wedge \mathrm{SO/U}_+ \wedge \mathrm{SO/U}_+ \xrightarrow[1 \wedge \oplus]{} \mathrm{S}^1 \wedge \mathrm{SO/U}_+ .$$

Under j', μ maps to $\wedge$ on $\Omega^\infty\mathrm{MU}(\infty)$ (since on spheres, $\wedge$ and composition are stably homotopic).

(1.7) NOTES. (i) So far as Hyp_* and J^S_* are concerned, it is sufficient to limit attention to

$$\begin{tikzcd}
\Omega(\mathrm{S}^1 \wedge \mathrm{SO/U}) \arrow[d] \arrow[dr, "j'"] & \\
& \Omega^\infty\mathrm{MU}(\infty) \\
Q(\mathrm{SO/U}) \arrow[ur, "j^S"'] &
\end{tikzcd}$$

and discard the disjoint base point, since $\pi_*(QS^0)$ is torsion (in positive dimensions) and so maps to zero in MU_*.

(ii) Since $(SO/U, \oplus)$ is an infinite loop space (e.g. see [5]) there are retractions

$$\begin{tikzcd}
\Omega(S^1 \wedge SO/U) \arrow[d] \arrow[dr, "r'"] & \\
Q(SO/U) \arrow[r, "r^S"'] & SO/U
\end{tikzcd}$$

which exhibit $\pi_*(SO/U)$ as a direct summand of $\Omega^U_{*,1}$ and of $\pi^S_*(SO/U)$. Thus $\operatorname{Im} J_*$ is a direct summand of Hyp_* and J^S_*.

The geometrical significance of these facts will become clear in section 3.

To conclude, we explain how the complementary summand to $\pi_*(SO/U)$ in $\Omega^U_{*,1}$ (and hence to $\operatorname{Im} J_*$ in Hyp_*) has an interesting interpretation.

For any topological group G (and in fact for any loop space), G acts on the join $G*G$ to give a principal G bundle

$$G \to G*G \to S^1 \wedge G$$

(see [7]). The projection is the hopf construction $H(m)$ on the group multiplication $m: G \times G \to G$.

Extending the sequence, we obtain a fibration

$$\Omega(G*G) \to \Omega(S^1 \wedge G) \xrightarrow{r} G$$

with splitting map $G \to \Omega(S^1 \wedge G)$ the standard inclusion.

Applying this to $G = SO/U$, r becomes r' of (1.7). We deduce

(1.8) PROPOSITION.

$$\Omega^U_{*,1} \cong \pi_*(SO/U) \oplus \pi_*(\Omega(SO/U * SO/U)) .$$

Hence, under J', we obtain a splitting of Hyp_*. It is convenient to label the second summand as $J^\perp_*$. So we have

(1.9) COROLLARY.

$$\mathrm{Hyp}_* = \operatorname{Im} J_* \oplus J^\perp_* .$$

2. The topology of U-hypersurfaces.

We now investigate some simple properties of our hypersurfaces.

Suppose that $M \subset S^{n+1}$ is an oriented hypersurface, which we may assume connected when studying $\Omega(S^1 \wedge SO/U)$. M determines a division of S^{n+1} into two compact submanifolds A and B, such that $S^{n+1}=A\cup B$ and $M=A\cap B$, their common boundary. One of these, say A, is distinguished by the orientation of M.

As CW complexes A and B have dimension $<n$, and they are n-duals in the sense of Spanier, admitting duality maps

$$\varrho\colon S^{n+1}\to S^1\wedge A\wedge B \quad \text{and} \quad \varrho'\colon S^1\wedge A\wedge B\to S^{n+1}\,.$$

Vice versa, given a reasonable embedding of a complex A' in S^{n+1}, we may thicken it into a regular neighbourhood A and take its boundary M, an oriented hypersurface. Then $B=S^{n+1}-\operatorname{int}(A)$ is an n-dual to A, so we shall label it henceforth as DA.

Note that the hypersurfaces described above are oriented (and, of course, framed) boundaries by A.

Now regard the Mayer–Vietoris cofibration

$$\begin{tikzcd}
A\vee DA \arrow[r, "i\vee i'"'] & A\cup DA \arrow[r, "c"'] \arrow[d, equal] & S^1\wedge(A\cap DA) \arrow[r] \arrow[d, equal] & S^1\wedge(A\vee DA) \\
 & S^{n+1} & S^1\wedge M &
\end{tikzcd}$$

where c is the Pontrjagin–Thom collapse. Both i and i' have images within $n+1$ discs in S^{n+1}, and so create an explicit nul-homotopy for $i\vee i'$. So there is a homotopy equivalence

$$h\colon S^{n+1}\vee(S^1\wedge A)\vee(S^1\wedge DA)\to S^1\wedge M\,.$$

Restricting h to S^{n+1} recovers c, i.e. a splitting for the suspension of the collapse $M\to S^n$ onto the top cell. Similarly, $h|S^1\wedge A$ splits the suspension of the boundary inclusion $j\colon M\to A$, whilst $h|S^1\wedge DA$ splits the suspension of $j'\colon M\to DA$ (up to sign).

Returning to the U structure on M, it is given as a map $M\to SO/U$, which may be thought of as twisting the given framing. Homotopic maps give equivalent U structures. Since the group $[M,SO/U]$ is exactly $KO^{-2}(M)$, we have to calculate the KO groups of M. Using the above splittings, we obtain

(2.1) LEMMA

$$KO^{-2}(M)\cong KO^{-2}(A)\oplus KO^{-2}(DA)\oplus KO^{-2}(S^n)\,.$$

Of course, such a formula holds for any (co) homology functor.

