MOTION OF LINKS IN THE 3-SPHERE

D. L. GOLDSMITH

Abstract.

A motion in M of a subspace N consists of an (ambient) isotopy of N through M which ultimately returns N to itself. Here we study the problem of determining all essentially different motions and the natural group structure on this set which is induced when two motions are multiplied by performing them on N in succession.

The aim of this paper is to calculate the group of motions of links in the 3-sphere. For "links with generalized axis", this is reduced to a calculation of isotopy classes of homeomorphisms of a surface punctured in a finite number of points. One example of a link with generalized axis is any torus link. Here, I give generators, and a complete set of defining relations for the group of motions of torus links in S^3.

Introduction.

A motion in M of a subspace N basically consists of an isotopy of N through M which ultimately returns N to itself. The problem which needs to be studied, is that of determining all essentially different ways in which this can be done, and then investigating the natural group structure on this set which is induced when two motions are multiplied by performing them on N in sucession. Clearly symmetries of N in M induce motions of N in M, and conversely, motions exhibit symmetries of the particular embedding of N in M.

This notion was probably first introduced with the fundamental group of a space, which, for a manifold, is the group of motions of a point in the space. If the subspace N which is being moved has several components, then the motion group takes into account not only the relationship of the subspace to the ambient space, but also the relationship of the components to each other, and their interaction during the motion. For example, although the motion group of a single point in a 2-disk is trivial, the motion group of two points in a 2-disk is Z, generated by the motion moving one point around the other, and in general, the motion group of n points in a 2-disk is the group of braids with n strings ([1] and [12]).

In fact, the concept of a motion originates from that of a braid. It was the idea of Hurwitz, and later of Fox ([16]), to envisage a braid as a continuous 1-parameter family of changing configurations of n distinct points in the xy-plane, where at each time t_0 the configuration is given by the intersection of the braid with the plane at height $z = t_0$. Later, David Dahm, a student of Fox, developed a general theory of motions "with compact support" of one space in another, and calculated the group of motions of a trivial link in Euclidean 3-space (see [12].

In this paper, we calculate generators and relations for the group of motions of all torus links in S^3. I prove:

THEOREM 8.7. *Generating motions for the group of motions of the type* (np, nq) *torus link, L, in* S^3, *are*:

1. $\sigma_i(p, q)$, $1 \leq i \leq n - 1$: *interchange the i-th and* $(i + 1)$-*st components*.
2. $\varrho_i(p, q)$, $1 \leq i \leq n$: *rotate the i-th component by* $2\pi/p$ *about its first axis of symmetry,* α.
3. f: *flip L by a* π-*rotation of* S^3, *which carries the link L to itself with reversed orientation*.

Theorem 8.7 also gives a complete set of defining relations for the motion group of L, but since this is a rather lengthy list, I will not include it in this introduction. It may help the reader to mention that most of the relations involving the generators ϱ_i, $1 \leq i \leq n$, and σ_i, $1 \leq i \leq n - 1$, are identical with the defining relations for the subgroup of 0-pure braids, in the classical group of braids with $n + 1$ strings, and that the element f, which is either of order two or four, depending on p and q, provides a normal extension of this subgroup by either 1, or $\mathbb{Z}/2\mathbb{Z}$, accordingly.

The torus links are a special case of the more general problem, which is also solved here, of finding a presentation for the group of motions of the link $L \cup \gamma$ in S^3, where γ is a fibred knot, and $L \subset S^3 - \gamma$ is any link which winds around the fibration, in the same manner that a closed braid winds around its axis. The knot is called a "generalized axis" for L.

Suppose the fibration $S^3 - \gamma \to S^1$ has fiber F, and monodromy $\psi: F \to F$. Suppose also that $L \cap F = P$ (a finite set of points) and $\psi: (F, P) \to (F, P)$. Our main result is that the group of motions $\mathcal{M}$ which takes L to L, and γ to γ, is defined by an exact sequence:

$$1 - \mathcal{H}_\psi(F, P)/[\psi], [2\pi] \to \mathcal{M} \to \mathbb{Z}/2\mathbb{Z} ,$$

where $\mathcal{H}_\psi(F, P)$ is the subgroup of the homeotopy group of the surface F punctured by P, which centralizes the element ψ. The motion $[2\pi]$ is defined in the text of the paper.

This paper is divided into eight sections:

Section 1 defines the group of motions of N in M. (For a more complete explantion of motion groups, see [20]).

Section 2 constructs a model for torus links, and establishes all the notation which is used in succeeding sections.

Section 3 computes the order of the motion $[2\pi]$ of L in S^3, and in R^3.

The concept of a generalized axis for a link is defined in Section 4, and it is proved that every component of a torus link is a generalized axis for the rest of the link.

In Section 5, I compute the homeotopy group $\mathscr{H}^+(S^3, L, \gamma)$ of orientation preserving homeomorphisms of the 3-sphere, which fix L and γ setwise. This group is shown to be isomorphic to $\mathscr{H}_\psi(F, P)$, which was mentioned earlier.

The group of motions $\mathscr{M}(S^3, L, \gamma)$ is computed in Section 6. Motions of L in $S^3 - \gamma$ are shown to extend braiding motions of the points P in the fiber surface F.

In Section 7, I compute the centralizer of the class $[\psi]$ of the monodromy map ψ for particular generalized axes of torus links.

The manifolds in this paper are assumed (without loss of generality) to carry P.L. and differentiable structures, and submanifolds are assumed to be both P.L. and differentiable submanifolds; thus, I will alternately make use of regular neighborhoods, and of tubular neighborhoods. However, homeomorphisms and isotopies are assumed to be in the topological category.

1. Motion groups.

The discussion of motion groups which follows, is preliminary to the calculation of groups of motions of links in S^3. For a detailed treatment, with extensive motivation, the reader is referred to "The Theory of Motion Groups" ([20]). Section 1 is meant to be a brief summary of the above paper, with proofs included if they are short.

Let M^m be an m-dimensional orientable manifold without boundary, and let $N_1, \ldots, N_n \subset M$ be compact subspaces.

NOTATION 1.1. $H(M)$ is the group of homeomorphisms of M, with the compact open topology.

$H_c(M)$ is the topological group of homeomorphisms $h \in H(M)$ which have compact support in M.

$H(M, N_{,1}, \ldots, N_n)$ is the subgroup of $h \in H(M)$ such that $h(N_i) = N_i$ for all $i = 1, \ldots, n$.

$H_c(M, N_1, \ldots, N_n)$ is the subgroup of $h \in H(M, N_1, \ldots, N_2)$ which have compact support in M.

$\mathscr{H}(M)$ is the group $\pi_0(\mathrm{H}_c(M); 1_M)$.

$\mathscr{H}^f(M)$ is the group $\pi_0(H(M); 1_M)$.

$\mathscr{H}(M, N_1, \ldots, N_n)$ is the group $\pi_0(H_c(M, N_1, \ldots, N_n); 1_M)$.

$\mathscr{H}^f(M, N_1, \ldots, N_n)$ is the group $\pi_0(H(M, N_1, \ldots, N_n); 1_M)$.

$\mathscr{H}^+(M)$ is the subgroup of $\mathscr{H}(M)$ represented by orientation preserving homeomorphisms.

$$\mathscr{H}^+(M, N_1, \ldots, N_n) = \mathscr{H}(M, N_1, \ldots, N_n) \cap \mathscr{H}^+(M).$$

DEFINITION 1.2. A motion of $(N_1, \ldots, N_n)$ in M is a path f in $H_c(M)$ such that $f(0) = 1_M$ and $f(1) \in H_c(M, N_1, \ldots, N_n)$. (We will denote $f(t)$ by f_t.)

DEFINITION 1.3. A *stationary motion of* $(N_1, \ldots, N_n)$ *in* M is a motion f such that for all $t \in [0,1]$, $1 \leq i \leq n$, $f_t(N_i) = N_i$.

DEFINITION 1.4. Let f, g be motions of N in M. f is equivalent to g (denoted $f \equiv g$) if $g^{-1} \circ f$ is homotopic (denoted $\simeq$), modulo endpoints, to a stationary motion. (g^{-1} and the operation $\circ$ are defined in 1.5.)

DEFINITION 1.5. A *relative fundamental group of a topological group G relative to a subgroup S of G and based at the unit 1_G of G*, denoted by $\pi_1(G, S; 1_G)$, is defined as follows:

A path f in G is a continuous map of the unit interval $[0, 1]$ into G; the image

of $t \in [0,1]$ under this map will be denoted f_t. We say a path in G is based at $e \in G$ if e is the initial endpoint of the path. A homotopy between two paths f and g in G whoch share the same initial and terminal endpoints, respectively, is a homotopy modulo end-points. Two paths f and g in G are multiplied by translating g until its initial endpoint coincides with the terminal endpoint of f; then the product, denoted $g \circ f$, takes on the values f_{2t} on the interval $0 \leq t \leq 1/2$, and $g_{2(t-1/2)} \circ [g_0]^{-1} \circ f_1$ on the interval $1/2 \leq t \leq 1$. The inverse f^{-1} of a path f is defined by $(f^{-1})_t = f_{1-t} \circ (f_1)^{-1}$.

Now consider the set of all 1_G-based paths in G whose terminal endpoints lie in the subgroups S. These are called 1_G-based relative loops in (G, S). An example is the trival loop, which is the constant path mapping all of $[0,1]$ to 1_G. Say that a 1_G-based relative loop in (G, S) is equivalent to the trivial loop if it is homotopic to a relative loop in the subspace S, and in general that two 1_G-based relative loops in (G, S), f and g, are equivalent if $g^{-1} \circ f$ is homotopic to a relative loop in S. Then the classes of relative 1_G-based loops in (G, S) which are equivalent by this equivalence relation form a group under the multiplication induced by multiplication of paths, and this group is called the relative fundamental group of $(G, S, 1_G)$.

REMARK. If f, g and 1_G-based, relative loops in (G, S), let $g \cdot f$ denote the path $(g \cdot f)_t = g_t \circ f_t$. Then $g \cdot f \simeq g \circ f$.

THEOREM 1.6. *The set of equivalence classes of motions of* $(N_1, \ldots, N_n)$ *in* M, *with multiplication induced by composition* $\circ$, *forms a group.* (*We will denote this group by* $\mathcal{M}(M, N_1, \ldots, N_n)$.)

PROOF. Clearly $\mathcal{M}(M, N_1, \ldots, N_n)$ is the relative fundamental group $\pi_1(H_c(M), H_c(M, N_1, \ldots, N_n); 1_M)$.

PROPOSITION 1.7. *Let* f, g *be motions of* N *in* M. *Then* $f \equiv g$ *if and only if* $f \simeq f'$, *where* f' *is a motion of* $(N_1, \ldots, N_n)$ *in* M *such that for all* $t \in [0,1]$, $1 \leq i \leq n$, $f'_t(N_i) = g_t(N_i)$.

PROOF. See Proposition 2.5 of [20].

EXAMPLE 1.8. The group of motions $\mathcal{M}(M, p)$ of a point p in a manifold M is the fundamental group $\pi_1(M; p)$ based at p.

EXAMPLE 1.9. The group of motions $\mathcal{M}(M^m, P_n)$ of n distinct points $P_n = \{p_1, \ldots, p_n\} \subset M^m$ in M^m, is the n-braid group $B_n(M^m)$. (See [7].) If $m > 2$, this is just $\oplus_{i=1}^n \pi_1(M; p_i)$. (See [12].)

PROOFS OF 1.8, 1.9. See 2.7–2.10 in [20].

DEFINITION 1.10. The homomorphism

$$\partial: \mathscr{M}(M, N_1, \ldots, N_n) \to \mathscr{H}^+(M, N_1, \ldots, N_n)$$

is defined $\partial([f]) = [f_1]$, where f is a motion of $(N_1, \ldots, N_n)$ in M.

THEOREM 1.11. *The following sequence is exact:*

$$\pi_1(H_c(M, N_1, \ldots, N_n); 1_M) \to \pi_1(H_c(M); 1_M) \to \mathscr{M}(M, N_1, \ldots, N_n)$$

$$\xrightarrow{\partial} \mathscr{H}^+(M, N_1, \ldots, N_n) \xrightarrow{i} \mathscr{H}^+(M) .$$

PROOF. This is just the long exact sequence for relative homotopy groups.

COROLLARY 1.12. *If $M = \mathbf{R}^n$, then the sequence:*

$$1 \to \mathscr{M}(M, N_1, \ldots, N_n) \xrightarrow{\partial} \mathscr{H}^+(M, N_1, \ldots, N_n) \to 1$$

is exact.

PROOF. $H_c(\mathbf{R}^3)$ is contractible, by the Alexander-Tietze theorem (see [6], [25]).

COROLLARY 1.13. *If $M = S^n$, the sequence:*

$$\mathbf{Z}_2 \to \mathscr{M}(M, N_1, \ldots, N_n) \xrightarrow{\partial} \mathscr{H}^+(M, N_1, \ldots, N_n) \to 1$$

is exact.

PROOF. $\pi_1(H(S^n); 1_{S^n}) = \mathbf{Z}_2$ (see [17]). It is generated by a 2π rotation of S^n, which I will denote by $[2\pi] \in \pi_1(H(S^n); 1_{S^n})$.

$$\pi_0(H(S^n); 1_{S^n}) = \mathscr{H}^+(S^n) = 1 .$$

THEOREM 1.14. *If $h \in H_c(M)$, then*

$$\mathscr{M}(M, N_1, \ldots, N_n) \simeq \mathscr{M}(M, h(N_1), \ldots, h(N_n)) .$$

PROOF. h defines a homeomorphism

$$(H_c(M), H_c(M, N_1, \ldots, N_n), 1_M) \to (H_c(M), H_c(M, h(N_1), \ldots, h(N_n)), 1_M) ,$$

by $h' \to hh'h^{-1}$ whenever $h' \in H_c(M)$. Since $\mathscr{M}(M, N_1, \ldots, N_n)$ $= \pi_1(H_c(M), H_c(M, N_1, \ldots, N_n); 1_M)$, Theorem 1.14 follows.

COROLLARY 1.15. *If there is an ambient isotopy from the n-tuple* $(N_1, \ldots, N_n)$ *to an n-tuple* $(N'_1, \ldots, N'_n)$ *of subspaces* $N_i \subset M$, *then*

$$\mathscr{M}(M, N_1, \ldots, N_n) \simeq \mathscr{M}(M, N'_1, \ldots, N'_n) .$$

2. Torus links.

This section will not only assemble specific models for torus links, but it will establish much of the notation which will be used in the chapters which follow.

NOTATION 2.1. Let $\mathbf{R}^2 \simeq \mathbf{C}^1 = \{re^{i\theta} : 0 \leq r < \infty, \ \theta \in \mathbf{R}\}$ be identified with the conplex plane, and let $D^2 = \{re^{i\theta} : 0 \leq r \leq 1\}$ and $S^1 = \{e^{i\theta} : \theta \in \mathbf{R}\}$ be the complex unit disk and the complex unit circle, respectively.

