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HARDY SPACES, A_∞, AND SINGULAR INTEGRALS ON CHORD-ARC DOMAINS

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Introduction.

Let Λ be an unbounded Jordan curve in the complex plane, and denote by D the region to one side of Λ. If Λ is locally rectifiable, we will denote arc length on Λ by σ. An arc along Λ with endpoints z_1 and z_2 will be denoted (z_1, z_2). Λ is said to be a *chord-arc curve* and D a chord-arc domain if there is a constant C such that for all points z_1, z_2 of Λ, $\sigma(z_1, z_2) \leqq C|z_1 - z_2|$. The non-tangential approach region to a boundary point $z \in \Lambda$ is given by

$$\Gamma_\alpha(z) = \Gamma_\alpha(z, D) = \{z' \in D : |z'-z| \leqq (1+\alpha)\operatorname{dist}(z', \Lambda)\} .$$

(The particular value of $\alpha > 0$ will not be relevant.) The non-tangential maximal function of a function F on D is

$$N_\alpha(F)(z) = \sup\{|F(z')| : z' \in \Gamma_\alpha(z)\}, \qquad z \in \Lambda .$$

The Hardy spaces for $0 < p < \infty$ are

$$H^p(D, d\sigma) = \{F : F \text{ holomorphic in } D \text{ and } N_\alpha(F) \in L^p(\Lambda, d\sigma)\} .$$

Our first goal is to study $H^p(D, d\sigma)$ and to deduce estimates for singular integrals on the curve Λ. In doing so we extend results of Kenig [12] and Coifman and Meyer [4] from Lipschitz domains to chord-arc domains. The underlying philosophy is that chord-arc curves are very similar to Lipschitz curves. In fact, a chord-arc curve is the bi-Lipschitzian image of a straight line (see Proposition 1.13). Moreover, the conformal mapping from the half-plane to a chord-arc domain is essentially as well-behaved as the conformal mapping to a Lipschitz domain. Our second goal is to show that in some sense the chord-arc condition is also necessary to the theorems in question. This is expressed by the close connection between the chord-arc condition and the condition A_∞ defined below.

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We have reversed the usual progression by specializing from n dimensions to two dimensions results of our earlier work [11]. We have done so because a clearer, more detailed understanding of the two dimensional case can give us further guidance in $\mathbf{R}^n$ (se section 5).

For many purposes a weaker condition than chord-arc, namely Ahlfors' three point condition, suffices. Λ satisfies the *three point condition* if there is a constant C such that for any three points z_1, z_2, z_3 on Λ with $z_3 \in (z_1, z_2)$, $|z_1 - z_3| \leqq C|z_1 - z_2|$. Ahlfors showed that the three point condition is equivalent to the existence of a global K-quasiconformal mapping f of the plane with $f(\Sigma) = D$ ($\Sigma = \{w : \operatorname{Im} w > 0\}$, K is the dilatation constant of f.) For this reason (and the equivalence of the half plane with the disk) the domain D is called a *K-quasicircle.*

Curves satisfying the three point condition need not be rectifiable. The appropriate measure on Λ is an unbounded version of harmonic measure defined as follows. For any Jordan domain D as above there is a conformal mapping $\Phi: \Sigma \to D$ such that $\Phi(\infty) = \infty$. To specify Φ uniquely, suppose that $\Phi(i) = z_0$ for some particular $z_0 \in D$. Φ extends to a homeomorphism of $\bar{\Sigma}$ onto $\bar{D}$. In fact, when D is a quasicircle, Φ extends to a global quasiconformal mapping $\mathbf{C} \to \mathbf{C}$, (see [15, p. 98]). *Harmonic measure* ω on Λ (at z_0) is defined as the measure for which $\omega(\Phi(E)) = |E|$. (E denotes a Borel subset of $\mathbf{R}$; $|E|$ denotes Lebesgue measure of E.) The Hardy spaces for quasicircles with respect to harmonic measure were examined in [11].