So we can now investigate the contribution made by each summand of (3.1) to the bordism class of M, first in $\Omega^{U}_{*,1}$ and then in MU_*.

For any twist $\delta\in KO^{-2}(M)$ we shall write these bordism classes as $\{M,\delta\}\in\Omega^{U}_{n,1}$ and $[M,\delta]=J'\{M,\delta\}\in MU_n$.

(2.2) PROPOSITION.

(i) $\{M,\alpha\} = \{M,\alpha'\} = 0 \quad \forall\,\alpha \in \mathrm{KO}^{-2}(A),\ \alpha' \in \mathrm{KO}^{-2}(DA)$

(ii) $\{M,\sigma\} \in \pi_n(\mathrm{SO/U}) \quad \forall\,\sigma \in \mathrm{KO}^{-2}(\mathrm{S}^n)$.

PROOF. (i) If we twist the framing on M by α, it bounds A with framing twisted by α. Similarly for DA.

(ii) M is diffeomorphic to the connected sum $M \# \mathrm{S}^n$, and σ may be displayed trivially on M^n and as σ on S^n. Thus the new U structure is equivalent to retaining the original on M and twisting S^n by σ. But the original framing on M bounds A, so

$$\{M^n,\sigma\} = \{\mathrm{S}^n,\sigma\} .$$

(2.3) COROLLARY. *With the data above, in* MU_n

(i) $[M,\alpha] = [M,\alpha'] = 0$

(ii) $[M,\sigma] = J(\sigma)$.

The above observations give substance to the splittings of (1.7), for if M is initially a sphere then $\mathrm{KO}^{-2}(M) \cong \mathrm{KO}^{-2}(\mathrm{S}^n)$ in (2.1). Furthermore, the splitting is the same if M is merely framed, since the top cell is still stably trivially attached.

Thus when investigating Hyp_*, our interest should centre on twistings δ which are non-zero in both $\mathrm{KO}^{-2}(A)$ and $\mathrm{KO}^{-2}(DA)$, yet which vanish on the top cell. Such U-hypersurfaces constitute the summands $\pi_*(\Omega(\mathrm{SO/U} * \mathrm{SO/U}))$ and $J_*^{\perp}$ of (1.9). They can be obtained directly from a representative $f\colon \mathrm{S}^n \to \Omega(\mathrm{SO/U} * \mathrm{SO/U})$ as follows.

Adjoint f, and regard the join as the space

$$(C(\mathrm{SO/U}) \times \mathrm{SO/U}) \cup (\mathrm{SO/U} \times C(\mathrm{SO/U})) ,$$

where the cones are thought of as attached to $\mathrm{SO/U} \times \mathrm{SO/U}$ in opposite directions. Now make the map transverse to $\mathrm{SO/U} \times \mathrm{SO/U}$, giving a hypersurface $M \subset \mathrm{S}^{n+1}$ equipped with a map

$$\alpha \times \alpha' \colon M \to \mathrm{SO/U} \times \mathrm{SO/U} .$$

By construction, α extends over A and α' extends over DA.

Enthusiasts may verify that this process yields a Pontrjagin–Thom isomorphism between the geometry and the homotopy theory.

Our programme is to begin by restricting the defining complex A, and hence the hypersurface M, to be of a simple type. Note that if A is a point (i.e. a 0 cell complex), M will be a sphere and so represent an element of $\mathrm{Im}\, J_*$; whilst if A

is a wedge of spheres (i.e. 1 cell complexes), M will be a connected sum of products of spheres, and so represent an element in $R\,\mathrm{Im}\,J_*$.

Therefore we propose to study the situation when A is either a (wedge of) 2 cell complexes, or else is *stably* a wedge of spheres. As we shall see, both these choices yield interesting, and apparently new information.

3. Sample calculations.

In this section we set up the calculational procedures we shall adopt in section 4. But first we note some general facts.

(3.1) Proposition. $R\,\mathrm{Im}\,J_*\otimes\mathrm{Q}\cong J^S_*\otimes\mathrm{Q}$ *as subrings of* $\mathrm{MU}_*\otimes\mathrm{Q}$.

Proof. The following diagram of Hurewicz homomorphisms is commutative (e.g. see [8])

$$\begin{tikzcd}
 & \pi_*(\mathrm{SO/U}) \arrow[r, "J"] \arrow[dl] \arrow[d, "h_2"] & \mathrm{MU}_* \arrow[d, "h_3"] \\
\pi^S_*(\mathrm{SO/U}) \arrow[r, "h_1"'] & H_*(\mathrm{SO/U}) \arrow[r, "i_*"'] & H_*(\mathrm{BU})
\end{tikzcd}$$

(and i_* is a monomorphism of Pontrjagin rings). Thus i_*h_2 has image $h_3(\mathrm{Im}\,J_*)$.

In addition, h_2 maps $\pi_*(\mathrm{SO/U})$ onto multiples of ring generators for $H_*(\mathrm{SO/U})$. Thus when we apply $\otimes\mathrm{Q}$ to the above diagram, we observe that

$$\begin{aligned}(h_3\otimes 1)(R\,\mathrm{Im}\,J_*\otimes\mathrm{Q}) &= (i_*\otimes 1)(H_*(\mathrm{SO/U})\otimes\mathrm{Q})\\ &= (i_*h_1\otimes 1)(\pi^S_*(\mathrm{SO/U})\otimes\mathrm{Q})\,.\end{aligned}$$

But both $h_1\otimes 1$ and $h_3\otimes 1$ are isomorphisms.

