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THE STRUCTURE OF AN ALMOST ARTINIAN MODULE

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Let R be a commutative ring with identity. Call an R-module *almost Artinian* if every proper homomorphic image is Artinian. If A is an almost Artinian R-module that is not Artinian, then it must be the case that either (1) R is a Noetherian domain of dimension one (equivalently every almost but not Artinian R-module is faithful) or (2) R has a non-zero prime ideal P such that R/P is a Noetherian domain of dimension one (equivalently every almost but not Artinian R-module is not faithful). It is also the case that $\mathrm{Ann}_R(A)$ is a prime ideal of R and A is isomorphic to a submodule of the quotient field of $R/\mathrm{Ann}_R(A)$. Therefore to study the structure of A it is sufficient to assume that R is a Noetherian domain of dimension one. The structure of A when R is a Dedekind ring along with the above results appear in [1].

The purpose of this paper is to give the structure of A in the general case, where R is a Noetherian domain of dimension one. Matlis in [3] studied the theory of an Artinian divisible module over a local Cohen–Macaulay ring of dimension one, and we will use results from this theory. In what follows, let Q denote the quotient field of R, and for each maximal ideal M of R let $R(M^\infty)$ denote the M-primary submodule of Q/R. That is, the set of elements in Q/R that are annihilated by some power of M.

THEOREM. *Let R be a Noetherian domain of dimension one and let A be an almost but not Artinian R-module. Then there is an ideal I of R, a non-zero element $y \in I$, and a finite set $M_1,\ldots,M_n$ of maximal ideals of R such that*

$$A \cong I(1/y)+G_1+\ldots+G_n$$

where each G_i is an R-submodule of Q containing R with the property that G_i/R is M_i-primary and there is a series of modules

$$R = D_{i_0} \subset D_{i_1} \subset \ldots \subset D_{i_{m_i}} = G_i$$

such that each factor D_j/D_{j-1} is divisible and has no proper non-zero divisible submodules.

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PROOF. Since A is isomorphic to a submodule of Q, we can assume without loss of generality that $R \subseteqq A \subseteqq Q$. Since A/R is Artinian, there exist maximal ideals $M_1, \ldots, M_n$ of R such that $A/R = P_1 \oplus \ldots \oplus P_n$, where each P_i is the M_i-primary part of A/R [2, Theorem 1]. It follows that each P_i is an Artinian module over the local, one dimensional, Noetherian domain R_{M_i}. Let G_i/R be the largest divisible R_{M_i}-submodule of P_i. Then $P_i/(G_i/R)$ has no non-zero divisible submodules, and must therefore be a finitely generated R_{M_i}-module [3, Theorem 5.1]. Hence there is a finitely generated R_{M_i}-submodule F_i/R of P_i such that $P_i = F_i/R + G_i/R$. Since G_i/R is an Artinian divisible R_{M_i}-module, it follows from [3, Theorem 5.5] that there is a composition series of divisible R_{M_i}-modules

$$0 = D_{i_0}/R \subset D_{i_1}/R \subset \ldots \subset D_{i_{m_i}}/R = G_i/R$$

such that the factor modules (isomorphic to D_j/D_{j-1}) have no proper non-zero divisible submodules. Since the R_{M_i}-submodules and the R-submodules of $R(M_i^\infty)$ are identical, it follows that F_i/R is a finitely generated R-module. Also each D_j/D_{j-1} is a divisible R-module with no proper non-zero divisible R-submodules. Let $F = F_1 + \ldots + F_n$. Since each F_i/R is finitely generated it follows that F is a finitely generated R-submodule of Q. Therefore there is an ideal I of R and a non-zero element $y \in I$ such that $F = I(1/y)$. Since A is the sum of the F_i and the G_i, the proof is finished.

REMARK. The series $R = D_{i_0} \subset D_{i_1} \subset \ldots \subset D_{i_{m_i}} = G_i$ in the theorem is unique in the following sense [3, Theorem 5.10]: Any other series from R to G_i has the same length, and there is a one to one correspondence between the sets of factors such that the corresponding factors are homomorphic images of each other.

R is said to be *analytically irreducible* if for each maximal ideal M, the completion of R in the M-adic topology is an integral domain. This is equivalent to the property that every proper R_M-submodule of Q/R_M is finitely generated [3, Theorem 7.1]. A Dedekind ring is analytically irreducible. The structure of an almost but not Artinian module is much simpler for this class of rings.

COROLLARY. *Let R be an analytically irreducible Noetherian domain of dimension one and let A be an almost but not Artinian R-module. Then there is an ideal I of R, a non-zero element $y \in I$, and a finite set $M_1, \ldots, M_k$ of maximal ideals such that*

$$A \cong I(1/y) + G(M_1) + \ldots + G(M_k)$$

where $G(M) = \{x \in Q \mid M^r x \subseteqq R \text{ for some } r \geqq 0\}$.

PROOF. As in the proof of the theorem we can assume that $R \subseteqq A \subseteqq Q$. Let $M_1, \ldots, M_n$ be the maximal ideals of R such that $A/R = P_1 \oplus \ldots \oplus P_n$, where each P_i is M_i-primary. Since $P_i \subseteqq R(M_i^\infty)$ and $R(M_i^\infty) \cong Q/R_{M_i}$, it follows that either P_i is finitely generated or $P_i = R(M_i^\infty) = G(M_i)/R$. After some renumbering of the indices the result follows.

REFERENCES

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