Let the 3-sphere be the union $\mathbf{C} \times S^1 \cup S^1 \times \mathbf{C}$ of solid tori, with the identifications $(re^{i\theta}, e^{i\varphi}) \sim (e^{i\theta}, (1/r)e^{i\varphi})$ if $r > 0$. Further, define:

$T = S^1 \times S^1 \subset S^3$ is the standard torus.

α is the circle $(0, e^{i\varphi})$, $0 \leq \varphi \leq 2\pi$, in S^3.

β is the circle $(e^{i\theta}, 0)$, $0 \leq \theta \leq 2\pi$, in S^3.

$\pi_p \colon \mathbf{C} \to \mathbf{C}$ is the map $re^{i\theta} \to re^{ip\theta}$.

$\pi_{p,q} \colon S^3 \to S^3$ is induced by $\pi_p \times \pi_q$.

(Note that $\pi_{p,q}$ is a branched covering of S^3 by S^3, whose branch curves are β with branching index q, and α with branching index p.)

$r_t \colon \mathbf{C} \to \mathbf{C}$ is rotation by $2\pi t$.

$r_{s,t} \colon S^3 \to S^3$ is the rotation induced by $r_s \times r_t$.

$i \colon \mathbf{R}^2 \times S^1 \to S^3$ is inclusion.

$P_n = \{$complex nth roots of unity$\}$.

$$L_{(np, nq)} = \bigcup_{k=1}^{n} i\left\{\left(\frac{1}{k} r_{qt/p}(P_p), e^{2\pi i t}\right), 0 \leq t \leq 1\right\}, \quad \text{where g.c.d. } (p,q) = 1.$$

We will need to use maps with compact support in section 4 on generalized axes. Therefore, define:

$\varphi_t \colon \mathbf{C} \to \mathbf{C}$ is the rotation with compact support given by:

$$\varphi_t = \begin{cases} r_t(z) & \text{if } |z| \leq 1 \\ r_{(1-s)t} & \text{if } |z| \geq 1, \text{ and } s = \min\{|z| - 1, 1\} \end{cases}$$

REMARK 2.2. The diagram

$$\begin{array}{ccc} L_{(np, nq)} & \to & S^3 \\ \downarrow \pi_{p,q} & & \downarrow \pi_{p,q} \\ L_{(n, n)} & \longrightarrow & S^3 \end{array}$$

is commutative.

DEFINITION 2.3. A torus knot of type (p, q), where g.c.d. $(p, q) = 1$, is a simple closed curve on the standard torus $T \subset S^3$, which winds p times in the direction of the longitude $1 \times S^1$, and q times in the direction of the meridian $S^1 \times 1$.

DEFINITION 2.4. A torus link of type (np, nq), where g.c.d. $(p, q) = 1$, is a collection of n disjoint, parallel torus knots of type (p, q) on the torus $T \subset S^3$ (see Figure 2.5).

Figure 2.5. $L_{(6, 8)}$.
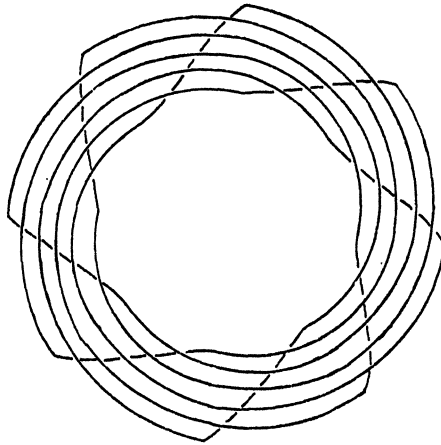
REMARK. Note that $L_{(np, nq)}$ is a torus link of type (np, nq).

NOTATION 2.6. The components of $L_{(n, n)}$ will be labeled:

$$C_k = i \left\{ \left(\frac{1}{k} e^{2\pi i t}, e^{2\pi i t} \right) : 0 \leqq t \leqq 1 \right\}, \quad 1 \leqq k \leqq n .$$

The components of $L_{(np, nq)}$ will be labeled:

$$K_k = i \left\{ \left(\frac{1}{k} r_{qt/p}(P_p), e^{2\pi i t} \right) : 0 \leqq t \leqq 1 \right\}, \quad 1 \leqq k \leqq n .$$

3. The motion $\{2\pi\}$.

Let $L \subset S^3$ be a link in the 3-sphere. The motion $\{2\pi\}$ which rotates S^3 one full rotation, generates the kernel of the map $\partial: \mathcal{M}(S^3, L) \twoheadrightarrow \mathcal{H}^+(S^3, L)$, defined in 1.10. In this section, I calculate this important subgroup.

NOTATION 3.1. $\{2\pi\}$ is the 1-based loop in $H(S^3)$ which generates

$\pi_1(H(S^3); 1_{S^3})$. Its equivalence class is denoted $[2\pi] \in \mathcal{M}(S^3, L)$.

Let $* \in S^3 - L$ be a point on the fixed point set of the rotation $\{2\pi\}$; let $D \subset S^3 - L$ and $D' \subset D$ be regular neighborhoods of $*$ which are left invariant by $\{2\pi\}$. There is a motion of L in $\mathbf{R}^3$ which coincides with $\{2\pi\}$ on $S^3 - D$, and which is constant on $D' - *$. We will denote this motion by $\{2\pi\}$ as well, and its equivalence class by $[2\pi] \in \mathcal{M}(\mathbf{R}^3, L)$. Note that $\partial([2\pi]) = \{2\pi\}_1$ has its support in $D - D'$.

THEOREM 3.2. $[2\pi] \in \mathcal{M}(\mathbf{R}^3, L)$ is trivial if and only if L is the trivial link.

PROOF. Suppose $1 = [2\pi] \in \mathcal{M}(\mathbf{R}^3, L)$. Then there is a stationary motion $f_t \in H_c(\mathbf{R}^3, L)$ such that $f_1 = \partial([2\pi]) = \{2\pi\}_1 \in H_c(\mathbf{R}^3, L)$.

Let N be a closed tubular neighborhood of L in S^3; let $N' \subset \operatorname{int} N$ be a sufficiently small, closed tubular neighborhood of L, that for all t, $f_t(N') \subset N$. By [21], we may assume $f_1 | N' = \operatorname{id}$.

By [13], there is a motion g of L in $\mathbf{R}^3$ such that for all t, $g_t | (\mathbf{R}^3 - N) = \operatorname{id}$ and $g_t | N' = f_t | N'$. Let $h_t = g_t^{-1} \circ f_t$. Then h is a motion of L in $\mathbf{R}^3$ satisfying for all t, $h_t | N' = \operatorname{id}$; also, $f = g \cdot h$.

If L is not the trivial link, there is some component K_i of L such that $\pi_1(N_i - K_i) \xrightarrow{i^*} \pi_1(S^3 - L)$ is injective, where $N_i \subset N$, $N'_i \subset N'$, are the components of N, N', respectively, containing K_i, and where the inclusion-induced map i_* is obtained by connecting N_i to the basepoint of $\mathbf{R}^3$.

Now any homeomorphism $\varrho \in H_c(\mathbf{R}^3, L)$ such that $\varrho(N_i) = N_i$ and

$$\varrho_*(i_*[\pi_1(N_i - K_i)]) = i_*[\pi_1(N_i - K_i)] ,$$

which induces the *identity* automorphism

$$\sigma_* : \pi_1(\mathbf{R}^3 - L) \to \pi_1(\mathbf{R}^3 - L) ,$$

must induce the *identity* automorphism

$$(\varrho | N_i - K_i)_* : \pi_1(N_i - K_i) \to \pi_1(N_i - K_i)$$

as well. Since $\{2\pi\}_1 = g_1 \circ h_1$ and h_1 both induce the identity automorphism of $\pi_1(\mathbf{R}^3 - L)$, so does g_1. Furthermore, g_1 satisfies all of the same conditions as ϱ, above. Therefore, $(g_1 | (N_i - K_i))_* : \pi_1(N_i - K_i) \to \pi_1(N_i - K_i)$ is the identity automorphism.

Now by Claim 5.25, $g_1 | (N_i - N'_i) \simeq \operatorname{id}$, rel. boundary. Consequently, $\{2\pi\}_1 \simeq \operatorname{id}$ in $H_c(\mathbf{R}^3)$ by an isotopy which is constant on N'_i. This is impossible, by [17].

So L must be the trivial link.

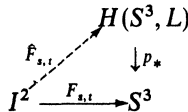We will now investigate the motion $[2\pi] \in \mathcal{M}(S^3, L)$.

NOTATION 3.3. Let $p_* \colon H(S^3) \to S^3$ be the map $p_*(h) = h(*)$. The map p_* is a fiber bundle, with fiber $p_*^{-1}(*) = H(R^3)$.

In general, SO (n) denotes the space of rotations of S^{n-1}. The restriction $p_* | \text{SO}(n) \colon \text{SO}(n) \to S^{n-1}$ is a fiber bundle with fiber $[p_* | \text{SO}(n)]^{-1}(*)$ $= \text{SO}(n-1)$.

LEMMA 3.4. *Let the 1_{S^3}-based loop $f_t \in H(S^3, L)$ be a stationary motion of L in S^3 (see Definition 1.3). Suppose $p_* \circ f \simeq *$ in $S^3 - L$.*
Then $f \simeq f'$ in $H(S^3, L)$ such that for all t, $f_t'(*) = *$.

PROOF. Let $F_{s,t} \in S^3 - L$ be a homotopy from $F_0 = p_* \circ f$ to $F_1 = *$. Then in the commutative diagram:

$$
\begin{array}{ccc}
 & & H(S^3, L) \\
 & \nearrow^{\hat{F}_{s,t}} & \downarrow^{p_*} \\
I^2 & \xrightarrow{\ F_{s,t}\ } & S^3
\end{array}
$$

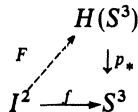there is a lift $\hat{F}_{s,t}$ of $F_{s,t}$ which satisfies

(1) $\hat{F}_0 = f$
(2) $F_{s,0} = F_{s,1} = 1_{S^3}$.

Then $f' = F_1$ is the desired loop in $H(S^3, L)$.

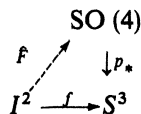LEMMA 3.5. *Let $f \colon [0,1] \times [0,1] \to S^3$ satisfy*

(1) $f(s,t) = *$ *if $(s,t) \in \partial I^2$*
(2) $f(s,t) = *$ *if $\frac{1}{2} \leq s \leq 1$.*

Then in the commutative diagram

$$
\begin{array}{ccc}
 & & H(S^3) \\
 & \nearrow^{F} & \downarrow^{p_*} \\
I^2 & \xrightarrow{\ f\ } & S^3
\end{array}
$$

there is a lift $F \colon [0,1] \times [0,1] \to H(S^3)$ satisfying $F(s,t) = 1_{S^3}$ if $(s,t) \in \partial I^2$.

PROOF. In the commutative diagram

$$
\begin{array}{ccc}
 & & \text{SO}(4) \\
 & \nearrow^{\hat{F}} & \downarrow^{p_*} \\
I^2 & \xrightarrow{\ f\ } & S^3
\end{array}
$$

f lifts to a map $\hat{F} \colon I^2 \to \text{SO}(4)$ satisfying

(1) $\hat{F}_{s,t} = 1_{S^3}$ if $t = 0, 1$, or $s = 0$

(2) $\hat{F}_{s,t} = \hat{F}_{s',t}$ if $\frac{1}{2} \leq s, s'$.

Now f represents an element $[f] \in \pi_2(S^3; *)$. According to the homotopy exact sequence for a fibration:

$$\ldots \to \pi_2(SO(3)) \to \pi_2(SO(4)) \to \pi_2(S^3; *) \xrightarrow{L} \pi_1(SO(3); 1_{S^3}) \to \ldots,$$

$[\hat{F}_1] = L([f])$. Since $\pi_2(S^3) \simeq 1$, then L is the trivial homomorphism, and $\hat{F}_1 \simeq 1_{S^3}$ (rel. endpoints) in $SO(3) \subset H(\mathbf{R}^3) \subset H(S^3, *)$.

Let $G_{s,t} : [\frac{1}{2}, 0] \times [0, 1] \to H(S^3)$ be a homotopy from $G_{\frac{1}{2}} = \hat{F}_1 \ (= \hat{F}_s$ for all $s \geq \frac{1}{2})$ to $G_1 = 1_{S^3}$. Define a homotopy $F_{s,t} \in H(S^3, *)$ by

$$F_{s,t} = \begin{cases} \hat{F}_{s,t} & \text{if } 0 \leq s \leq \frac{1}{2} \\ G_{s,t} & \text{if } \frac{1}{2} \leq s \leq 1 \,. \end{cases}$$

Then $F_{s,t}$ is the desired lift of f.

LEMMA 3.6. *Let f be a 1_{S^3}-based loop in $H(S^3, L)$. Then $[p_* \circ f]$ $\in$ Center $(\pi_1(S^3 - L; *))$.*

PROOF. Let α_s be the path $f_{s,t}(*)$, $t \in [0, 1]$, in S^3. For each $*$-based loop γ in $S^3 - L$, $\alpha_s[f_t(\gamma)]\alpha_s^{-1}$ is a homotopy in $S^3 - L$ from γ to $\alpha_1 \gamma \alpha_1^{-1}$ (rel. $*$). Since $\alpha_1 = p_* \circ f$, this completes the proof.

THEOREM 3.7. *Let L be one of the following links:*

(1) $L_{(np, nq)}$

(2) $L_{(np, nq)} \cup \alpha$

(3) $L_{(np, nq)} \cup \beta$

(4) $L_{(np, nq)} \cup \alpha \cup \beta$,

where $L_{(np, nq)}$ is the torus link of type (np, nq) and $\alpha, \beta \subset S^3 - L_{(np, nq)}$ are the axes described in Notation 2.1, and where $p + q$ is odd, or $n = 0$.
   Then $1 = [2\pi] \in \mathcal{M}(S^3, L)$. Otherwise, $[2\pi] \in \mathcal{M}(S^3, L)$ has order two.

PROOF. Suppose $[2\pi]$ is trivial in $\mathcal{M}(S^3, L)$. Let $F : [0, 1] \times [0, 1] \to H(S^3)$ be a homotopy from $F_0 = \{2\pi\}$ to a stationary motion F_1 of L in S^3.
   By Lemma 3.6, $[p_* \circ F_1] \in$ Center $(\pi_1(S^3 - L))$. There are two cases to consider:

CASE 1: $\pi_1(S^3 - L)$ has trivial center.

In this case, $p_* \circ F_1 \simeq *$ in $S^3 - L$. By Lemma 3.4, $F_1 \simeq f'$ in $H(S^3, L)$, such that for all $t, f_t'(*) = *$. Let $\hat{F}_{s,t} \in H(S^3, L)$ be the homotopy from $\hat{F}_0 = F_1$ to $\hat{F}_1 = f'$.