(3.2) Corollary.

$$R\,\mathrm{Im}\,J_*\otimes\mathrm{Q} = \mathrm{Hyp}_*\otimes\mathrm{Q} = J^S_*\otimes\mathrm{Q}\,.$$

Proof. Apply (3.1) to (1.5).

Thus the problem of relating the three subrings of MU_* described in (1.5) is entirely concerned with subgroups of maximal rank in a free abelian group. In other words the issues are of divisibility, and as such may be investigated by taking quotients.

The "smallest" subring involved is $R\,\mathrm{Im}\,J_*$, and $\mathrm{MU}_*/R\,\mathrm{Im}\,J_*$ is theoretically determinable by KU theory. Passing to BP may well be a fruitful method for organising the algebra. On the other hand we know of no sure method for computing J^S_*, the "largest" subring.

Note from (1.3) that none of $R\,\mathrm{Im}\,J_*$, Hyp_*, J_*^S are summands of MU_*. And we shall see below that Hyp_* is *not* polynomial.

To put these matters into perspective, consider the stabilisation map $\mathrm{st}\colon \Omega(S^1 \wedge \mathrm{SO/U}) \to Q(\mathrm{SO/U})$. It gives

$$\begin{tikzcd}
\Omega^{\mathrm{U}}_{*,1}\otimes \mathrm{Q} \arrow[r, "\mathrm{st}_*\otimes 1"] \arrow[d, "J'\otimes 1"'] & \pi^S_*(\mathrm{SO/U})\otimes \mathrm{Q} \arrow[d, "h_1\otimes 1", "\cong"'] \\
\mathrm{Hyp}_*\otimes \mathrm{Q} \arrow[r, "\cong"'] & H_*(\mathrm{SO/U})\otimes \mathrm{Q}
\end{tikzcd}$$

Thus the free part of $\Omega^{\mathrm{U}}_{*,1}$ consists of two summands. Firstly those elements which survive under st_*, say Φ; and secondly $\mathrm{Ker}\,(\mathrm{st}_*)$. Then J' is a monomorphism on Φ, so Hyp_* is the image of faithful representation of Φ. This alone suggests that our divisibility problems will be intractible!

To make more precise computations, we must recall some notation pertaining to MU_*. We may write MU_* as a polynomial algebra $\mathrm{Z}[x_1, x_2, \ldots, x_k, \ldots]$, where x_k has real dimension $2k$. Several authors have given procedures for choosing the x_k's, but no canonical choice has emerged.

Over Q, the requirement for a generator is simple, viz. $s_k(x_k) \neq 0$, where $s_k \in H^{2k}(\mathrm{BU})$ is the additive characteristic class corresponding to the symmetric polynomial $\sum t_i^k$ in $H^{2k}(\times_i \mathrm{CP}^\infty)$ (e.g. see [10]).

For example, amongst the results needed to prove (1.3) is the following. Let $[\mathrm{S}^{4k+2}]$ be the U bordism class represented by a generator of $\pi_{4k+2}(\mathrm{SO/U}) \cong \mathrm{Z}$. Then

$$s_{2k+1}[\mathrm{S}^{4k+2}] = \begin{pmatrix} 2(2k+1)! & k \text{ even} \\ (2k+1)! & k \text{ odd} \end{pmatrix} = \varkappa(k), \text{ say} .$$

Thus $[\mathrm{S}^{4k+2}]$ is an acceptable choice for x_{4k+2} over Q. We deduce

(3.3) NOTE. All the rings of (3.2) may be described as

$$\mathrm{Q}[[\mathrm{S}^2], [\mathrm{S}^6], \ldots, [\mathrm{S}^{4k+2}], \ldots] .$$

Thus our overall strategy is to describe our (integral) bordism classes in MU_* as rational combinations of these sphere classes.

We start by recalling that

$$H^*(\mathrm{BU}; \mathrm{Q}) \cong \mathrm{Q}[s_1, s_2, \ldots, s_k, \ldots]$$

and that in dimension $2n$ a basis is given by monomials

$$s_w = (s_1)^{w_1}(s_2)^{w_2} \ldots (s_n)^{w_n} \quad \text{where } \sum_{i=1}^{n} i w_i = n .$$

We write $W(n)$ for the set of such sequences $(w_1, w_2, \ldots, w_n)$, so that a U-bordism class y_n is determined by the Chern numbers $\{s_w(y_n);\ w \in W(n)\}$.

Equivalently, and dually, we have to evaluate the image of y_n under the Hurewicz isomorphism

$$\underline{h}\otimes 1\colon \mathrm{MU}_*\otimes \mathrm{Q} \to H_*(\mathrm{MU};\mathrm{Q}) \cong H_*(\mathrm{BU};\mathrm{Q})\,.$$

But we may write

$$H_*(\mathrm{BU};\mathrm{Q}) \cong \mathrm{Q}\{s^w;\ \ w\in W(n)\,\forall\, n\}$$

where s^w is the basis element dual to s_w. Moreover $\underline{h}\otimes 1$ identifies $[\mathrm{S}^{4k+2}]$ with $\varkappa(k)s^{\Delta(2k+1)}$, where $\Delta(i)$ is the sequence $(0,\ldots,0,1)$ with $i-1$ zeros. By (3.2) and (3.3), these elements are ring generators for the image of $H_*(\mathrm{SO/U};\mathrm{Q})$ in $H_*(\mathrm{BU};\mathrm{Q})$.