The connection between arc length and harmonic measure that makes it possible to study the Hardy spaces with respect to arc length is known as A_∞. We say that σ and ω are *A_∞-equivalent* if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any arc $I \subset \Lambda$ and Borel set $E \subset I$, $\sigma(E)/\sigma(I) < \delta$ implies $\omega(E)/\omega(I) < \varepsilon$. Coifman and Fefferman [3] proved that A_∞ is an equivalence relation and that it is equivalent to the apparently weaker statement where all $\varepsilon > 0$ is replaced by a single $\varepsilon < 1$. Thus we can refer to a single pair (ε, δ) with $\varepsilon < 1$ and $\delta > 0$ as *A_∞ constants* for the A_∞-equivalence. The crucial theorem is due to Laurentiev [14]:

THEOREM. *If D is a chord-arc domain, then arc-length and harmonic measure on ∂D are A_∞-equivalent. The A_∞ constants depend only on the chord-arc constant of D.*

We will prove this theorem in the next section. Two versions of a converse are proved in Section 4.

As in [11], it is worthwhile to remember the simple feature shared by the properties under consideration. The constants associated to Lipschitz domains, quasicircles, and the three point, chord-arc, and A_∞ conditions are all invariant under the change of scale $z \to rz$.

We would like to thank Professor F.W. Gehring for suggesting the proof of Proposition 1.1 based on quasiconformal mapping given here.

1. Geometric properties of the conformal mapping Φ.

Our first observation is that Φ preserves non-tangential approach regions:

PROPOSITION 1.1. *If D is a K-quasicircle, then for any $\alpha > 0$ there exist $\beta, \gamma > 0$ depending only on α, K such for any $x \in \mathbf{R} = \partial\Sigma$,*

$$\Gamma_\beta(\Phi(x), D) \subset \Phi(\Gamma_\alpha(x, \Sigma)) \subset \Gamma_\gamma(\Phi(x), D) .$$

This is an easy consequence of the fact that Φ extends to a global quasiconformal mapping and

THEOREM 1.2. (Distortion Theorem [20]). *Let f be a K-quasiconformal mapping of the plane to itself. There is a constant C depending only on K such that for any $z \in \mathbf{C}$, $r > 0$, there exists $r' > 0$ such that*

$$B(f(z), r') \subset f(B(z, r)) \subset B(f(z), Cr') ,$$

where $B(z, r) = \{z' : \ |z - z'| < r\}$.

The distortion theorem says that f preserves comparative distance in the following sense: If z_1, z_2, z_3 satisfy $|z_1 - z_2| \leqq |z_2 - z_3|$, then $|f(z_1) - f(z_2)| \leqq C|f(z_2) - f(z_3)|$. The same is true for f^{-1}.

A useful companion to 1.2 is the following estimate: There exists a number $\tilde{M}$ such that if $C < \tilde{M}$, and dist $(B(z_1, s), B(z_2, r)) \leqq Cs$, and $s \leqq Cr$, then

$$\text{diam}\{f(B(z_1, s))\} \leqq \tilde{M} C^\alpha \,\text{diam}\{f(B(z_2, r))\} , \tag{1.2'}$$

(see [20]). As a consequence of (1.2) and (1.2′), we deduce that there exists a constant M depending only on K such that

$$\text{if } |z_1 - z_2| \leqq \frac{1}{M}|z_2 - z_3| , \tag{1.2''}$$

then

$$|f(z_1) - f(z_2)| \leqq \tfrac{1}{2}|f(z_2) - f(z_3)| .$$

By iterating these inequalities we find that there exists a constant M depending only on K such that

$$(1.3)\quad |z_1-z_2| \leqq M^{-k}|z_2-z_3| \quad \text{implies } |f(z_1)-f(z_2)| \leqq C2^{-k}|f(z_2)-f(z_3)| .$$

The same is true for f^{-1}. A corollary of 1.3 is that f and f^{-1} are Hölder continuous. There is $\alpha>0$ depending only on K such that if $z \in B$, where B is the unit disc, and $w \in \mathsf{C}$,

$$(1.4)\qquad C^{-1}\min(|z-w|^{\alpha}, |z-w|^{1/\alpha}) \leqq \frac{|f(z)-f(w)|}{\operatorname{diam} f(B)}$$

$$\leqq C\max(|z-w|^{\alpha}, |z-w)^{1/\alpha})$$

estimates 1.3 and 1.4 will be used frequently. Proposition 1.1 can also be proved directly using harmonic measure.