Define a homotopy $G: [0,1] \times [0,1] \to H(S^3)$ from $G_0 = \{2\pi\}$ to $G_1 = f'$ by:

$$G_{s,t} = \begin{cases} F_{(4s,\,t)} & \text{if } 0 \leq s \leq \frac{1}{4} \\ \hat{F}_{(4[s-\frac{1}{4}],\,t)} & \text{if } \frac{1}{4} \leq s \leq \frac{1}{2} \\ \hat{F}_{(1,\,t)} & \text{if } \frac{1}{2} \leq s \leq 1 \ . \end{cases}$$

Then $p_* \circ G$ satisfies hypotheses (1), (2) of Lemma 3.5. By Lemma 3.5, there is a homotopy $\hat{G}_{s,t} \in H(S^3)$ such that $\hat{G}_{(s,t)} = 1_{S^3}$ if $(s,t) \in \partial I^2$, and $p_* \circ \hat{G} = p_* \circ G$. Let $H = G \circ \hat{G}^{-1}$.

Now H is a homotopy from $H_0 = \{2\pi\}$ to $H_1 = f'$ (where for all t, $f'_t \in H(S^3, L)$) such that for all $s,t \in [0,1]$, $H_{s,t}(*) = *$. By the Main Theorem of [13], we can find $\hat{H}: [0,1] \times [0,1] \to H_c(S^3 - *, L)$ such that for all $s,t \in [0,1]$, $\hat{H}_{s,t}|L = H_{s,t}|L$.

Finally, $\hat{H}_{s,t} \in H_c(S^3 - *)$ is a homotopy from $\hat{H}_0$ to a stationary motion $\hat{H}_{1,t} \in H_c(S^3 - *, L)$ of L in $S^3 - *$. Since $\hat{H}_{0,t}|L = \{2\pi\}_t|L$ for all t, then by Proposition 1.7, $\hat{H}_0$ and $\{2\pi\}$ are equivalent motions of L in $S^3 - *$. Therefore $1 = [\hat{H}_0] = [2\pi] \in \mathcal{M}(S^3 - *, L)$. But then L must be the trivial link in S^3, by Theorem 3.2.

CASE 2. $\pi_1(S^3 - L)$ has non-trivial center.

According to [24], L must be one of the links (1)–(4), with no restriction on $p + q$.

Let $S^3 = C \times S^1 \cup S^1 \times C/\sim$, as in Notation 2.1. Define the rotation $r_{s,t} : S^3 \to S^3$ by

$$r_{s,t}(z_1, z_2) = (z_1 e^{2\pi s i}, z_2 e^{2\pi t i})$$

if $(z_1, z_2) \in C \times S^1$ or $(z_1, z_2) \in S^1 \times C$. By Notation 2.1, $r_{t,t}(L_{(n,n)}) = L_{(n,n)}$ for all t; by Remark 2.2, $\pi_{p,q}(L_{(np,nq)}) = L_{(n,n)}$. Since

$$\pi_{p,q} \circ r_{qt,\,pt} = r_{pqt,\,pqt} \circ \pi_{p,q} \ ,$$

it follows that $r_{qt,\,pt}(L_{(np,nq)}) = L_{(np,nq)}$ for all t.

Now, the 1_{S^3}-based loop $r_{qt,\,pt} \in H(S^3, L_{(np,nq)}, \alpha, \beta), \bar{} \ t \in [0,1]$, is a *stationary motion* of $(L_{np,nq}) \cup \alpha \cup \beta$ in S^3. Let $R_{q,p}$ be the motion $R_{q,p}(t) = r_{qt,\,pt}$. Then $R_{q,p} \simeq [2\pi]^{p+q}$. If $p + q$ is odd, then (since $\pi_1(H(S^3); 1_{S^3}) \simeq Z_2$ is generated by $[2\pi]$, $1 = [2\pi] \in \mathcal{M}(S^3, L_{(np,nq)} \cup \alpha \cup \beta)$.

If $p + q$ is even, then $R_{q,p}$ is homotopic (rel. endpoints) to the constant loop 1_{S^3}. Let $* \in S^3 - L$. It is a well-known fact that if $n \neq 0$, the $*$-based loop $r_{qt,\,pt}(*)$, $t \in [0,1]$, represents a generator of Center $(\pi_1(S^3 - L_{(np,nq)}; *))$. Thus for some k, $p_* \circ F_1 \circ R^k_{q,p} \simeq *$ in $S^3 - L_{(np,nq)}$. In this case, let $F'_1 = F_1 \circ R^k_{q,p}$. Since $R_{q,p} \simeq 1_{S^3}$, then $F_1 \simeq F'_1$ in $H(S^3)$. Now proceed as in Case 1, for F'. We conclude that either L is the trivial link, or $[2\pi] \in \mathcal{M}(S^3, L)$ has order two.

This leaves the case when $n = 0$; thus, L is one of the links $\alpha, \beta, \alpha \cup \beta$. Clearly, in this case the motion $\{2\pi\}$ of L in S^3 is a stationary motion of L; therefore $1 = [2\pi] \in \mathcal{M}(S^3, L)$. This completes the proof.

4. A generalized axis for a link.

DEFINITION 4.1. A fibered knot γ is a *generalized axis* for a link $L \subset S^3 - \gamma$ if L intersects each fiber of some fibration, say $\pi: S^3 - \gamma \to S^1$, transversely, in a finite number of points.

Let $F = \pi^{-1}(1)$ be the fiber of the fibration $\pi: S^3 - \gamma \to S^1$, $\bar{F} = \text{Closure }(F)$ (so that $\partial \bar{F} = \gamma$) and suppose $F \cup L = P_n$ (a set of n distinct points). By the theory of fibrations, there is a surjective map

$$f: F \times I \to S^3 - \gamma$$

satisfying:
   (1) $f: F \times (0, 1) \to S^3 - F$ is a homeomorphism.
   (2) $f_0: F \to F$ is the identity and $f_1: F \to F$ is a homeomorphism with compact support.
   (3) f extends to the closure $\bar{F} \times [0, 1]$, and $f: \gamma \times [0, 1] \to \gamma$ is projection (where $\gamma = \partial \bar{F}$).

The map $f_1: F \to F$ is usually called the *monodromy* of the fibration. Note that f_1 actually fixes the set P_n.

DEFINITION 4.2. The map $f_1: (F, P_n) \to (F, P_n)$ is a *monodromy* for the generalized axis γ.

REMARK 4.3. If γ is unknotted, then $F = \mathbf{R}^2$ and γ is commonly called an *axis* for L.

REMARK 4.3. If γ is a generalized axis for L, then γ is also a generalized axis for any sublink of L, and the monodromy map is the same for the sublink as for the link.

In the following examples, all of the notation is explained in 2.1.

EXAMPLE 4.4. The circle β is an axis for the link $L_{(np, nq)} \cup \alpha$. A monodromy map is

$$\varphi_{q/p}: \left(\mathbf{R}^2, \left[\bigcup_{k=1}^{n} \frac{1}{k} P_p \right] \cup 0 \right) \to \left(\mathbf{R}^2, \left[\bigcup_{k=1}^{n} \frac{1}{k} P_p \right] \cup 0 \right).$$

EXAMPLE 4.5. *Let L be a torus link with $n > 1$ components. Then any one of the components is a generalized axis for the rest of the link.*

For the proof of Example 4.5, we need the following:

CLAIM 4.6. *There is an isotopy of $C_1 \cup \beta$ in $S^3 - [\bigcup_{i=2}^n C_i] - \alpha$ which interchanges C_1 and β.* (We will denote the end of this isotopy by:

$$g: (S^3, C_1, C_2, \ldots, C_n, \alpha, \beta) \to (S^3, \beta, C_2, \ldots, C_n, \alpha, C_1))$$

PROOF. Obvious.

REMARK 4.7. The map $g \circ \pi_{p,q}: S^3 \to S^3$ is a branched covering of the 3-sphere branched along the curve α with branching index p, and along the curve C_1 with branching index q.

DEFINITION 4.8. Let $\pi: F \to \mathbf{R}^2 - \{0, 1\}$ be the covering space corresponding to the representation $\pi_1(\mathbf{R}^2 - \{0, 1\}) \to \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}$, which is the composition:

$$\pi_1 \xrightarrow{\text{Ab.}} H_1(\mathbf{R}^2 - \{0, 1\}; \mathbf{Z}) \simeq \mathbf{Z} \oplus \mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \oplus \mathbf{Z}/q\mathbf{Z}$$

(where the generators $0 \oplus 1$, $1 \oplus 0$ of H_1 are represented by small circles about the points $0, 1$, respectively, and the last map is the quotient map).

NOTATION 4.9. Let $\pi: F \to \mathbf{R}^2 - \{0, 1\}$ be as in Definition 4.8.

$$\hat{P}_k = \pi^{-1}\left(\frac{1}{k}\right), \quad k = 1, \ldots, n.$$
$$A = \pi^{-1}(0).$$

PROOF OF 4.5. By the last remark in Section 2, we may assume $L = L_{(np, nq)}$ with components $K_1, \ldots, K_n$. It is sufficient to show that K_1 is a generalized axis for $K_2 \cup \ldots \cup K_n \subset S^3 - K_1$.

By Remark 2.2 and Notation 2.6, the diagram:

$$
\begin{array}{ccc}
K_1, \ldots, K_n, \alpha, \beta & \to & S^3 \\
\downarrow{\scriptstyle \pi_{p,q}} & & \downarrow{\scriptstyle \pi_{p,q}} \\
C_1, \ldots, C_n, \alpha, \beta & \to & S^3
\end{array}
$$

is commutative. Composing vertically with the map g defined in Claim 4.6, we obtain the diagram:

$$
\begin{array}{ccc}
K_1, K_2, \ldots, K_n, \alpha, \beta & \to & S^3 \\
\downarrow{\scriptstyle g \circ \pi_{p,q}} & & \downarrow{\scriptstyle g \circ \pi_{p,q}} \\
\beta, C_2, \ldots, C_n, \alpha, C_1 & \to & S^3
\end{array}
$$

This restricts to the commutative diagram:

$$K_2, \ldots, K_n \rightarrow S^3 - K_1$$
$$\downarrow g \circ \pi_{p,q} \qquad \downarrow g \circ \pi_{p,q}$$
$$C_2, \ldots, C_n \rightarrow S^3 - \beta$$

Since β is an axis for the link $C_2 \cup \ldots \cup C_n$, and since the branch set $\alpha \cup C$ of the branched covering $g \circ \pi_{p,q}$ is transverse to the fibration, it follows easily that K_1 is a generalized axis for $K_2 \cup \ldots \cup K_n$. (Proof: Pull back the fibration.)

REMARK 4.10. The proof of 4.5 actually shows that K_1 is a generalized axis for $K_2 \cup \ldots \cup K_n \cup \alpha \cup \beta$.

REMARK 4.11. The monodromy for K_1 in Remark 4.10 is the pullback Σ of the monodromy of the axis β for the link $C_1 \cup \ldots \cup C_n \cup \alpha$:

$$F, \left[\bigcup_{i=1}^{n} \hat{P}_i \right] \cup A \xrightarrow{\;\Sigma\;} F, \left[\bigcup_{i=1}^{n} \hat{P}_i \right] \cup A$$
$$\downarrow \pi \qquad\qquad\qquad \downarrow \pi$$
$$\mathsf{R}^2, \left[\bigcup_{k=1}^{n} \frac{1}{k} \right] \cup 0 \xrightarrow{\;\varphi_1\;} \mathsf{R}^2, \left[\bigcup_{k=1}^{n} \frac{1}{k} \right] \cup 0$$

5. The Group $\mathscr{H}^+(S^3, L, \gamma)$.

Let γ be a generalized axis for the link $L \subset S^3 - \gamma$. By Corollary 1.13 there is an exact sequence

$$\mathsf{Z}_2 \rightarrow \mathscr{M}(S^3, L, \gamma) \rightarrow \mathscr{H}^+(S^3, L, \gamma) \rightarrow 1 \; .$$

As our object is to eventually compute the motion group $\mathscr{M}(S^3, L, \gamma)$, it is clearly necessary to calculate the group $\mathscr{H}^+(S^3, L, \gamma)$.

Since Section 5 is rather technical, I will summarize it briefly. It is divided into three sections, as indicated below:

The group $\mathscr{H}(S^3 - \gamma, L)$.

The main theorem of this section is Theorem 5.20, which says that $\mathscr{H}(S^3 - \gamma, L)$ is isomorphic to a group of surface homeomorphisms. More precisely:

THEOREM 5.20. $J: \mathscr{H}_\psi(F, P) \rightarrow \mathscr{H}(S^3 - \gamma, L)$ *is an isomorphism.*

(Here $S^3 - \gamma$ fibers over the circle with fiber F, $P = F \cap L$, $\psi \in H_c(F, P)$ is a monodromy map for the generalized axis γ of $L \subset S^3 - \gamma$, and $\mathscr{H}_\psi(F, P) \subset \mathscr{H}(F, P)$ is the centralizer of the element $[\psi] \in \mathscr{H}(F, P)$.)

The proof of Theorem 5.20 is contained in a sequence of propositions. Rather than listing them here, may I suggest that the reader review the statements of Propositions 5.7, 5.9, 5.10, Construction 5.12, Proposition 5.14, and Proposition 5.16, to get the thread of the argument.

The map e: $\mathscr{H}(S^3 - \gamma, L) \to \mathscr{H}^+(S^3, L, \gamma)$.

In this section, a homomorphism is constructed from $\mathscr{H}(S^3 - \gamma, L)$ to the group which we wish to know. (e is just the map induced by extension). The main theorem is:

THEOREM 5.26. *The sequence*

$$1 \to \langle [\psi], [\tau] \rangle \to \mathscr{H}_\psi(F, P) \overset{J}{\cong} \mathscr{H}(S^3 - \gamma, L) \overset{e}{\longrightarrow} \mathscr{H}^+(S^3, L, \gamma)$$

is exact.

The group $\mathscr{H}^+(S^3, L, \gamma)$.

The group $\mathscr{H}^+(S^3, L, \gamma)$ is shown to be normal extension of a quotient group of the group $\mathscr{H}_\psi(F, P)$ of surface homeomorphisms, by the trivial group or $\mathbf{Z}_2$. The main theorem is:

THEOREM 5.28. *The sequence*

$$1 \to \langle [\psi], [\tau] \rangle \to \mathscr{H}_*(F, P) \overset{e \circ J}{\longrightarrow} \mathscr{H}^+(S^3, L, \gamma) \to \mathbf{Z}_2$$

is exact.

Now I will proceed with Section 5.

NOTATION. Let $f: F \times [0, 1] \twoheadrightarrow S^3 - \gamma$ be the map in Definition 4.1, and let $\psi = f_1$ be a monodromy for γ. (See Definition 4.2). Let $L = K_1 \cup \ldots \cup K_n$ have components K_i, and let $P_i = F \cap K_i$. Index the points in $P_i = \{p_{i1}, \ldots, p_{i, n(i)}\}$ so that $f(p_{ik} \times 1) = p_{i, k+1}$ if $1 \leq k < n(i)$. Let N be a closed, regular neighborhood of P in F; thus, N consists of a number of disjoint 2-disks N_{ij}, $1 \leq i \leq n$, $1 \leq j \leq n(i)$, such that $p_{ij} \in N_{ij}$. Then $f(N_i \times [0, 1])$ is a regular neighborhood of K_i in $S^3 - \gamma$. Finally, let $l_{ij} \subset F$ be a polygonal arc from p_{ij} to γ.