Now given a twisting $\delta\colon M \to \mathrm{SO/U}$, we note that the new normal structure on the hypersurface M is given by the composition

$$M \xrightarrow[\delta]{} \mathrm{SO/U} \xrightarrow[\iota]{} \mathrm{BU}\,.$$

which we label $\tilde{\delta}$. Hence the s classes of $[M,\delta]$ arise by evaluating monomials $s_w(\tilde{\delta})$ on the fundamental class $\sigma \in H_*(M)$. Such calculations are aided by the fact that the diagram

$$\begin{tikzcd}
\mathrm{KO}^{-2}(M) \arrow[rr, "c"] \arrow[dr, "i^*"'] & & \mathrm{KU}^{-2}(M) \arrow[dl, "\zeta"] \\
& \mathrm{KU}^{0}(M) &
\end{tikzcd}$$

is commutative, where ζ is the Bott isomorphism. So $s_w(\tilde{\delta}) = s_w(\zeta c\delta)$ may be read off from $\mathrm{ch}\,(c\delta)$.

The most systematic way of organising this information is to utilise the scheme laid out in [9].

(3.4) LEMMA. *Let M^n be framed, and $\delta \in \mathrm{KO}^{-2}(M)$, and let $\Psi\colon \mathrm{KO}^{-2}(M) \to \mathrm{MU}_n$ be given by $\Psi(\delta) = [M,\delta]$. Then Ψ may be factorised as*

$$\mathrm{KO}^{-2}(M) \xrightarrow[J_u]{} \mathrm{MU}^0(M_+) \xrightarrow[D]{} \mathrm{MU}_n\,,$$

where $J_u = 1 + \mathcal{J}_u$ is induced by $j\colon \mathrm{SO/U} \to \Omega^\infty \mathrm{MU}(\infty)$ (and is exponential) whereas $D = \langle\,\cdot\,,\Sigma\rangle$ is the duality homomorphism induced by the fundamental class $\Sigma \in \mathrm{MU}_n(M_+)$.

Our plan is to use the characteristic class calculations discussed earlier in order to trace this factorisation of Ψ in (co)homology, using the Hurewicz and Boardman maps

$$\underline{h}\colon \mathrm{MU}_*(M) \to H_*(\mathrm{BU}_+)\otimes H_*(M)\otimes \mathrm{Q}$$

$$\bar{h}\colon \mathrm{MU}^*(M) \to H_*(\mathrm{BU}_+)\otimes H^*(M)\otimes \mathrm{Q}\,.$$

To this end, we need the following useful formula:

(3.5) FORMULA. Let $\delta \in \mathrm{KO}^{-2}(M)$. Then

$$\bar{h}\tilde{J}_u(\delta) = \sum_v \varepsilon_v \mathrm{S}^v \otimes \mathrm{ch}_v(\exp\tilde{\delta})\,.$$

Here $\exp\tilde{\delta}$ is to be interpreted as the usual power series in the (nilpotent) ring $\mathrm{KU}^0(M)\otimes\mathrm{Q}$. Also, v ranges over the non-decreasing elements in the set V consisting of all sequences of odd integers, so if $v=(v(1),\dots,v(t))$, $\mathrm{ch}_v(\tilde{\delta}^t)=\mathrm{ch}_{v(1)}(\tilde{\delta})\dots\mathrm{ch}_{v(t)}(\tilde{\delta})$. Similarly, $\mathrm{S}^v=[\mathrm{S}^{2v(1)}]\dots[\mathrm{S}^{2v(t)}]$ whilst $\varepsilon_v=\varepsilon(v(1))\dots\varepsilon(v(t))$, where

$$\varepsilon(v(i)) = \begin{cases}1\\ \frac{1}{2}\end{cases} \quad \text{as } v(i) \equiv \begin{cases}3(4)\\ 1(4)\end{cases}.$$

PROOF. In $H^*(\mathrm{BU};\mathrm{Q})$, ch_{2k+1} is dual to $(2k+1)!\,s^{\Delta(2k+1)}$. Thus the linear terms in $\bar{h}\tilde{J}_u(\delta)$ have the form

$$\sum_k \varepsilon(2k+1)[\mathrm{S}^{2k+1}]\otimes\mathrm{ch}_{2k+1}(\tilde{\delta})\,.$$

Considering product terms in the same fashion, remark that $\mathrm{ch}_v(\tilde{\delta}^t)$ has value $t!/\varepsilon_v$ on $[\mathrm{S}^{2v(1)}]\dots[\mathrm{S}^{2v(t)}]$, and zero on other monomials of the same dimension. Thus a typical non-linear term of $\bar{h}\tilde{J}_u(\delta)$ is $\mathrm{S}^v\otimes(1/t!)\,\mathrm{ch}_v(\tilde{\delta}^t)$. Now sum over t.

This formula neatly captures the idea that J_u is exponential: applied to a sum it yields

$$\bar{h}\tilde{J}_u(\delta_1+\delta_2) = \sum_{v,w} \varepsilon_{v,w}\mathrm{S}^{v,w}\mathrm{ch}_v(\exp\tilde{\delta}_1)\mathrm{ch}_w(\exp\tilde{\delta}_2)\,.$$

To crystallise these ideas, we conclude with an interesting application: it requires two lemmas.