PROPOSITION 1.5. *If D is a K-quasicircle, then the image under Φ of a vertical line $\{x+iy\colon y>0\}$ is a chord-arc curve with constant depending only on K.*

PROOF: First, recall the well-known estimate

$$(1.6)\qquad |\Phi'(z)| \operatorname{dist}(z, \mathsf{R}) \cong \operatorname{dist}(\Phi(z), \Lambda)$$

(see [17, page 22]). (Here $A \cong B$ means $C^{-1}A \leqq B \leqq CA$, where C depends only on K. Elsewhere in the paper, $A \cong B$ will mean the same thing, but with K replaced by the constant(s) appropriate to the context.) To prove 1.6, note that 1.3 implies that if $r=\operatorname{dist}(z, \mathsf{R})$, then Φ maps $B(z, \frac{1}{2}r)$ into a ball of radius Cr', where $r'=\operatorname{dist}(\Phi(z), \Lambda)$. By Schwarz' lemma, $|\Phi'(z)| \leqq Cr'/r$. The same argument applied to Φ^{-1} gives the lower bound.

To prove the proposition, it is enough to consider integrals of the form

$$\int_0^y |\Phi'(x+it)|\, dt ,$$

and show that they are comparable to $\operatorname{dist}(\Phi(x+iy), \Lambda)=r$. Since Φ extends to a global quasiconformal mapping, the image under Φ of the vertical line satisfies the three point condition with a constant depending only on the dilatation constant K. It is then easy to see that for sufficiently large M, if we choose z_k so that $z_k=\Phi(x+iy_k)$ and $\operatorname{dist}(z_k, \Lambda)=M^{-k}r$, the $\{y_k\}$ decrease, $y_k/y_{k+1} \leqq C$ and for $y_{k+1} \leqq t \leqq y_k$,

$$\operatorname{dist}(\Phi(x+it), \Lambda) \cong M^{-k}r .$$

It follows from 1.6 that

$$\int_0^y |\Phi'(x+it)|\, dt = \sum_{k=0}^{\infty} \int_{y_{k+1}}^{y_k} |\Phi'(x+it)|\, dt \leqq C \sum_{k=0}^{\infty} M^{-k}r = Cr .$$

Let $w_1 = \Phi(x+i\eta_1)$, $w_2 = \Phi(x+i\eta_2)$. Then, we will denote

$$L(w_1) = \Phi(\{(x+iy) :\ y \geqq \eta_1\}),$$
$$\text{and} \quad L(w_1, w_2) = \Phi\{(x+iy) :\ \eta_1 \leqq y \leqq \eta_2\}.$$

By 1.5, if we use the arc-length parametrization, we see that if $z \in D$, $\gamma > 0$,

$$\int_{L(z)} \operatorname{dist}(\zeta, \Lambda)^{-1-\gamma} d\sigma(\zeta) \cong \operatorname{dist}(z, \Lambda)^{-\gamma}. \tag{1.7}$$

If $z_0 \in \Lambda$, $z \in L(z_0)$ and $\gamma < 2$, then

$$\int_{L(z_0, z)} \operatorname{dist}(\zeta, \Lambda)^{1-\gamma} d\sigma(\zeta) \cong \operatorname{dist}(z, \Lambda)^{2-\gamma}. \tag{1.8}$$

We will now prove Laurentiev's theorem (see Introduction). The core of the proof is a lemma corresponding to the case where I is replaced by ∂B, where B is the unit disk.

LEMMA 1.9. *Let Ψ be a conformal mapping of the unit disk B onto a domain Ω such that $\Psi(0)=0$, $B \subset \Omega$ and $\sigma(\partial\Omega) < C$. For any $\varepsilon > 0$ there exists $\delta > 0$ depending only on C and ε such that if $E \subset \partial B$, $\sigma(\Psi(E)) < \delta$ implies $|E| < \varepsilon$.*

The proof can be found in [14] or [11].