In order to compute $\mathscr{H}^+(S^3, L, \gamma)$, I will first compute $\mathscr{H}(S^3 - \gamma, L)$:

The group $\mathscr{H}(S^3 - \gamma, L)$.

NOTATION. Let $h \in H_c(S^3 - \gamma, L)$. By [11] we may assume that h is a differentiable homeomorphism, and that N, $f(N \times [0, 1])$, are differentiable

neighborhoods of P and L, respectively. Then, by the uniqueness of differentiable tubular neighborhoods ([21]) and the Main Theorem of [13], there is an isotopy $h_t \in H_c(S^3 - \gamma, L)$ from $h_0 = \mathrm{id}$ to h_1 such that $h_1 \circ h$ ($f(N \times [0,1])) = f(N \times [0,1])$; in particular, we may assume that for all $k = 1, \ldots, n(i)$, $t \in [0,1]$, there exist k', t' with

$$f(N_{ik} \times t) \xrightarrow{h_1 \circ h} f(N_{jk'} \times t'),$$

where j depends on i. By reindexing, we may assume that $h_1 \circ h(p_{i1}) = p_{j1}$ and, therefore, that for all $k = 1, \ldots, n(i)$, $h_1 \circ h(p_{ik}) = p_{jk}$. (A consequence of this is that $n(i) = n(j)$.) Finally, it is easy to arrange that

$$f(N_{ik} \times t) \xrightarrow{h_1 \circ h} f(N_{jk} \times t), \quad \text{for all } t.$$

By replacing h by $h_1 \circ h$, we will assume, henceforth, that h satisfies:

  (1) $h(p_{ik}) = p_{jk}$ for some j depending on $i, k = 1, \ldots, n(i)$.
  (2) $\forall t, \forall k, h(f[N_{ik} \times t]) = f[N_{jk} \times t]$, for j as in (1).

DEFINITION 5.1. The infinite cyclic covering space $p: F \times R^1 \to S^3 - \gamma$ is the composition

$$F \times R^1 \to F \times [0,1] \xrightarrow{f} S^3 - \gamma,$$

where the first map is $(x, t) \to (x, t (\mathrm{mod}\, 1))$.

DEFINITION 5.2. $r_t: F \times R^1 \to F \times R^1$ is the strong deformation retract from $F \times R^1$ to $F \times 0$ defined by $r_t(x, s) = (x, (1-t)s)$.

DEFINITION 5.3. Let λ be a relative loop in $(S^3 - \gamma, F)$. The *linking number* $\ell(\lambda, \gamma)$ *of* λ *with* γ is defined as follows:
   Choose any path $\lambda' \subset F$ joining $\lambda(1)$ to $\lambda(0)$. Define $\ell(\lambda, \gamma) = \ell_{S^3}(\lambda\lambda', \gamma)$, where the right side of the equation is the linking number in S^3 of the loop $\lambda\lambda'$ with γ ($\ell(\lambda, \gamma)$ is well-defined because for every loop $\gamma' \subset F$, $\ell_{S^3}(\gamma', \gamma) = 0$).

EXAMPLE 5.4. *Let* $\lambda \subset F$ *be an arc from* γ *to* γ. *If* $h \in H_c(S^3 - \gamma)$, *then* $\ell(h(\lambda), \gamma) = 0$.

PROOF. h preserves linking numbers in S^3.

CLAIM 5.5. *Let* λ *be a relative loop in* $(S^3 - \gamma, F)$. *Then* λ *lifts to a relative loop in* $(F \times R^1, F \times 0)$, *under the infinite cyclic covering projection* $p: F \times R^1 \to S^3 - \gamma$ *(in Definition 5.1), if and only if* $\ell(\lambda, \gamma) = 0$.

PROOF. Left to the reader.

CLAIM 5.6. *Let* $h \in H_c(S^3 - \gamma, L)$. *Then* $h \simeq h'$ *in* $H_c(S^3 - \gamma, L)$, *such that for all* i, j, $\ell(h'(\ell_{ij}), \gamma) = 0$.

PROOF. We are assuming that h satisfies (1), (2), above. It is easy to obtain $h \simeq h'$ in $H_c(S^3 - \gamma, L)$ such that for all i, $\ell(h'(\ell_{i,1}), \gamma) = 0$, by sliding L along itself. Now, for all i, k, $l(h'(\ell_{ik}), \gamma) = 0$:

In general, let $\lambda_{\ell, m}$ be the segment of K_ℓ from $p_{\ell, m}$ to $p_{\ell, 1}$. Since h' preserves linking numbers in S^3,

$(*)$ $$\ell(h'[l_{ik} \lambda_{ik} l_{i,1}^{-1}], \gamma) = \ell(l_{ik} \lambda_{ik} l_{i,1}^{-1}, \gamma) .$$

But

$$h'[l_{ik} \lambda_{ik} l_{i,1}^{-1}] = [h'(l_{ik})] \lambda_{jk} [h'(l_{i,1}^{-1})];$$

so the left side of $(*)$ *is*:

$$\ell(h'(l_{ik}), \gamma) + \ell(\lambda_{jk}, \gamma) + \ell(h'(l_{i,1}^{-1}), \gamma) = \ell(h'(l_{ik}), \gamma) + n(j) - 1 .$$

The right side of $(*)$ is:

$$\ell(l_{ik}, \gamma) + \ell(\lambda_{ik}, \gamma) + \ell(l_{i,1}^{-1}, \gamma) = n(i) - 1 .$$

(We are applying Example 5.4 in both of these calculations.) Since $n(i) = n(j)$ by a previous argument, we conclude that $\ell(h'(l_{ik}), \gamma) = 0$.

PROPOSITION 5.7. *Let* $h \in H_c(S^3 - \gamma, L)$. *Then* $h \simeq h'$ *in* $H_c(S^3 - \gamma, L)$ *such that* $h'(F) = F$.

PROOF. We may assume that h satisfies properties (1), (2), and by Claim 5.6,

(3) $\forall i, k$, $\ell(h(l_{ik}), \gamma) = 0$.

Choose disjoint, simple arcs $\lambda_1, \ldots, \lambda_m \subset F - \bigcup_{i,k} l_{i,k}$ from γ to γ, such that $F - [\bigcup_{i,k} l_{i,k}] - [\bigcup_{i=1}^m \lambda_i]$ is an open 2-cell. By Example 5.4 $\ell(h(\lambda_i), \gamma) = 0$, $i = 1, \ldots, n$. By Claim 5.5, $h(l_{ik})$, $1 \leq i \leq n$, $1 \leq k \leq n(i)$, and $h(\lambda_i)$, $1 \leq i \leq m$, lift to relative loops in $(F \times R^1, F \times 0)$ under the infinite cyclic covering projection $p: F \times R^1 \to S^3 - \gamma$ in Definition 5.1. Thus, in the commutative diagram:

$$\begin{array}{ccc} & F \times R^1 & \\ \overset{\hat{h}}{\nearrow} & & \downarrow p \\ F \xrightarrow{\ h|F\ } & & S^3 - \gamma \end{array}$$

a lift $\hat{h}$ of $h | F$ exists.

Let $r_t: F \times R \to F \times R$ be the strong deformation retract from $F \times R$ to $F \times 0$, in Definition 5.2. Then $p \circ r_t \circ \hat{h}: F \to S^3 - \gamma$ is a homotopy from $h(F)$ to F. In fact, $p \circ r_t \circ h | [F - \text{int}(N)]$ is a homotopy rel. boundary, by property (2). By

Corollary 5.5 of [26], there is an isotopy $h_t \in H_c(S^3 - \gamma, L)$ of $h_0 = $ id to h_1 such that $h_1 \circ h(F) = F$.

Let $h' = h_1 \circ h$.

CLAIM 5.8. *Let $S^1 \times [0,1]$ be the annulus, where $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. If $h \in H(S^1 \times [0,1])$ satisfies $(h \,|\, S^1 \times \{0,1\}) = $ id, then h is isotopic, rel. boundary, to p^k for some $k \in \mathbf{Z}$, where $p: S^1 \times [0,1] \to S^1 \times [0,1]$ is the map $p(z,t) = (ze^{2\pi ti}, t)$.*

PROOF. Well-knowm.

PROPOSITION 5.9. *Let $h \in H_c(S^3 - \gamma, L)$ satisfy $h \,|\, F = 1_F$. Then $h \simeq $ id in $H_c(S^3 - \gamma, L)$.*

PROOF. Define $\varphi: F \times [0,1] \to F \times [0,1]$ by:

$$\varphi(x) = \begin{cases} f^{-1} \circ h \circ f(x) & \text{if } x \in F \times (0,1) \\ x & \text{if } x \in F \times \{0,1\} . \end{cases}$$

It is sufficient to show that $\varphi \simeq $ id in $H(F \times I, P \times I)$, rel. boundary.

Choose points $q_{ij} \in \partial N_{ij}$. Since h preserves linking numbers in S^3, $\varphi(q_{ij} \times [0,1]) \simeq q_{ij} \times [0,1]$, modulo endpoints, in the annulus $(\partial N_{ij}) \times [0,1]$. Then Claim 5.8 implies that $\varphi \cong \varphi'$ in $H(F \times I, P \times I)$, rel. boundary, where $\varphi' \,|\, ((\partial N_{ij}) \times [0,1]) = $ id, and $\varphi'(N_{ij} \times t) = N_{ij} \times t$ for all $t \in [0,1]$.

Now use the Alexander–Tietze theorem ([6], [25]) to isotop $\varphi' \,|\, (N_{ij} \times t)$ to the identity, rel. p_{ij} and ∂N_{ij}, simultaneously for all i, j, t. As a result, $\varphi' \simeq \varphi''$ in $H(F \times I, P \times I)$, rel. boundary, such that $\varphi'' \,|\, (N \times [0,1]) = $ id.

Finally, $\varphi'' \,|\, ([F - \text{int}(N)] \times [0,1])$ is isotopic to the identity, rel. boundary, by Theorem 7.1 of [26].

PROPOSITION 5.10. *Suppose $h \in H_c(S^3 - \gamma, L)$ and $h \simeq $ id in $H_c(S^3 - \gamma, L)$. If $h(F) = F$, then $h \,|\, F \simeq 1_F$ in $H_c(F, P)$.*

PROOF. The automorphism $h_*: \pi_1(S^3 - L - \gamma) \to \pi_1(S^3 - L - \gamma)$ induced by h is the identity, because $h \simeq $ id in $H_c(S^3 - \gamma, L)$. Since the inclusion-induced homomorphism $i_*: \pi_1(F - P) \to \pi_1(S^3 - L - \gamma)$ is injective, the automorphism $(h \,|\, F - P)_* : \pi_1(F - P) \to \pi_1(F - P)$ induced by $h \,|\, F - P$ must also be the identity automorphism. Now by a theorem about surface homeomorphisms, $h \,|\, F \simeq 1_F$ in $H_c(F, P)$.

CLAIM 5.11. *Suppose $\psi' \in H_c(F, P)$ is a homeomorphism such that $\psi' \simeq \psi$ in $H_c(F, P)$. Then ψ' is another monodromy for the generalized axis γ of $L \subset S^3 - \gamma$.*

PROOF. Let $g_t \in H_c(F, P)$ be an isotopy from $g_0 = 1_F$ to $g_1 = \psi^{-1} \circ \psi'$. Let $G: F \times [0, 1] \to F \times [0, 1]$ be defined by $G(x, t) = (g_t(x), t)$.
Then the composition:

$$F \times [0, 1] \xrightarrow{\ G\ } F \times [0, 1] \xrightarrow{\ f\ } S^3 - \gamma$$

satisfies conditions (1)–(3) of Definition 4.1. Also, if we set $f' = f \circ G$, then $f'_1 = \psi'$. Therefore ψ' is a monodromy for γ.

CONSTRUCTION 5.12. *Let $g \in H_c(F, P)$, and suppose $g \circ \psi \circ g^{-1} \simeq \psi$ in $H_c(F, P)$. Then there is a homeomorphism $h_g \in H_c(S^3 - \gamma, L)$ such that $h_g \mid F = g$.*

PROOF. By Claim 5.11, we may take $g \circ \psi \circ g^{-1}$ to be another monodromy for the generalized axis α of $L \subset S^3 - \gamma$. Thus, there is a map $f': F \times [0, 1] \to S^3 - \gamma$ (defined in 5.10) satisfying conditions (1)–(3) of Definition 4.1, such that $f'_1 = g \circ \psi \circ g^{-1}$.
Now, there is a unique homeomorphism h_g which makes the diagram

$$
\begin{array}{ccc}
F \times [0, 1] & \xrightarrow{\ g \times \mathrm{id}\ } & F \times [0, 1] \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f'} \\
S^3 - \gamma & \xrightarrow{\ h_x\ } & S^3 - \gamma
\end{array}
$$

commute. Then $h_g \in H_c(S^3 - \gamma, L)$, and $h_g \mid F = g$.

REMARK 5.13. In Claim 5.11, the map $f': F \times [0, 1] \to S^3 - \gamma$ with monodromy $f'_1 = \psi'$, depends continuously on the isotopy between ψ and ψ'. Therefore, in Construction 5.12, the map $g \to h_g$ depends continuously on the isotopy between ψ and $g \circ \psi \circ g^{-1}$.

PROPOSITION 5.14. *Suppose $h \in H_c(S^3 - \gamma, L)$ satisfies $h \mid F \simeq 1_F$ in $H_c(F, P)$. Then $h \simeq h'$ in $H_c(S^3 - \gamma, L)$ such that $h' \mid F = 1_F$.*

PROOF. Let $g_t \in H_c(F, P)$ be an isotopy from $g_0 = 1_F$ to $g_1 = h^{-1}$. Note that for all $t \in [0, 1]$, $g_t \circ \psi \circ g_t^{-1} \simeq \psi$ in $H_c(F, P)$. (The isotopy is $g_{st} \circ \psi \circ g_{st}^{-1}$, $s \in [0, 1]$.) Then by Remark 5.12, $h_{g_t} \in H_c(S^3 - \gamma, L)$ is an isotopy from the identity to h_{g_1}. Thus, $h \circ h_{g_t} \in H_c(S^3 - \gamma, L)$ is an isotopy from h to $h' = h \circ h_{g_1}$. Now $h' \mid F = h \circ h^{-1} = 1_F$.

CLAIM 5.15. *Let the homeomorphism $\varphi: F \times [0, 1] \to F \times [0, 1]$ be level preserving on $\partial(F \times [0, 1])$. Then φ is isotopic, rel. $\partial(F \times [0, 1])$, to a level preserving homeomorphism $\varphi': F \times [0, 1] \to F \times [0, 1]$.*

PROOF. This is essentially Lemma 3.5 of [26].