(3.6) LEMMA. *Suppose X is stably a finite wedge of spheres. Then in* $\mathrm{KU}^0(X)$, *x^t is divisible by $t!$ $\forall x$ or t* (i.e. "*$\exp x$ is integral*").

PROOF. Let ψ^p be the Adams operation for the prime p, so that $\psi^p(x)\equiv x^p(p)$ in the free abelian group $\mathrm{KU}^0(X)$. Using the stable structure of X, $\psi^p(x)\equiv 0(p)$, whence $p\,|\,x^p$ for any prime. So $t!\,|\,x^t$.

Note that replacing $\mathrm{KU}^0(X)$ by $H^*(X; \mathbf{Z})$, and ψ^p by the Steenrod power (or square) gives the same result for integral cohomology.

(3.7) LEMMA. *With X as in (3.6), and x in the image of $c\colon \mathrm{KO}^{-2}(X) \to \mathrm{KU}^{-2}(X)$, then $\mathrm{ch}_v(x^t)$ is divisible by $t!/\varepsilon_v$ $\forall\, v \in V$.*

PROOF. We can rewrite x as $2x'+x''$, where x' incorporates all the components of ch_* in dimensions $\equiv 2(8)$. Thus

$$x^t = 2^t(x')^t + \ldots + \binom{t}{k} 2^{t-k}(x')^{t-k}(x'')^k + \ldots + (x'')^t$$

and by (3.6) $(t-k)! \mid (x')^{t-k}$, and $k! \mid (x'')^k$.

The result now follows by a simple induction.

Hence we can prove

(3.8) THEOREM. *Suppose that the hypersurface $M \subset S^{n+1}$ is stably a wedge of spheres. Then*

$$\Psi(M) \subset R\,\mathrm{Im}\,J_n\,.$$

PROOF. Let $\delta \in \mathrm{KO}^{-2}(M)$, so that

$$\underline{h}\Psi(\delta) = \langle \bar{h}\bar{J}_u(\delta), \underline{h}\Sigma\rangle \quad \text{in } H_{2n}(\mathrm{BU})\,.$$

Now the splitting of M discussed in section 2 ensures that $\underline{h}\Sigma = 1 \otimes \sigma$ in $H_*(\mathrm{BU}_+) \otimes H_*(M)$, so by (3.5),

$$\underline{h}\Psi(\delta) = \sum \langle \mathrm{ch}_v(\exp \bar{\delta}), \sigma\rangle \varepsilon_v S^v\,,$$

summing over all v with $\sum v(i) = n$.

But from (3.7), $\varepsilon_v \mathrm{ch}_v(\exp\bar{\delta}) \in H^{2n}(M; \mathbf{Z})$, so $\underline{h}\Psi(\delta)$ is an integral combination of S^v's. So $\Psi(\delta) \in R\,\mathrm{Im}\,J_*$.

(3.9) NOTES. (i) We have shown that $R\,\mathrm{Im}\,J_*$ consists precisely of those classes representable on hypersurfaces which are stably a wedge of spheres.

(ii) The given proof, with minimal modification, applies to a framed M of arbitrary codimension.

4. 2 cell complexes.

We now concentrate on hypersurfaces defined by 2 cell complexes.

Let $x \in \mathrm{Hyp}_{2n}$ be represented by $M \subset S^{2n+1}$, with defining complex $A = S^a \cup_0 e^b$. We shall see below that it suffices to restrict a and b by $a \geqq 2$ and

$b \leqq 2n-2$. According to [3], we may then assume that $DA \simeq S^{2n-b} \cup_{\varphi} e^{2n-a}$, where $s^{2n-b-1}\theta \simeq (-1)^{ab} s^{a-1}\varphi$.

(4.1) LEMMA. *Let X be one of A, DA (or M) as above. Then* $\mathrm{MU}_*(X)$ *and* $\mathrm{MU}^*(X)$ *are free modules on* 2 *(or* 5*) generators over* MU_*, *if* $b > a+1$.

PROOF. Since θ is stably torsion, the result it true for A and DA (for MU_* is free). It follows for M from (2.1).

As both an illustration and a general explanation of our method, we now examine a typical case;

(4.2) EXAMPLE. Let $\nu \in \pi_7(S^4)$ be the Hopf map. Choose an embedding $A = S^6 \cup_\nu e^{10} \subset S^{21}$ (say by suspending a smooth embedding $HP^2 \subset S^{19}$). Then DA is given by $S^{10} \cup_\nu e^{14}$. Thus

$$M = S^6 \cup e^{10} \cup e^{10} \cup e^{14} \cup e^{20}$$

$$S^1 \wedge M \simeq (S^7 \cup_\nu e^{11}) \vee (S^{11} \cup_\nu e^{15}) \vee S^{21} .$$

It is advantageous to consider A and DA as Thom complexes of the "adjoint bundles" $\nu\colon S^4 \to \mathrm{BSO}(6)$ and $\nu\colon S^4 \to \mathrm{BSO}(10)$ respectively (for ν lies in the classical $\mathrm{Im}\, J$). Being Spin bundles, these are KO orientable, so we have

$$\mathrm{KO}^{-2}(A) \cong \mathrm{KO}(S^4_+) \cong \mathrm{Z} \oplus \mathrm{Z} .$$

$$\mathrm{KO}^{-2}(DA) \cong \mathrm{KO}^{-4}(S^4_+) \cong \mathrm{Z} \oplus \mathrm{Z} .$$

We may choose generators α_1, β_1 for $\mathrm{KO}^{-2}(A)$ and α_2, β_2 for $\mathrm{KO}^{-2}(DA)$ related by

$$\alpha_2 = x\alpha_1 \quad \text{and} \quad y\beta_1 = x\beta_2 ,$$

where x generates $\mathrm{KO}_4 \cong \mathrm{Z}$, y generates $\mathrm{KO}_8 \cong \mathrm{Z}$, and $x^2 = 4y$.