PROPOSITION 5.16. *Let $h \in H_c(S^3 - \gamma, L)$ be such that $h(F) = F$. If $g = h \mid F$, then $g \circ \psi \circ g^{-1} \simeq \psi$ in $H_c(F, P)$.*

PROOF. Let $\varphi \colon F \times [0, 1] \to F \times [0, 1]$ be defined by

$$\varphi(x) = \begin{cases} f^{-1} \circ h \circ f(x) & \text{if } x \in F \times (0, 1) \\ [g(x)] \times 0 & \text{if } x \in F \times 0 \\ [\psi^{-1} \circ g \circ \psi(x)] \times 1 & \text{if } x \in F \times 1. \end{cases}$$

Then $\varphi \in H_c(F \times [0, 1], P \times [0, 1])$.

We have assumed (by property (2) which h satisfies) that φ is already level preserving on $N \times [0, 1]$. Let $\hat{F} \subset F - P$ be a compact submanifold such that $\partial \hat{F} \subset N \cup A$, where A is a tubular neighborhood of γ in S^3, and $\varphi \mid (F - \hat{F}) \times [0, 1] = \text{id}$; let $\hat{\varphi}$ be the restriction $\hat{\varphi} = \varphi \mid \hat{F} \times [0, 1]$.

Now, $\hat{\varphi}$ is level-preserving on $\partial(\hat{F} \times [0, 1])$, so by Claim 5.15, there is an isotopy of $\hat{\varphi}$, constant on $\partial(\hat{F} \times [0, 1])$, to a level-preserving homeomorphism. Thus, there is an isotopy $\varphi \simeq \varphi'$ in $H_c(F \times [0, 1], P \times [0, 1])$, constant on $\partial(F \times [0, 1])$, to a level preserving homeomorphism $\varphi' \colon F \times [0, 1] \to F \times [0, 1]$.

Let $\varphi'_t \colon F \to F$ be the restriction of φ' to level t (thus, $\varphi'(x, t) = (\varphi'_t(x), t)$ for $(x, t) \in F \times [0, 1]$). Then $\varphi'_t \in H_c(F, P)$ is an isotopy between $\varphi'_0 = g$ and $\varphi'_1 = \psi^{-1} \circ g \circ \psi$; hence $\psi \circ \varphi'_t \circ g^{-1}$ is an isotopy in $H_c(F, P)$ from ψ to $g \circ \psi \circ g^{-1}$.

NOTATION 5.17. The centralizer of $[h] \in \mathscr{H}(X, Y)$ will be denoted $\mathscr{H}_h(X, Y)$.

DEFINITION 5.18. $J \colon \mathscr{H}_\psi(F, P) \to \mathscr{H}(S^3 - \gamma, L)$ is defined by $J([g]) = [h_g]$, where h_g is defined in Construction 5.12.

The map J is well-defined, by the following remark:

REMARK 5.19. Suppose $g \simeq g'$ in $\mathscr{H}(F, P)$ and $h'_{g'} \in H_c(S^3 - \gamma, L)$ satisfies $h'_{g'} \mid F = g'$. Then $[h'_{g'} \circ h_g^{-1} \mid F] \simeq 1_F$ in $\mathscr{H}(F, P)$; so by Proposition 5.9, $h'_{g'} \circ h_g^{-1} \simeq \text{id}$ in $H_c(S^3 - \gamma, L)$. Thus, $[h'_{g'}] = [h_g] \in \mathscr{H}(S^3 - \gamma, L)$.

It is also easy to check that J is a homomorphism.

THEOREM 5.20. $J \colon \mathscr{H}_\psi(F, P) \to \mathscr{H}(S^3 - \gamma, L)$ *is an isomorphism (where $\psi \in H_c(F, P)$ is a monodromy for the generalized axis γ of $L \subset S^3 - \gamma$).*

PROOF. By Proposition 5.10, J is injective. By Propositions 5.9, 5.7, 5.16, and Construction 5.12, J is surjective.

The map $e: \mathscr{H}(S^3 - \gamma, L) \to \mathscr{H}^+(S^3, L, \gamma)$.

DEFINITION 5.21. The homomorphism $e: \mathscr{H}(S^3 - \gamma, L) \to \mathscr{H}^+(S^3, L, \gamma)$ is induced by extension.

DEFINITION 5.22. Let $\tau \in H_c(F, P)$ be a Lickorish twist (see [8]) about a simple, closed curve on F, isotopic in F to γ.

REMARK 5.23. $[\tau]$ is in the center of the group $\mathscr{H}(F, P)$.

DEFINITION 5.24. Let G be a group, and let $g_1, \ldots, g_n \in G$. Then $\langle g_1, \ldots, g_n \rangle$ G is the smallest subgroup containing g_i, $i = 1, \ldots, n$.

Recall that $S^1 = \{z \in \mathbf{C} : |z| = 1\}$. Let $T = S^1 \times S^1$ be the torus. Define $\delta: [0, 1] \to [0, 1]$ by:

$$\delta(t) = \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{4} \\ 2(t - \frac{1}{4}) & \text{if } \frac{1}{4} \leq t \leq \frac{3}{4} \\ 1 & \text{if } \frac{3}{4} \leq t \leq 1 . \end{cases}$$

Define $r_{s,t}: T \to T$ to be $r_{s,t}(z_1, z_2) = (z_1 e^{2\pi s i}, z_2 e^{2\pi t i})$. Let $R_i \in H_c(T \times (0, 1))$, $i = 1, 2$ be the homeomorphisms

$$R_1(x, t) = (r_{\delta(t), 0}(x), t), \qquad R_2(x, t) = (r_{0, \delta(t)}(x), t) .$$

CLAIM 5.25. *Let* $T = S^1 \times S^1$ *be the torus. Then* $\mathscr{H}(T \times (0, 1)) \simeq \mathbf{Z} \oplus \mathbf{Z}$ *is freely generated by* $[R_i]$, $i = 1, 2$.

PROOF. Well-known.

THEOREM 5.26. *The sequence*

$$1 \to \langle [\psi], [\tau] \rangle \to \mathscr{H}_\psi(F, P) \stackrel{J}{\cong} \mathscr{H}(S^3 - \gamma, L) \stackrel{e}{\longrightarrow} \mathscr{H}^+(S^3, L, \gamma) \text{ is exact .}$$

PROOF. We must show that kernel $(e \circ J) = \langle [\psi], [\tau] \rangle$.

Let $h \in H_c(S^3 - \gamma, L)$ be such that $[h] \in$ kernel (e). Then there is an isotopy $h_t \in H(S^3, L, \gamma)$ from $h_0 = 1_{S^3}$ to $h_1 = h$. Let A be an open, tubular neighborhood of γ in $S^3 - L$, such that h has its support in $S^3 - A$; let $A' \subset A$ be a sufficiently small closed tubular neighborhood of γ, such that for all t, $h_t(A') \subset A$. Let $f_t \in H(S^3, \gamma, L)$ be an isotopy which has its support in A, from $f_0 = 1_{S^3}$ to f_1, such that for all $t, f_t | A' = h_t | A'$. (The extension f_t of the isotopy $h_t | A'$ exists by the Main Theorem of [13].

Now, $g_t = f_t^{-1} \circ h_t \in H_c(S^3 - \gamma, L)$ is an isotopy from $g_0 = \mathrm{id}$ to $g_1 \in H_c(S^3 - \gamma, L)$. Also, $h = h_1 = f_1 \circ g_1$; so $h \simeq f_1 \mid (S^3 - \gamma)$ in $H_c(S^3 - \gamma, L)$.

Finally, $(f_1 \mid S^3 - \gamma) \in H_c(S^3 - \gamma, L)$ has its support in the thickened torus $A - A'$. By Claim 5.25, $\mathscr{H}(A - A') \simeq \mathbf{Z} \oplus \mathbf{Z}$. The free generators map to $[h_\tau]$ and $[h_\psi]$ under the homomorphism $\mathscr{H}(A - A') \to \mathscr{H}(S^3 - \gamma, L)$ induced by extension. (τ is defined in Definition 5.22, and h_g is defined in Construction 5.12).

Thus, kernel $(e \circ J) = \langle [\psi], [\tau] \rangle$.

The group $\mathscr{H}^+(S^3, L, \gamma)$.

DEFINITION 5.27. $\mathscr{H}^+(S^3, L, \gamma) \to \mathscr{H}(\gamma) = \mathbf{Z}_2$ is the homomorphism induced by restriction.

THEOREM 5.28. *The sequence*

$$1 \to \langle [\psi], [\tau] \rangle \to \mathscr{H}_\psi(F, P) \xrightarrow{e \circ J} \mathscr{H}^+(S^3, L, \gamma) \to \mathbf{Z}_2$$

is exact.

PROOF. It only remains to show that if $h \in H^+(S^3, L, \gamma)$ satisfies $1 = [h \mid \gamma] \in \mathscr{H}(\gamma)$, then $[h] \in \mathrm{image}\ (J)$. (The exactness of the rest of the sequence is proved in Theorem 5.26.)

If $h \mid \gamma \simeq 1_\gamma$, then $h \simeq h'$ in $H^+(S^3, L, \gamma)$ such that $h' \mid A = 1_A$ (where A is the tubular neighborhood of γ defined in the proof of Theorem 5.26. This is an application of Theorem 10, Chapter IV, Section 6, of [21].) Then $(h' \mid (S^3 - \gamma)) \in H_c(S^3 - \gamma, L)$, and $[h'] = J([h' \mid (S^3 - \gamma)])$.

6. The group of motions $\mathscr{M}(S^3, L, \gamma)$.

The main theorem of this section is Theorem 6.8, in which the group of motions of the link $L \subset S^3 - \gamma$ with generalized axis γ, is computed to be a $\mathbf{Z}_2$, or trivial, normal extension of a quotient group of the surface homomorphism group $\mathscr{H}_\psi(F, P)$. This provides presentations for many motion groups of links, since generators and defining relations are now known for all of the surface homeomorphism groups (also called homeotopy groups).

NOTATION 6.1. Let $A \subset S^3 - L$ be a closed, tubular neighborhood of γ. The subspace $(S^3 - A, L) \subset (S^3 - \gamma, L)$ is a strong deformation retract of the space $(S^3 - \gamma, L)$; therefore, the inclusion i induces an isomorphism $i_*: \mathscr{H}(S^3 - A, L) \to \mathscr{H}(S^3 - \gamma, L)$.

Let $* \in A - \gamma$ be a point, and let $E: \mathscr{H}(S^3 - A, L) \to \mathscr{H}(S^3 - *, L, \gamma)$ be induced by extension.

Let $\mathscr{M}(S^3 - *, L, \gamma) \to \mathscr{M}(S^3, L, \gamma)$ be induced by extension.

Recall (Corollary 1.12) that $\partial: \mathscr{M}(S^3 - *, L, \gamma) \to \mathscr{H}(S^3 - *, L, \gamma)$, defined by $\partial([f]) = [f_1]$, where f is a motion of (L, γ) in $S^3 - *$, is an isomorphism.

Finally, let $\{2\pi\}$ be the motion of S^3 which rotates S^3 one full rotation. Thus, $[2\pi] \in \pi_1(H(S^3); 1_{S^3}) \simeq \mathbb{Z}/2\mathbb{Z} = \mathbb{Z}_2$ is the generator.

DEFINITION 6.2. The homomorphism $K: \mathscr{H}(S^3 - \gamma, L) \to \mathscr{M}(S^3, L, \gamma)$ is the composition:

$$\mathscr{H}(S^3 - \gamma, L) \xrightarrow[\simeq]{i_*^{-1}} \mathscr{H}(S^3 - A, L) \xrightarrow{E} \mathscr{H}(S^3 - *, L, \gamma) \xrightarrow[\simeq]{\partial^{-1}}$$

$$\to \mathscr{M}(S^3 - *, L, \gamma) \to \mathscr{M}(S^3, L, \gamma) .$$

The image of K.

Let $\mathscr{M}(S^3, L, \gamma) \to \mathscr{H}(\gamma)$ be the composition $\mathscr{M}(S^3, L, \gamma) \xrightarrow{\partial} \mathscr{H}^+(S^3, L, \gamma) \to \mathscr{H}(\gamma)$, where the last map is induced by restriction. Clearly, $\mathscr{H}(\gamma) \simeq \mathbb{Z}_2$.

PROPOSITION 6.3. *The sequence*

$$\mathscr{H}(S^3 - \gamma, L) \xrightarrow{K} \mathscr{M}(S^3, L, \gamma) \to \mathscr{H}(\gamma) = \mathbb{Z}_2$$

is exact.

PROOF. Let f be a motion of (L, γ) in S^3 such that $f_1 | \gamma \simeq 1_\gamma$. Then there is an isotopy $g_t \in H(S^3, L, \gamma)$ from $g_0 = 1_{S^3}$ to g_1, such that $g_1 | A = f_1 | A$, for the closed tubular neighborhood $A \subset S^3 - L$ of γ (by [21]). Now, g is a stationary motion of (L, γ) in S^3, so $f' = f \circ g^{-1}$ is a motion of (L, γ) in S^3 which is equivalent to f. But $f_1 \circ g_1^{-1} | (S^3 - \gamma) \in H_c(S^3 - \gamma, L)$, since $f_1 \circ g_1^{-1} | A = 1_A$; so $[f_1' | (S^3 - \gamma)]$ $\xrightarrow{K} [f''] \in \mathscr{M}(S^3, L, \gamma)$, where f'' is a motion of (L, γ) in S^3 such that $f_1'' = f_1'$.

By Theorem 1.11, the sequence

$$\pi_1(H(S^3); 1_{S^3}) \to \mathscr{M}(S^3, L, \gamma) \xrightarrow{\partial} \mathscr{H}^+(S^3, L, \gamma)$$

is exact. Since $\partial[f'] = \partial[f'']$, either $f' \equiv f''$ or $f' \equiv \{2\pi\} \circ f''$.

Finally, $[2\pi] = K([h_\psi])$ (see Proposition 6.7). So either $K([f_1' | (S^3 - \gamma)]) = [f']$ or $K([h_\psi] \cdot [f_1' | (S^3 - \gamma)]) = [f']$. Since $[f'] = [f]$, this completes the proof.

The kernel of K.

Let $\tau, \psi \in H_c(F, P)$ and $\langle [\psi], [\tau] \rangle \subset \mathscr{H}(F, P)$ be defined as in Definition 5.22 and the introduction to Section 5, and in Definition 5.24.

NOTATION 6.4. Let A be an open tubular neighborhood of γ in $S^3 - L$, such that h_t has its support in A, and let $A' \subset A$ be a sufficiently small, closed tubular

neighborhood of γ, that $h_\tau \,|\, A' = h_\psi \,|\, A' = 1_{A'}$. Let $* \in F \cap A'$ be an interior point, and let $B \subset A' - \gamma$ be a ball centered at $*$.

Let $\tau_t \in H_c(A, L)$ be an isotopy from 1_A to $h_\tau \,|\, A$. (We may assume $\tau_1 \,|\, (A - \gamma) \in H_c(A - \gamma)$.) Let $\psi_t \in H_c(A, L)$ be positive rotation of A' about γ. (We may assume $\psi_1 \,|\, (A - \gamma) \in H_c(A - \gamma)$.)