To apply (3.5), we must evaluate ch on the image of each element in $\mathrm{KO}^{-2}(\cdot)$. Since complexification preserves Thom isomorphisms, we deduce with the help of [1] that

$$\mathrm{ch}_3(\tilde{\alpha}_1) = t_1, \quad \mathrm{ch}_5(\tilde{\alpha}_1) = \tfrac{1}{12} u_1$$

$$\mathrm{ch}_3(\tilde{\beta}_1) = 0, \quad \mathrm{ch}_5(\tilde{\beta}_1) = 2u_1 ,$$

where t_1, u_1 are respective generators of $H^6(A)$, $H^{10}(A)$ (with Z coefficients). The 1/12 arises as the e invariant $e_{\mathrm{C}}(\nu)$.

Substituting in (3.5) gives in $H_*(\mathrm{BU}_+) \otimes H^*(M)$

$$\bar{h}\tilde{J}_u(\lambda\alpha_1 + \mu\beta_1) = 1 \otimes 1 + \lambda[S^6] \otimes t_1 + (\tfrac{1}{24}\lambda + \mu)[S^{10}] \otimes u_1$$

$\forall\,\lambda, \mu \in \mathbf{Z}$. Of course, products vanish in $KO^*(A)$.

Similarly, we can write

$$\bar{h}\tilde{J}_u(\lambda'\alpha_2+\mu'\beta_2) = 1\otimes 1+\lambda'[S^{10}]\otimes t_2+(\tfrac{1}{6}\lambda'+\mu')[S^{14}]\otimes u_2\,.$$

Now by duality, products in $H^*(M)$ are given by

$$t_1u_2 = -u_1t_2 = g_{20} \qquad \text{(e.g. see [3])}$$

where $g_{20} \in H^{20}(M)\cong \mathbf{Z}$ is the dual of $\sigma \in H_{20}(M)$.

So if $\delta=\lambda\alpha_1+\mu\beta_1+\lambda'\alpha_2+\mu'\beta_2 \in KO^{-2}(M)$, then

$$\bar{h}\tilde{J}_u(\delta) = 1\otimes 1+\{\lambda(\tfrac{1}{6}\lambda'+\mu')[S^6][S^{14}]-\lambda'(\tfrac{1}{24}\lambda+\mu)[S^{10}]^2\}\otimes g_{20}$$

modulo the linear terms in t_1, u_1, t_2, and u_2.

Moreover from the splitting of (2.1),

$$\underline{h}\Sigma = 1\otimes\sigma \quad \text{ in } H_*(BU_+)\otimes H_*(M)\,.$$

Thus $\forall\,\lambda, \mu, \lambda', \mu' \in \mathbf{Z}$

$$\begin{aligned}\underline{h}\Psi(\delta) &= \langle \bar{h}\tilde{J}_u, \underline{h}\Sigma\rangle\\ &= \lambda(\tfrac{1}{6}\lambda'+\mu')[S^6][S^{14}]-\lambda'(\tfrac{1}{24}\lambda+\mu)[S^{10}]^2\,.\end{aligned}$$

Clearly we have unearthed elements of Hyp_{20} which do not lie in $R\,\mathrm{Im}\,J_*$. For example

$$\lambda = \lambda' = 1, \quad \mu = \mu' = 0 \quad \text{gives} \quad \tfrac{1}{24}(4[S^6][S^{14}]-[S^{10}]^2)$$

$$\lambda = 1,\ \lambda' = 6,\ \mu = 0,\ \mu' = 1 \quad \text{gives} \quad \tfrac{1}{4}[S^{10}]^2\,.$$

(4.3) Corollary. Hyp_* *is not polynomial.*

(4.4) Corollary. *The quotients*

$$\mathrm{Hyp}_{20}/R\,\mathrm{Im}\,J_{20} \hookrightarrow MU_{20}/R\,\mathrm{Im}\,J_{20}$$

have $\mathbf{Z}_{24}$ *as a subgroup.*

Proof. This is a simple algebraic consequence of the formulae arising in (4.2).

We now propose to let θ range over all possible $\pi_{b-1}(S^a)$. It is therefore convenient to write A_θ for $S^a \cup_\theta e^b$, and M_θ for the hypersurface defined by some embedding $A_\theta \subset S^n$. Before we begin calculating, it is also helpful to divide these complexes A into various types, according to how c and ch work.

(4.5) DEFINITIONS. We shall say that A_θ has *type* A if both cells are of dimension $\equiv 2$ (4), and write

$$A_\theta = S^{2c} \cup_\theta e^{2d} \quad \text{with } c, d \equiv 1 \ (2) .$$

We shall say that A_θ has *type* B if only the top cell has dimension $\equiv 2(4)$, and if θ is not stably $\mu_1 \in \pi^S_{8*+1}$ (where $d_{\mathrm{R}}(\mu_1) = 1 \in \mathrm{Z}_2$: see [1]). If θ is stably μ_1 we shall say that A_θ has *type* B^μ. In either case, we write

$$A_\theta = S^a \cup_\theta e^{2d} \quad \text{with } d \equiv 1(2) .$$

Similarly, we shall say that A_θ has *type* C if only the bottom cell has dimension $\equiv 2(4)$, and θ is not stably μ_1 or $\mu_2 \in \pi^S_{8*+2}$. Otherwise, A_θ has *type* C^μ. In either case, we write

$$A_\theta = S^{2c} \cup_\theta e^b \quad \text{with } c \equiv 1(2) .$$

Next we note two lemmas, easily proved.