Let $\#(\tau)$ be the number of signed rotations which B executes in S^3 during τ_t; let $\#(\psi)$ be the number of signed rotations which B executes in S^3 during ψ_t.

CLAIM 6.5. *Under the map* $\mathscr{H}(A - \gamma) \to \mathscr{H}(S^3 - \gamma, L)$ *induced by extension,*

(1) $[\tau_1 \,|\, (A - \gamma)] \to [h_\tau]$ *and*
(2) $[\psi_1 \,|\, (A - \gamma)] \to [h_{\psi^{-1}}]$.

PROOF. (1) follows by definition. For (2) consider the extension $h \in H_c(S^3 - \gamma, L)$ of $\psi_1 \in H_c(A - \gamma)$. It is necessary to exhibit an isotopy $H_t \in H_c(S^3 - \gamma, L)$ from $h_{\psi^{-1}}$ to h.

Let $F' = F \cap A'$, and let $F_A = F - A$. Define $\hat{H}_t \colon F \times [0, 1] \to F \times [0, 1]$ such that $\hat{H}_t \,|\, (F_A \times [0, 1])$ is the map

$$\hat{H}_t(x, s) = \begin{cases} (\psi^{-1}(x), s + t) & \text{if } s + t \leq 1, \ x \in F_A \\ (x, s + t - 1) & \text{if } s + t > 1, \ x \in F_A, \end{cases}$$

and *such that* $\hat{H}_t \,|\, (F' \times [0, 1]) = \mathrm{id}$.

Define $H_t \colon S^3 - \gamma \to S^3 - \gamma$ by

$$H_t(x) = \begin{cases} f \circ \hat{H}_t \circ f^{-1}(x) & \text{if } x \in S^3 - F \\ f \circ \hat{H}_t(x, 0) & \text{if } x \in F. \end{cases}$$

It is possible to define $\hat{H}_t$ so that $H_t \in H_c(S^3 - \gamma)$ varies continuously with t. Now $H_0 = [h_{\psi^{-1}}]$ and $H_1 = [h]$.

CLAIM 6.6. $\#(\tau) = \#(\psi) = 1 \pmod 2$.

PROOF. $\#(\psi) = 1$ by definition of ψ_t.
$\#(\tau) = 1 \pmod 2$ by the Roller Coaster Lemma ([14]).

(The tracks of the roller coaster are two parallel, simple closed curves in A', which are isotopic in F, to γ. The roller coaster car may be thought of as the 3-ball B. Send the car once around the tracks; the number of rotations which it makes in S^3 is $\#(\tau)$.)

PROPOSITION. 6.7. *In the exact sequence*

$$1 \to \mathrm{kernel}\ (K) \hookrightarrow \mathscr{H}(S^3 - \gamma, L) \xrightarrow{K} \mathscr{M}(S^3, L, \gamma),$$

$\mathrm{kernel}\ (K) \subset J(\langle [\psi], [\tau] \rangle)$.

$$[\tau] \xrightarrow{K \circ J} [2\pi] \in \mathcal{M}(S^3, L, \gamma), \qquad [\psi] \xrightarrow{K \circ J} [2\pi] \in \mathcal{M}(S^3, L, \gamma).$$

PROOF. Let $e: \mathcal{H}(S^3 - \gamma, L) \to \mathcal{H}(S^3, L, \gamma)$ be induced by extension. Then e is the same as the composition

$$\mathcal{H}(S^3 - \gamma, L) \xrightarrow{K} \mathcal{M}(S^3, L, \gamma) \xrightarrow{\partial} \mathcal{H}^+(S^3, L, \gamma).$$

(See Definition 6.2.) But by Theorem 5.26, kernel $(e) = J(\langle[\psi], [\tau]\rangle)$. This proves the first part of the theorem.

Examination of the map $\partial^{-1}: \mathcal{H}(S^3 - *, L, \gamma) \to \mathcal{M}(S^3 - *, L, \gamma)$ reveals that

$$[h_\tau] \xrightarrow{K} [2\pi]^{\#(\tau)} \in \mathcal{M}(S^3, L, \gamma), \qquad [h_\psi] \xrightarrow{K} [2]^{\#(\psi)} \in \mathcal{M}(S^3, L, \gamma).$$

Now by Claim 5.6, $\#(\tau) = \#(\psi) = 1 \pmod 2$. Since $1 = [2\pi]^2 \in \mathcal{M}(S^3, L, \gamma)$ (see Notation 6.1), this completes the proof.

THEOREM 6.8. *There is an exact sequence*

$$1 \to \mathcal{H}_\psi(F, P)/(*) \to \mathcal{M}(S^3, L, \gamma) \to \mathcal{H}(\gamma) = \mathbf{Z}_2,$$

where (*) *consists of the relations*:
 (1) $[\tau] = [\psi]$
 (2) $[\tau]^2 = 1$
 (3) $[\tau] = 1$ *if and only if* $L \cup \gamma$ *is a link of the type listed in* (1)–(4) *of Theorem* 3.7.

PROOF. By Proposition 6.3, the sequence:

$$1 \to \text{kernel } (K \circ J) \to \mathcal{H}_\psi(F, P) \xrightarrow{K \circ J} \mathcal{M}(S^3, L, \gamma) \to \mathbf{Z}_2$$

is exact. By Proposition 6.7, kernel $(K \circ J) \subset \langle[\psi], [\tau]\rangle$, where

$$[\tau] \xrightarrow{K \circ J} [2\pi] \quad \text{and} \quad [\psi] \xrightarrow{K \circ J} [2\pi].$$

Since $[2\pi]$ is the generator of $\pi_1(H(S^3); 1_{S^3})$, $1 = [2\pi]^2 \in \mathcal{M}(S^3, L, \gamma)$. Thus far, relations (1) and (2) are accounted for.

To complete the proof, we apply Theorem 3.7.

7. Homeotopy groups of surfaces as branched covering spaces.

It is apparent from Theorem 6.8, that an understanding of surface homeomorphism groups is necessary in order to compute motion groups of links. The main tools in this section are Claims 7.2, 7.3, which compare the symmetric subgroup (with respect to covering translations) of the homeotopy group $\mathcal{H}(F)$ of a branched covering space $F \to T_{0,n}$, to the braid group of the branch points in $T_{0,n}$, and $\mathcal{H}(T_{0,n})$. (Here, $T_{0,n}$ is the n-punctured 2-sphere).

Applications are made to the particular branched covering spaces defined in Section 2 (on torus links), which will then be used in Section 8. Finally, presentations are given for the n-braid groups of the disk and the annulus.

NOTATION 7.1. Let $\mathscr{H}^f(M, N)$ denote $\pi_0(H(M, N))$. (This is the analogue of $\mathscr{H}(M, N)$, without compact support required for maps or isotopies. The letter f stands for "free".)

As in Notation 5.17, if $[g] \in \mathscr{H}^f(M, N)$, then let $\mathscr{H}^f_g(M, N)$ denote the centralizer of $[g]$ in $\mathscr{H}^f(M, N)$.

CLAIM 7.2. *Let $T_{0,n}$ be the 2-sphere with $n \geq 1$ points removed. Let $\pi: F \to T_{0,n}$ be an m-fold, cyclic branched covering space, with branch points $p_1, \ldots, p_k$, where $k \geq 1$. Assume that either $n \geq 2$ or $k \geq 2$. Let p_i have branching index $m_i > 1$, where $m_i \neq m_j$ if $i \neq j$. Finally, let $t \in H(F, p_1, \ldots, p_k)$ generate the cyclic group of covering translations.*
Then the map

$$\mathscr{H}^f(T_{0,n}, p_1, \ldots, p_k) \to \mathscr{H}^f_t(F)/[t]$$

induced by lifting, is an isomorphism.

PROOOF. This follows from Theorem 8.2 in [27]. (See also [10], [23].)

In Claim 7.3, let $\varphi_1: \mathbf{R}^2 \to \mathbf{R}^2$ be as defined in Notation 2.1.

CLAIM 7.3. *Let $\pi: F \to \mathbf{R}^2$ be an m-fold, cyclic branched covering of $\mathbf{R}^2$, with branch points $p_1, \ldots, p_k \in \mathbf{R}^2$. Let p_i have branching index $m_i > 1$, where $m_i \neq m_j$ if $i \neq j$. Let $X \subset \mathbf{R}^2 - \{p_1, \ldots, p_k\}$ be a collection of $n \geq 1$ points, and let $\hat{X} = \pi^{-1}(X) \subset F$. We will assume that $X \cup \{p_1, \ldots, p_k\} \subset \operatorname{int}(D)$, where D is the unit disk in $\mathbf{R}^2$. Finally, let $\hat{\varphi} \in H_c(F, \hat{X})$ be the unique lift of $\varphi_1 \in H_c(\mathbf{R}^2)$ which has compact support in F.*
Then the map

$$\mathscr{H}(\mathbf{R}^2, X, p_1, \ldots, p_k) \to \mathscr{H}_{\hat{\varphi}}(F, \hat{X})$$

induced by lifting, is an isomorphism.

PROOF. Let the homomorphism $H_c(\mathbf{R}^2, X, p_1, \ldots, p_k) \to H_c(F, \hat{X})$ map each $h: \mathbf{R}^2 \to \mathbf{R}^2$ to its *unique* lift $\hat{h} \in H_c(F, \hat{X})$ which has compact support in F. Clearly, this induces a homomorphism $\mathscr{H}(\mathbf{R}^2, X, p_1, \ldots, p_k) \to \mathscr{H}(F, \hat{X})$.

Next, note that $\hat{\varphi}$ is isotopic, in $H(F, \hat{X})$, to t, where t generates the cyclic group of covering translations of $\pi: F \to \mathbf{R}^2$. Let $\hat{t} = t \mid (F - \hat{X})$. Then, by Claim

7.2, the map

$$\mathcal{H}^f(\mathbf{R}^2 - X, p_1, \ldots, p_k) \to \mathcal{H}^f_{\hat{f}}(F - \hat{X})/[\hat{f}]$$

induced by lifting, is an isomorphism. This may be reformulated, the map

$$\mathcal{H}^f(\mathbf{R}^2, X, p_1, \ldots, p_k) \to \mathcal{H}^f_{\hat{\phi}}(F, \hat{X})/[\hat{\phi}]$$

induced by lifting, is an isomorphism.

In the commutative diagram:

$$
\begin{array}{ccccc}
1 & & 1 & & 1 \\
\downarrow & & \downarrow & & \downarrow \\
\langle[\varphi_1]\rangle & \xrightarrow{\ \simeq\ } & \langle[\hat{\phi}]\rangle & \longrightarrow & \langle[\hat{\phi}]\rangle \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{H}(\mathbf{R}^2, X, p_1, \ldots, p_k) & \to & \mathcal{H}_{\hat{\phi}}(F, \hat{X}) \to & & \mathcal{H}^f_{\hat{\phi}}(F, \hat{X}) \\
\downarrow & & & & \downarrow \\
\mathcal{H}^f(\mathbf{R}^2, X, p_1, p_k) & & \xrightarrow{\ \simeq\ } & & \mathcal{H}^f_{\hat{\phi}}(F, \hat{X})/[\hat{\phi}] \\
\downarrow & & & & \downarrow \\
1 & & & & 1
\end{array}
$$

the vertical sequences are exact, and all the horizontal arrows, except for $\mathcal{H}(F, \hat{X}) \to \mathcal{H}^f(F, \hat{X})$ and $\langle[\hat{\phi}]\rangle \to \langle[\hat{\phi}]\rangle$, are induced by lifting. The isomorphism $\langle[\varphi_1]\rangle \to \langle[\hat{\phi}]\rangle$ is a consequence of the hypotheses $n \geq 1$, $k \geq 1$.

It follows from diagram chasing that the map $\mathcal{H}(\mathbf{R}^2, X, p_1, \ldots, p_k) \to \mathcal{H}_{\hat{\phi}}(F, \hat{X})$ is injective, and the composition $\mathcal{H}(\mathbf{R}^2, X, p_1, \ldots, p_k) \to \mathcal{H}_{\hat{\phi}}(F, \hat{X}) \to \mathcal{H}^f_{\hat{\phi}}(F, \hat{X})$ is surjective. Now the sequence

$$1 \to \langle[\hat{\phi}^m]\rangle \to \mathcal{H}_{\hat{\phi}}(F, \hat{X}) \to \mathcal{H}^f_{\hat{\phi}}(F, \hat{X}) \to 1$$

is exact. Let $g \in \mathcal{H}_{\hat{\phi}}(F, \hat{X})$. Then there is $g' \in \mathcal{H}(\mathbf{R}^2, X, p_1, \ldots, p_k)$ such that $g' \to [\hat{\phi}^m]^s g$, for some integers s. Then $[\hat{\phi}^{-ms}]g' \to g$, and, therefore, the map $\mathcal{H}(\mathbf{R}^2, X, p_1, \ldots, p_k) \to \mathcal{H}_{\hat{\phi}}(F, \hat{X})$ is surjective.

Applications.

In Application 7.4, let the *pq*-fold, cyclic branched covering space $\pi: F \to \mathbf{R}^2$, branched about $0 \in \mathbf{R}^2$ with branching index p, and about $1 \in \mathbf{R}^2$ with branching index q, be defined in Definition 4.8. (We think of $\mathbf{R}^2$ as being the complex plane). Let $\hat{P}_k$, A be as in Notation 4.9, and let $\Sigma \in H_c(F, \hat{P}_1, \hat{P}_2, \ldots, \hat{P}_n, A)$ be defined as in Remark 4.11.

APPLICATION 7.4. *Suppose $q > 1$. Then the map*

$$\mathcal{H}^f_{\Sigma}\left(F, \hat{P}_1, \bigcup_{k=2}^{n} \hat{P}_k\right)\Big/[\Sigma] \to \mathcal{H}^f_{\Sigma}\left(F, \bigcup_{k=2}^{n} \hat{P}_k\right)\Big/[\Sigma]$$

induced by restriction is an isomorphism.

PROOF. CASE 1 $(p > 1)$. In the commutative diagram:

$$\mathscr{H}^f_\Sigma\left(F, \hat{P}_1, \bigcup_{k=2}^{n} \hat{P}_k\right)\Big/[\Sigma] \to \mathscr{H}^f_\Sigma\left(F, \bigcup_{k=2}^{n} \hat{P}_k\right)\Big/[\Sigma]$$

$$\mathscr{H}^f\left(\mathbf{R}^2, 1, \bigcup_{k=2}^{n} \frac{1}{k}, 0\right),$$

the vertical arrows are induced by lifting, and the horizontal arrow, by restriction. The vertical arrows are isomorphisms by Claim 7.2 (where $T_{0,n} = \mathbf{R}^2 - \bigcup_{k=2}^{n} 1/k$). The precise conclusion of Claim 7.2 is that the maps

$$\mathscr{H}^f(T_{0,n}, 1, 0) \to \mathscr{H}^f\left(F - \bigcup_{k=2}^{n} \hat{P}_k, \hat{P}_1\right)\Big/[\Sigma]$$

and

$$\mathscr{H}^f(T_{0,n}, 1, 0) \to \mathscr{H}^f\left(F - \bigcup_{k=2}^{n} \hat{P}_k\right)\Big/[\Sigma]$$

induced by lifting, are isomorphisms.) Application 7.4 follows.

CASE 2 $(p = 1)$. In the commutative diagram:

$$\mathscr{H}^f_\Sigma\left(F, \bigcup_{k=2}^{n} \hat{P}_k, \hat{P}_1\right)\Big/[\Sigma] \to \mathscr{H}^f_\Sigma\left(F, \bigcup_{k=2}^{n} \hat{P}_k\right)\Big/[\Sigma]$$

$$\mathscr{H}^f\left(\mathbf{R}^2, \bigcup_{k=2}^{n} \frac{1}{k}, 1\right),$$

the vertical arrows are induced by lifting, and the horizontal arrow is induced by restriction. As in Case 1, the vertical arrows are isomorphisms, which gives the desired conclusion.

In Application 7.5, let $\pi_p: \mathbf{R}^2 \to \mathbf{R}^2$ be the p-fold cyclic covering branched about $0 \in \mathbf{R}^2$ with index p, and let $\varphi_t: \mathbf{R}^2 \to \mathbf{R}^2$, $t \in \mathbf{R}$, be defined in Notation 2.1. Note that $\pi_p((1/k)P_p) = 1/k$.

APPLICATION 7.5. *If $p > 1$, the map:*

$$\mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) \to \mathscr{H}_{\varphi_{q/p}}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right)$$

induced by lifting, is an isomorphism.

PROOF. The diagram:

$$R^2 \xrightarrow{\varphi_{q/p}} R^2$$
$$\downarrow^{\pi_p} \qquad \downarrow^{\pi_p}$$
$$R^2 \xrightarrow{\varphi_q} R^2$$

commutes; thus, there is a homomorphism:

(*) $$\mathscr{H}_{\varphi_q}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) \to \mathscr{H}_{\varphi_{q/p}}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p, 0\right)$$

induced by lifting.

Now, φ_1 generates Center $(\mathscr{H}(R^2, \bigcup_{k=1}^{n} 1/k, 0))$. (See Theorem 7.10). Therefore,

$$\mathscr{H}_{\varphi_q}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) = \mathscr{H}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right).$$

If $p > 1$, because g.c.d. $(p, q) = 1$, there exists an integer k such that $kq = 1$ (mod p). Then

$$(\varphi_{q/p})^k = \varphi_{pk'/p}\varphi_{1/p} = \varphi_{k'}\varphi_{1/p},$$

for some integer k'. Since φ_1 generates Center $(\mathscr{H}(R^2, \bigcup_{k=1}^{n} (1/k)P_p))$, this implies that

$$\mathscr{H}_{\varphi_{q/p}}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right) = \mathscr{H}_{\varphi_{1/p}}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right).$$

Finally, in the commutative diagram:

$$\mathscr{H}_{\varphi_{1/p}}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right) = \mathscr{H}_{\varphi_{q/p}}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right)$$
$$\downarrow^{\simeq} \qquad\qquad \downarrow^{\simeq}$$
$$\mathscr{H}_{\varphi_1}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) = \mathscr{H}_{\varphi_1}\left(R^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right),$$

the vertical arrows are the homomorphisms (*), for $q = 1$ and q. Since $p > 1$, the vertical arrow on the left is an isomorphism, by Claim 7.3. Application 7.5 is immediate.

The braid groups of the disk and the annulus.

NOTATION 7.6. Let $B_n(R^2), B_n(R^2 - \{0\})$ denote the Artin n-braid group (see [1]–[5]) and the n-braid group of the annulus, respectively

REMARK. It was proved in [20] that:

$$B_n(\mathbf{R}^2) = \mathscr{M}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}\right) \quad \text{and} \quad B_n(\mathbf{R}^2 - \{0\}) = \mathscr{M}\left(\mathbf{R}^2 - \{0\}, \bigcup_{k=1}^{n} \frac{1}{k}\right).$$
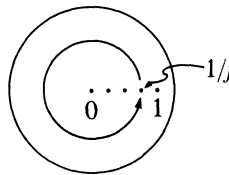
DEFINITION 7.7. Let $\sigma_j \in B_n(\mathbf{R}^2)$ and $\sigma_j \in B_n(\mathbf{R}^2 - \{0\})$, $j = 1, \ldots, n-1$, be represented by the braid (motion) which interchanges $1/j$ and $1/(j+1)$, as in Figure 7.8.

Let $\varrho_j \in B_n(\mathbf{R}^2 - \{0\})$, $j = 1, \ldots, n$, be represented by the braid (motion) which winds the point $1/j$ about $0 \in \mathbf{R}^2$ as in Figure 7.8. (We will let σ_j and ϱ_j denote both the motion, and its equivalence class.)

Figure 7.8. The braids σ_j, $j = 1, \ldots, n-1$.
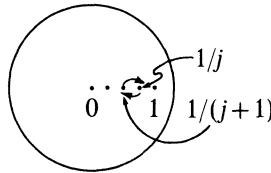
The braids ϱ_j, $j = 1, \ldots, n$.

THEOREM 7.9. (Due to E. Artin). *The group* $B_n(\mathbf{R}^2)$ *is:*

$$\{\sigma_i, \, i = 1, \ldots, n-1 \, : \, \sigma_i \leftrightarrow \sigma_j \text{ if } |i-j| \geqq 2, \, \sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1}$$

$$\text{if } 1 \leqq i \leqq n-2\} \, .$$

PROOF. See [5].

THEOREM 7.10. *The group* $B_n(\mathbf{R}^2 - \{0\})$ *is:*

$$\sigma_i \leftrightarrow \sigma_j \qquad\qquad \text{if } |i-j| \geqq 2 \, ,$$

$$\sigma_i \sigma_{i+1} \sigma_i = \sigma_{i+1} \sigma_i \sigma_{i+1} \qquad\qquad \text{if } 1 \leqq i \leqq n-2 \, ,$$

$$\varrho_i \leftrightarrow \varrho_j \qquad\qquad \text{if } 1 \leqq i, \, j \leqq n \, ,$$

$$\varrho_i \leftrightarrow \sigma_j \qquad\qquad \text{if } j \neq i-1 \, ,$$

$$\varrho_{j+1} = \sigma_j \ldots \sigma_2 \sigma_1 \varrho_1 \sigma_1 \sigma_2 \ldots \sigma_j \qquad\qquad \text{if } 1 \leqq j \leqq n-1\} \, .$$

PROOF. A presentation for this group is given in [8], and the above presentation may be deduced from this, or any other known presentation.

8. Motions of torus links in S^3.

As an application of Theorem 6.8, I will now calculate the group $\mathcal{M}(S^3, L_{(np, nq)})$ of motions of the torus link $L_{(np, nq)}$ of type (np, nq), in S^3. I will give generators and a complete set of defining relations for this group.

Recall the definition of $L_{(np, nq)}$, the axes α and β for $L_{(np, nq)}$, and the map

$$\varphi_{q/p} \colon \left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p \right) \to \left(R^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p \right)$$

from Notation 2.1.

The key theorem.

To increase the readability of the argument which follows, I will present the proof of Theorem 8.1 in its most natural order, which is backwards.

THEOREM 8.1. *Suppose $q > 1$, $n > 0$. Then the map*

$$\mathcal{M}(S^3, L_{(np, nq)}, \beta) \to \mathcal{M}(S^3, L_{(np, nq)})$$

induced by restriction is an isomorphism.

PROOF. In the commutative diagram

$$
\begin{array}{ccccccccc}
1 & \to & \langle [2\pi] \rangle & \hookrightarrow & \mathcal{M}(S^3, L_{(np, nq)}) & \xrightarrow{\partial} & \mathcal{H}^+(S^3, L_{(np, nq)}) & \to & 1 \\
& & \uparrow \simeq & & \uparrow & & \uparrow \simeq & & \\
1 & \to & \langle [2\pi] \rangle & \hookrightarrow & \mathcal{M}(S^3, L_{(np, nq)}, \beta) & \xrightarrow{\partial} & \mathcal{H}^+(S^3, L_{(np, nq)}, \beta) & \to & 1 \ ,
\end{array}
$$

the vertical arrows are induced by restriction, and the rows are exact, by Corollary 1.13.

The vertical arrow on the left is an isomorphism, by Theorem 6.7. The vertical arrow on the right is an isomorphism by Lemma 8.2. By the Five Lemma, the map $\mathcal{M}(S^3, L_{(np, nq)}, \beta) \to \mathcal{M}(S^3, L_{(np, nq)})$ induced by restriction, is an isomorphism.

LEMMA 8.2. *If $q > 1$, $n > 0$, the homomorphism*

$$\mathcal{H}^+(S^3, L_{(np, nq)}, \beta) \to \mathcal{H}^+(S^3, L_{(np, nq)})$$

induced by restriction, is an isomorphism.

PROOF. Let $L_{(np, nq)}$ have components $K_1, \ldots, K_n$. Let the homomorphisms

$$\mathcal{H}^+(S^3, L_{(np, nq)}), \ \mathcal{H}^+(S^3, L_{(np, nq)}, \beta) \to S(n)$$

to the group $S(n)$ of permutations of n objects, map each homomorphism h to its permutation of the components $K_1, \ldots, K_n$.

Now, in the commutative diagram which follows, the rows are exact, and the vertical arrows are induced by restriction:

$$1 \to \mathscr{H}^+(S^3, K_1, \ldots, K_n) \to \mathscr{H}^+(S^3, L_{(np, nq)}) \to S(n)$$
$$\uparrow \qquad\qquad\qquad \uparrow \qquad\qquad \uparrow$$
$$1 \to \mathscr{H}^+(S^3, K_1, \ldots, K_n, \beta) \to \mathscr{H}^+(S^3, L_{(np, nq)}, \beta) \to S(n) .$$

It follows from the definition of the motions of $L_{(np, nq)}$ in S^3 in the next section, that the homomorphisms $\mathscr{H}^+(S^3, L_{(np, nq)})$, $\mathscr{H}^+(S^3, L_{(np, nq)}, \beta) \to S(n)$ are surjective. It is a consequence of Lemma 8.3, that the vertical arrow on the left is an isomorphism. Lemma 8.2 now follows from the Five Lemma.

In Lemma 8.3, let $L_{(np, nq)} = K_1 \cup \ldots \cup K_n$ have components K_i.

LEMMA 8.3. *Suppose* $q > 1$, $n > 0$. *Then the homomorphism*

$$\mathscr{H}^+\left(S^3, \bigcup_{i=2}^n K_i, K_1, \beta\right) \to \mathscr{H}^+\left(S^3, \bigcup_{i=2}^n K_i, K_1\right)$$

induced by restriction, is an isomorphism.

PROOF. Let $\pi: F \to \mathbf{R}^2$ be as in Definition 4.8, and P_k and A be defined in Notation 4.9. Let $\Sigma: F \to F$ be as in Remark 4.11.

By Remarks 4.10 and 4.11, the component K_1 is a generalized axis for the link $K_2 \cup \ldots \cup K_n$, with monodromy $\Sigma \in H_c(F, \hat{P}_2, \ldots, \hat{P}_n)$. By Theorem 5.28, the sequence:

$$1 \to \langle [\Sigma], [\tau] \rangle \to \mathscr{H}_\Sigma\left(F, \bigcup_{i=2}^n \hat{P}_i\right) \xrightarrow{e \circ J} \mathscr{H}^+\left(S^3, \bigcup_{i=2}^n K_i, K_1\right) \to \mathbf{Z}_2$$

is exact. Since

$$\mathscr{H}_\Sigma^f\left(F, \bigcup_{i=2}^n \hat{P}_i\right) = \mathscr{H}_\Sigma\left(F, \bigcup_{i=2}^n \hat{P}_i\right)\bigg/[\tau]$$

(see Notation 7.1 and Definition 5.22), we may rewrite the exact sequence:

$$(*) \qquad 1 \to \mathscr{H}_\Sigma^f\left(F, \bigcup_{i=2}^n \hat{P}_i\right)\bigg/[\Sigma] \to \mathscr{H}^+\left(S^3, \bigcup_{i=2}^n K_i, K_1\right) \to \mathbf{Z}_2 .$$

Again by Remark 4.11 and Corollary 4.13, the component K_1 is a generalized axis for the link $[\bigcup_{i=2}^n K_i] \cup \beta$, with monodromy $\Sigma \in H_c(F, \hat{P}_1, \ldots, \hat{P}_n)$. By

Theorem 5.28, the sequence:

$$1 \to \langle [\Sigma], [\tau] \rangle \to \mathscr{H}_\Sigma\left(F, \bigcup_{i=2}^{n} \hat{P}_i, \hat{P}_1\right) \to \mathscr{H}^+\left(S^3, \bigcup_{i=2}^{n} K_i, K_1, \beta\right) \to \mathbf{Z}_2$$

is exact. Rewriting this exact sequence using

$$\mathscr{H}_\Sigma^f\left(F, \bigcup_{i=2}^{n} \hat{P}_i, \hat{P}_1\right) = \mathscr{H}_\Sigma\left(F, \bigcup_{i=2}^{n} \hat{P}_i, P_1\right)\Big/[\tau],$$

we have:

$$(**) \quad 1 \to \mathscr{H}_\Sigma^f\left(F, \bigcup_{i=2}^{n} \hat{P}_i, \hat{P}_1\right)\Big/[\Sigma] \to \mathscr{H}^+\left(S^3, \bigcup_{i=2}^{n} K_i, K_1, \beta\right) \to \mathbf{Z}_2.$$

We now have a commutative diagram, in which the rows $(*)$, $(**)$, are exact, and the vertical arrows are induced by restriction:

$$(*) \qquad 1 \longrightarrow \mathscr{H}_\Sigma^f\left(F, \bigcup_{i=2}^{n} \hat{P}_i\right)\Big/[\Sigma] \longrightarrow \mathscr{H}^+\left(S^3, \bigcup_{i=2}^{n} K_i, K_1\right) \longrightarrow \mathbf{Z}_2$$

$$\uparrow \simeq \qquad\qquad\qquad \uparrow \qquad\qquad \uparrow \simeq$$

$$(**) \qquad 1 \to \mathscr{H}_\Sigma^f\left(F, \bigcup_{i=2}^{n} \hat{P}_i, \hat{P}_1\right)\Big/[\Sigma] \to \mathscr{H}^+\left(S^3, \bigcup_{i=2}^{n} K_i, K_1, \beta\right) \to \mathbf{Z}_2.$$

By the definition of the flip motion f in the next section (The motions of $L_{(np, nq)}$ in S^3), it follows that f_1 maps onto the generator of $\mathbf{Z}_2$; whereby the maps

$$\mathscr{H}^+\left(S^3, \bigcup_{i=2}^{n} K_i, K_1\right), \ \mathscr{H}^+\left(S^3, \bigcup_{i=2}^{n} K_i, K_1, \beta\right) \to \mathscr{H}(K_1) = \mathbf{Z}_2$$

are surjective. The vertical arrow on the left is an isomorphism by Application 7.4. Lemma 8.3 now follows from the Five Lemma.

The motions of $L_{(np, nq)}$ in S^3.