(4.6) LEMMA. *The composition*

$$\mathrm{KO}^{-2}(A_\theta) \xrightarrow[c]{} \mathrm{KU}^0(A_\theta) \xrightarrow[\mathrm{ch}]{} H^*(A_\theta; \mathrm{Q})$$

is trivial unless A_θ is of one of the above types. More, if of type B or B^μ, then $\mathrm{ch} \circ c$ is trivial on the bottom cell, and if of type C or C^μ, then $\mathrm{ch} \circ c$ is trivial on the top cell.

(4.7) LEMMA. *If A_θ has type* A, B *or* C, *then the cofibration of θ yields a splitting*

$$\mathrm{KO}^{-2}(A_\theta) \cong \mathrm{KO}^{-2}(S^a) \oplus \mathrm{KO}^{-2}(S^b) .$$

If A_θ has type B^μ, then projection onto the top cell induces $2\colon \mathrm{Z} \to \mathrm{Z}$ in $\mathrm{KO}^{-2}(\cdot)$; and if A_θ has type C^μ, inclusion of the bottom cell also induces $2\colon \mathrm{Z} \to \mathrm{Z}$ in $\mathrm{KO}^{-2}(\cdot)$.

Armed with these tools, we can now generalise (4.2).

(4.8) THEOREM. *Let A_θ be of type* A, *and suppose $A_\theta \subset S^{4n+1}$. Then we have that*

$$\Psi(M_\theta)/R \operatorname{Im} J_{4n} \subset \mathrm{Hyp}_{4n}/R \operatorname{Im} J_{4n}$$

is a finite cyclic group. A generator is

$$\tfrac{1}{2}e(4[S^{2c}][S^{4n-2c}] - [S^{2d}][S^{4n-2d}]) \ \mathit{if} \ \begin{cases} d - c \equiv 2(4) \\ 2n \equiv c - 1 \equiv 2(4) \end{cases}$$

and

$$qe([S^{2c}][S^{4n-2c}] - [S^{2d}][S^{4n-2d}]) \ \mathit{otherwise},$$

where

$$q = \tfrac{1}{2} \text{ if } \begin{cases} d-c \equiv 2(4) \\ c \equiv 3(4),\ 2n \equiv 0(4) \end{cases}, \qquad q = 2 \text{ if } \begin{cases} d-c \equiv 2(4) \\ 2n \equiv c-1 \equiv 0(4) \end{cases}$$

and $q=1$ *in all other cases. Here* $e=e_{\mathbf{C}}(\theta)$.

Note that if $a=0$, then $e=0$.

(4.9) COROLLARY. (i) *The quotients*

$$\mathrm{Hyp}_{4n}/R\,\mathrm{Im}\,J_{4n} \hookrightarrow MU_{4n}/R\,\mathrm{Im}\,J_{4n}$$

all have $\mathbf{Z}_{r\sigma}$ *as a subgroup, where* $r=1$ *or* 4 *and* σ *is the order of* $\mathrm{Im}\,J \subset \pi_{2d-1}(S^{2c})$, *with c, d, n as above.*

(ii) $\frac{1}{4}[S^{8a+2}]^2 \in \mathrm{Hyp}_* \quad \forall a > 0$.

PROOF. For (ii), embed $S^{8a-2} \cup_\nu e^{8a+2}$ in S^{16a+5}.

To consider the other possible types for A_θ, we refer to (4.6) and (4.7). Firstly, let A have type B^u or C^u: the following cases are of interest.

(4.10) LEMMA. *Let* $\theta \in \pi_{8a+2b+1}(S^{2b})$ *have* $d_{\mathbf{R}}(\theta)=1$, *and suppose* $A_\theta \subset S^{8n+5}$, *where* $2b \equiv 0$ *or* $2(8)$. *Then* $\Psi(M_\theta)/R\,\mathrm{Im}\,J_{8n+4} \cong \mathbf{Z}_2$, *generated by*

$$\tfrac{1}{2}[S^{2b}][S^{8n+4-2b}] \quad \text{if } 2b \equiv 2(8)$$

$$\tfrac{1}{2}[S^{8a+2b+2}][S^{8(n-a)-2b+2}] \quad \text{if } 2b \equiv 0(8)\,.$$

Again referring to (4.6) and (4.7) we can conclude

(4.11) LEMMA. *For all other types of* A_θ, *and all other dimensions excluded from* (4.8) *and* (4.10), $\Psi(M_\theta)/R\,\mathrm{Im}\,J_* = 0$.

Now (4.8) and (4.10) may be combined:

(4.12) THEOREM. *The subset of* $\mathrm{Hyp}_*/R\,\mathrm{Im}\,J_*$ *realisable on hypersurfaces defined by* 2 *cell complexes consists of the cyclic subgroups described in* (4.8).

PROOF. It remains only to show that (4.10) actually describes a subgroup of (4.8).

For this, suppose we have an embedding

$$S^{8k+2} \cup_\mu e^{8l+4} \subset S^{8n+5}\,.$$

Then $n>k$ (even if $\mu=\eta$, i.e. $k=l$). So $\Psi(M_\mu)$ is generated by $\frac{1}{2}[S^{8k+2}][S^{8n-8k+2}]$.

But from (4.2), we may embed $S^{8k+2} \cup_{\nu} e^{8k+6}$ in S^{8n+5} $(n>k)$ such that $\Psi(M_\nu) \ni \frac{1}{4}[S^{8k+2}][S^{8n-8k+2}]$.

As stated, (4.12) has several unsatisfactory aspects. For example, we can offer no general results as to the smallest values of c and d for which a given e invariant is realisable on $A_\theta = S^{2c} \cup_\theta e^{2d}$. Moreover, even given such knowledge, we cannot specify the least n for which A_θ embeds in S^{4n+1}.

So far as the first shortcoming is concerned, we may of course assume that θ is a generator of $\operatorname{Im} J \subset \pi^S_{4k-1}$, and that A_θ is the mapping cone in the smallest stable dimension available, i.e. $A_\theta = S^{4k+2} \cup_\theta e^{8k+2}$. To deal with the second drawback we recall the following lemma, which is based on the join construction.

(4.13) LEMMA. *Any complex* $S^a \cup_\theta e^b$ *may always be embedded in* S^{a+b+1}.

We may then assemble a weaker, but more systematic and memorable version of (4.12).

(4.14) THEOREM. *In dimensions of the form* $12d+4r \geqq 20$, *where* $r=1$, 2 *or* 3 *and* $d=1,2,\ldots$, *the quotients*

$$\mathrm{Hyp}_*/R \operatorname{Im} J_* \hookrightarrow \mathrm{MU}_*/R \operatorname{Im} J_*$$

all have

$$\mathbf{Z}_{\sigma(1)} \oplus \ldots \oplus \mathbf{Z}_{\sigma(i)} \oplus \ldots \oplus \mathbf{Z}_{\sigma(d)}$$

as a subgroup, where $\sigma(i)$ *is the order of* $\operatorname{Im} J \subset \pi^S_{4i-1}$.

Moreover, each summand may be realised on a hypersurface defined by a 2 *cell complex; and each general element on a hypersurface defined by a wedge of* 2 *cell complexes.*

5. Epilogue.

Although the main purpose of section 4 is to study $\mathrm{Hyp}_*/R \operatorname{Im} J_*$, we have deliberately included $\mathrm{MU}_*/R \operatorname{Im} J_*$ in the statement of our results, since we previously had little information concerning the latter quotient.

This may, in theory, be thoroughly investigated via $\mathrm{KU}_*(\mathrm{BU})$ and the Hattori–Stong theorem. Of course, a full scale analysis of the situation by these means is a formidable algebraic undertaking, involving a detailed study of the coaction primitives in $\mathrm{KU}_*(\mathrm{BU})$.

However, the formulae are clearly of interest, and we hope to examine them more closely in future, with the aid of recent calculations by Francis Clarke.

We conclude by noting that several simple questions on Hyp_* remain unanswered. Here are a sample two which have persistently eluded us.

(5.1) PROBLEMS.

(i) Does $\frac{1}{2}[S^{8k+2}][S^{8l+6}]$ lie in Hyp_* for any k, l?

(ii) Does $\frac{1}{4^r}[S^{8k_1+2}]\ldots[S^{8k_r+2}]$ lie in Hyp_* for all $k_1, \ldots, k_r$ and every r?

We can in fact partially answer (i) by proving that such classes are not representable on any hypersurface which is stably a wedge of 2 cell complexes. We can also partially answer (ii) by proving that such classes do indeed arise on hypersurfaces for a large range of integers $k_1, \ldots, k_r$ and r.

REFERENCES

1. J. F. Adams, *On the groups $J(X)$*, IV, Topology 5 (1966), 21–71.
2. A. J. Baker, Ph. D. Thesis, Manchester University, 1980.
3. G. Cooke, *Embedding certain complexes up to homotopy type in euclidean space*, Ann. of Math. 90 (1969), 144–156.
4. P. J. Eccles, *Filtering framed bordism by embedding codimension*, J. London Math. Soc. (2) 19 (1979), 163–169.
5. J. P. May, *E_∞ ring spaces and E_∞ ring spectra* (Lecture Notes in Mathematics 577), Springer-Verlag, Berlin - Heidelberg - New York, 1977.
6. J. Milnor, *On the cobordism ring and a complex analogue*, Amer. J. Math. 82 (1960), 505–521.
7. J. Milnor, *Construction of universal bundles* II, Ann. of Math. 63 (1956), 430–436.
8. N. Ray, *Bordism J-homomorphisms*, Illinois J. Math. 18 (1974), 290–309.
9. N. Ray, R. M. Switzer, and L. Taylor, *G structures, G bordism and universal manifolds*, Mem. Amer. Math. Soc. 193 (1977), 1–27.
10. R. M. Switzer, *Algebraic topology-homotopy and homology* (Die Grundlehren der Mathematischen Wissenschaften 212), Springer-Verlag, Berlin - Heidelberg - New York, 1975.
11. R. M. Wood, *Framing the exceptional Lie Group* G_2, Topology 15 (1976), 303–320.

MATHEMATICS DEPT.
THE UNIVERSITY
MANCHESTER M13 9PL
ENGLAND