1. $\sigma_i(p, q)$, $i = 1, \ldots, n-1$: interchange K_i with K_{i+1}.
2. $\varrho_i(p, q)$, $i = 1, \ldots, n$: rotate K_i by $2\pi/p$ about the axis α.

The motions $\sigma_i(p, q)$, $i = 1, \ldots, n-1$, are defined as follows (and the motions $\varrho_i(p, q)$ are defined similarly):

Let the motion σ_i, $i = 1, \ldots, n-1$, of $\bigcup_{k=1}^{n} 1/k$ in $\mathbf{R}^2 - \{0\}$ be given by $\sigma_{i,t}(p) \in H_c(\mathbf{R}^2, 0)$, $t \in [0, 1]$, where $\sigma_{i,t}(p)$ is the unique homeomorphism with compact support, which makes the diagram:

$$\begin{array}{ccc} \mathbf{R}^2 & \xrightarrow{\sigma_{i,t}(p)} & \mathbf{R}^2 \\ \downarrow \pi_p & & \downarrow \pi_p \\ \mathbf{R}^2 & \xrightarrow{\sigma_{i,t}} & \mathbf{R}^2 \end{array}$$

commute. (See Figure 8.4). (Clearly $\sigma_i(p)$ and $\varrho_i(p)$ are symmetric with respect to rotation by $2\pi/p$.)

Figure 8.4.
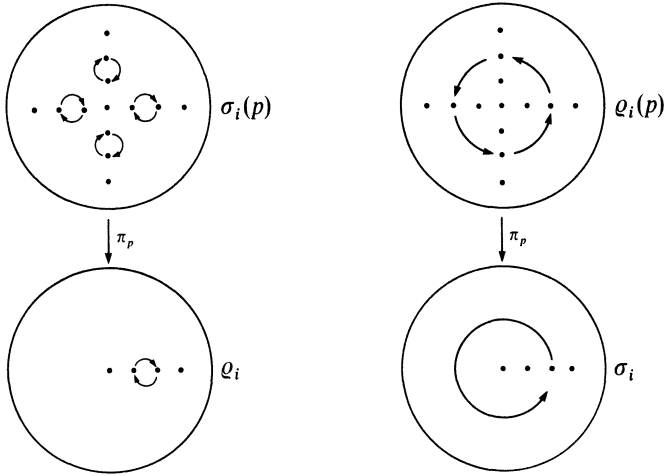
We wish to extend the motion $\sigma_i(p)$ of $[\bigcup_{k=1}^n (1/k)P_p]$ in $\mathbf{R}^2$, to a motion of the torus link $L_{(np,nq)}$ in $S^3 - \alpha - \beta$. Recall from Example 4.4, that β is an axis for the link $L_{(np,nq)} \cup \alpha$, with monodromy

$$\varphi_{q/p}: \left(\mathbf{R}^2, \left[\bigcup_{k=1}^n \frac{1}{k}P_p\right] \cup 0\right) \to \left(\mathbf{R}^2, \left[\bigcup_{k=1}^n \frac{1}{k}P_p\right] \cup 0\right).$$

Since the motion $\sigma_i(p)$ commutes with the monodromy $\varphi_{q/p}$ at every t, the extension is possible by elementary arguments. We will denote the extension by $\sigma_i(p,q)$.

REMARK 8.5. The motions $\sigma_i(p,q)$, $i=1,\dots,n-1$, $\varrho_i(p,q)$, $i=1,\dots,n$, are invariant under the rotation $r_{1/p,1/q}: S^3 \to S^3$ of the sphere (defined in Notation 2.1).

REMARK 8.6. Certain products of the motions ϱ_i, $i=1,\dots,n$, can be described as follows (for readability, I will exclude the pair (p,q) from the notation; but it must be kept in mind that when I write ϱ_i, I intend to write $\varrho_i(p,q)$):

1) $\varrho_1\varrho_2 \dots \varrho_n$: rotate all of $L_{(np,nq)}$ by $2\pi/p$ about α.
2) $(\varrho_1\varrho_2 \dots \varrho_n)^{-1}$: rotate all of $L_{(np,nq)}$ by $2\pi/q$ about β.

3. f: flip $L_{(np,nq)}$.

The motion f is a rotation of S^3 by π, which carries α to α^{-1}, β to β^{-1}, and $L_{(np, nq)}$ to itself with opposite orientation on every component. $f_1 \in H(S^3, L_{(np, nq)})$ is the map

$$(re^{i\theta}, se^{i\varphi}) \rightarrow (re^{-i\theta}, se^{-i\varphi}) ,$$

where $S^3 = \mathbf{C} \times S^1 \cup S^1 \times \mathbf{C}/(re^{i\theta}, e^{i\varphi}) \sim (e^{i\theta}, r^{-1}e^{i\varphi})$, as in Notation 2.1.

We are now ready to state the main theorem. In Theorem 8.7, let $\sigma_i(p, q)$, $\varrho_i(p, q)$ and f denote both the motion of $L_{(np, nq)}$ in S^3 and its equivalence class in $\mathcal{M}(S^3, L_{(np, nq)})$.

THEOREM 8.7. (The Main Theorem). *Let* $\mathcal{M}(S^3, L_{(np, nq)})$ *be the group of motions of the torus link* $L_{(np, nq)}$ *of type* (np, nq) *in* S^3.

Generating motions for this group are $\sigma_i(p, q)$, $i = 1, \ldots, n-1$, $\varrho_i(p, q)$, $i = 1, \ldots, n$, f *and* $[2\pi]$.

A complete set of defining relations are:

1. $\sigma_i(p, q) \leftrightarrow \sigma_j(p, q)$ *if* $|i-1| \geqq 2$.

2. $\sigma_i(p, q)\sigma_{i+1}(p, q)\sigma_i(p, q) = \sigma_{i+1}(p, q)\sigma_i(p, q)\sigma_{i+1}(p, q)$ *if* $1 \leqq i \leqq n-2$.

3. $\varrho_i(p, q) \leftrightarrow \varrho_j(p, q)$ *if* $1 \leqq i, j \leqq n$.

4. $\varrho_i(p, q) \leftrightarrow \sigma_j(p, q)$ *if* $j \neq i-1$.

5. $\varrho_{j+1}(p, q) = \sigma_j(p, q) \ldots \sigma_2(p, q)\sigma_1(p, q)\varrho_1(p, q)\sigma_1(p, q)\sigma_2(p, q) \ldots \sigma_j(p, q)$

    if $1 \leqq j \leqq n-1$.

6. $f^2 = [2\pi]$.

7. $(\varrho_1(p, q)\varrho_2(p, q) \ldots \varrho_n(p, q))^p = [2\pi] = (\varrho_1(p, q)\varrho_2(p, q) \ldots \varrho_n(p, q))^{-p}$.

8. $[2\pi]^{p+q} (\mathrm{mod}\ 2) = 1$.

9. $f^{-1}\sigma_i(p, q)f = \sigma_i^{-1}(p, q)$, $i = 1, \ldots, n-1$.

10. $f^{-1}\varrho_i(p, q)f = \varrho_i^{-1}(p, q)$, $i = 1, \ldots, n$.

11. (*if* $p = 1$) $\varrho_n = 1$.

12. (*if* $q = 1$) $\varrho_1 = 1$.

PROOF. Suppose $q > 1$. By Theorem 8.1, the homomorphism

$$\mathcal{M}(S^3, L_{(np, nq)}, \beta) \rightarrow \mathcal{M}(S^3, L_{(np, nq)})$$

induced by restriction, is an isomorphism. We will compute $\mathcal{M}(S^3, L_{(np, nq)}, \beta)$.

By Theorem 6.8, and Example 4.4, we have an exact sequence:

$$(*)\ 1 \rightarrow \langle [\varphi_{q/p}], [\tau] = [\sigma] \rangle \rightarrow \mathcal{H}_{\varphi_{q/p}}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}P_p\right) \xrightarrow{K} (S^3, L_{(np, nq)}, \beta) \twoheadrightarrow \mathbf{Z}/2$$

where $\mathbf{Z}/2\mathbf{Z} = \mathcal{H}(\beta)$, and the motion f maps to the generator of $\mathcal{H}(\beta)$.

CASE 1. $(p > 1)$. By Application 7.5, the homomorphism:

$$\mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) \xrightarrow{\simeq} \mathscr{H}_{\varphi_{q/p}}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right)$$

induced by lifting under the p-fold, cyclic branched covering

$$\pi_p \colon \left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right) \to \left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}\right),$$

branched about $0 \in \mathbf{R}^2$, is an isomorphism. The exact sequence $(*)$ then becomes:

$$1 \to \langle [\varphi_q], [\varphi_p] \rangle \to \mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) \to \mathscr{M}(S^3, L_{(np, nq)}, \beta) \to \mathbf{Z}/2\mathbf{Z} \ ,$$

where the map $(\mathbf{R}^2, \bigcup_{k=1}^{n} (1/k), 0) \to \mathscr{M}(S^3, L_{(np, nq)}, \beta)$ is the composition:

$$\mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) \to \mathscr{H}_{\varphi_{q/p}}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k} P_p\right) \to \mathscr{M}(S^3, L_{(np, nq)}, \beta) \ .$$

Now, $\mathscr{H}(\mathbf{R}^2, \bigcup_{k=1}^{n} (1/k), 0)$ is the n-braid group $B_n(\mathbf{R}^2 - \{0\})$ of the annulus. A presentation for the latter group is given in Theorem 7.10. Since, by the *definition of $\sigma_i(p, q)$ and $\varrho_i(p, q)$*, $\sigma_i \to \sigma_i(p, q)$ and $\varrho_i \to \varrho_i(p, q)$ under the map:

$$\mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}, 0\right) \to \mathscr{M}(S^3, L_{(np, nq)}, \beta) \ ,$$

it follows that the motions $\sigma_i(p, q)$, $i = 1, \ldots, n-1$, $\varrho_i(p, q)$, $i = 1, \ldots, n, f$ and $[2\pi]$ generate $\mathscr{M}(S^3, L_{(np, nq)}, \beta)$. This also accounts for the defining relations 1–5 of the subgroup, image (K).

Relation 6 follows from the definition of f. Relation 7 follows from Remark 8.6. Relation 8 is proved in Theorem 3.7. Together, relations 1–8 are a *complete* set of defining relations for the subgroup, image (K), by Theorem 6.8.

Relations 9, 10 are geometrically obvious. Since Relations 11, 12 do not apply, this completes the proof of this case; for, Relations 9, 10 define the particular normal extension of image (K), which defines the group $\mathscr{M}(S^3, L_{(np, nq)}, \beta)$ if $p, q > 1$.

CASE 2. $(p = 1)$. Since $[\varphi_1]$ generates the center of the group $\mathscr{H}(\mathbf{R}^2, \bigcup_{k=1}^{n} (1/k))$, the groups $\mathscr{H}(\mathbf{R}^2, \bigcup_{k=1}^{n} (1/k))$ and $\mathscr{H}_{\varphi_q}(\mathbf{R}^2, \bigcup_{k=1}^{n} (1/k))$ are identical. Thus, the exact sequence $(*)$ becomes:

$$1 \to \langle [\varphi_q], [\varphi_1] \rangle \to \mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^{n} \frac{1}{k}\right) \xrightarrow{K} \mathscr{M}(S^3, L_{(np, nq)}) \to \mathbf{Z}/2\mathbf{Z} \ .$$

Now, $\mathscr{H}(\mathbf{R}^2, \bigcup_{k=1}^n (1/k))$ is the n-braid group $B_n(\mathbf{R}^2)$ of the disk, a presentation for which appears in Theorem 7.9. Since, by the definition of $\sigma_i(p,q)$, $\sigma_i \to \sigma_i(p,q)$ under the map:

$$\mathscr{H}\left(\mathbf{R}^2, \bigcup_{k=1}^n \frac{1}{k}, 0\right) \to \mathscr{M}(S^3, L_{(np,nq)}, \beta) \, ,$$

it follows that $\sigma_i(p,q)$, $i=1,\ldots,n-1$, are generating motions for the group $\mathscr{M}(S^3, L_{(np,nq)}, \beta)$, and also that $\varrho_n = 1$. This also accounts for the defining relations 1 and 2 and 11, of the subgroup image (K). The rest of the proof is as in Case 1.

Now suppose $q = 1$. The homeomorphism $h(z_1, z_2) = (z_2, z_1)$ of $S^3 = \mathbf{C} \times S^1 \cup S^1 \times \mathbf{C}/\sim$ (as in Notation 2.1), carries α to β, β to α, and $L_{(np,nq)}$ to $L_{(nq,np)}$. If $p > 1$, we may apply Theorems 8.7 to the torus link $L_{(nq,np)}$. There is a transformation of the motions as a result of conjugation by the homeomorphism h, $(\mathscr{M}(S^3, L_{(nq,np)}) \to \mathscr{M}(S^3, L_{(np,nq)})$ is the transformation $[f_t] \to [h^{-1} \circ f_t \circ h]$, where f_t, $t \in [0,1]$, is a motion of $L_{(np,nq)}$ in S^3) which may be computed without too much difficulty. I leave it to the reader to verify that the resulting presentation of $\mathscr{M}(S^3, L_{(np,nq)})$ is equivalent to that of Theorem 8.7.

If $q = 1$, $p = 1$, then each K_i, $i = 1, \ldots, n$, is unknotted; in fact, $L_{(n,n)}$ is equivalent (ambient isotopic) to the link $L_{(n-1,n-1)} \cup \beta$ in S^3. Thus, setting $\sigma_{n-1}(p,q) = 1 = \varrho_n(p,q)$, Theorem 8.7 gives generators, and a complete set of defining relations for the motion group $\mathscr{M}(S^3, L_{(n-1,n-1)}, \beta)$.

It is easy to verify that relations 1–12 hold in $\mathscr{M}(S^3, L_{(n,n)})$, and that they include a complete set of defining relations for the subgroup $\mathscr{M}(S^3, \bigcup_{i=1}^{n-1} K_i, K_n)$, by the previous paragraph. Since the addition of the motion $\sigma_{n-1}(p,q)$ (where $p = q = 1$) allows for all possible permutations of the components $K_1, \ldots, K_n$, relations 1–12 must be a complete set of defining relations for $\mathscr{M}(S^3, L_{(n,n)})$.

COROLLARY 8.8. *The group* $\mathscr{M}(S^3, L_{(p,q)})$ *of motions in* S^3 *of the torus knot* $L_{(p,q)}$ *of type* (p,q), *is*

$$\mathscr{M}(S^3, L_{(p,q)}) = \{f : (f^2)^{p+q \,(\mathrm{mod}\,2)} = 1\} \; ;$$

where the generator f *rotates* S^3 *by* π, *taking* $L_{(p,q)}$ *to itself with opposite orientation.*

PROOF. Set $\sigma_i(p,q) = 1$ in Theorem 8.7.

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DEPARTMENT OF MATHEMATICS
THE UNIVERSITY OF MICHIGAN, USA

AND

DEPARTMENT OF MATHEMATICS
FAIRFIELD UNIVERSITY, FAIRFIELD
CONNECTICUT 06430, USA.