To reduce the full theorem to this lemma, consider the square

$$Q_I = \{x+iy :\ x \in I,\ 0 < y < |I|\}.$$

Let x_I be the midpoint of I and $w_I = \Phi(x_I + i|I|/2)$. Let $r = \operatorname{dist}(w_I, \Lambda)$. From 1.2, $B(w_I, C^{-1}r) \subset \Phi(Q_I) \subset B(w_I, Cr)$. By the chord-arc property for Λ, Propositions 1.5 and 1.6, $\sigma(\Phi(I)) \cong r$ and $\sigma(\partial\Phi(Q_I)) \leqq Cr$. By a change of scale in the domain of Φ by a factor $|I|^{-1}$ and in the range of Φ by a factor r^{-1}, we may as well assume $|I| = 1$ and $r = 1$. We are now faced with proving a lemma like 1.9 with the unit disk replaced by the unit square. This change is harmless. Harmonic measure for the square is well known; and arc length and it are mutually absolutely continuous.

We have considered "vertical" lines $L(z)$. Let us examine "horizontal" lines $\Lambda_t = \{\Phi(x+it)\ x \in \mathbf{R}\}$.

In the proof of Laurentiev's theorem we showed that, for any $t > 0$, if I is an interval with $|I| \approx ct$, and center x_I and we let $I' = \{x+it :\ x \in I\}$, then $|\Phi'(z)| \cong r/t$ for all $z \in I'$, where $r = \operatorname{dist}(\Phi(x_I + it), \Lambda)$. Using this fact, (1.2) and (1.2'), it is easy to show that if $J = [a, b]$, with $(b-a) \leqq ct$, then

$$\int_J |\Phi'(x+it)|\,dx \cong |\Phi(b+it)-\Phi(a+it)| .$$

Also in the proof of Laurentiev's theorem, we showed that for any x,

$$|\Phi'(x+it)| \cong \frac{1}{t}\int_x^{x+t} |\Phi'(s)|\,ds .$$

Therefore, if $J=[a,b]$, $(b-a)\geqq ct$,

$$\int_a^b |\Phi'(x+it)|\,dx \cong \int_a^b \left(\frac{1}{t}\int_x^{x+t} |\Phi'(s)|\,ds\right)dx =$$

$$= \int_a^{b+t} |\Phi'(s)|\,ds \cong \int_a^b |\Phi'(s)|\,ds ,$$

the last equivalence holding because of the A_∞ property of $|\Phi'(s)|$. Moreover, by (1.3), if c is large enough, (and $(b-a)\geqq ct$),

$$|\Phi(b+it)-\Phi(a+it)| \geqq \tfrac{1}{2}|\Phi(b)-\Phi(a)| .$$

By the chord arc property of Λ, it then follows that

$$|\Phi(b+it)-\Phi(a+it)| \geqq C\int_a^b |\Phi'(x+it)|\,dx .$$

It then follows that Λ_t is a chord-arc curve, with constant comparable to that of D.

COROLLARY 1.10. *Let D be a chord-arc domain. Denote $D_t=\{\Phi(x+iy): y>t\}$. $\sigma_t=$ arc length on $\partial D_t=\Lambda_t$. $\omega_t=$ harmonic measure for D_t. Then σ_t and ω_t are A_∞ equivalent with constants depending only on the chord-arc constant of D.*

The extent to which Λ_t are truly horizontal is measured by the following easy consequence of 1.3 and 1.4.

PROPOSITION 1.11. *Let D be a K-quasicircle. There is $\varepsilon>0$ and C depending only on K such that for $w\in\Lambda_t$ and $\zeta\in\Lambda$,*

$$\frac{\operatorname{dist}(w,\Lambda)}{|w-\zeta|} \leqq C\left(\frac{\operatorname{dist}(\zeta,\Lambda_t)}{|w-\zeta|}\right)^{\varepsilon} .$$

Using the chord-arc property for Λ_t we can also estimate integrals along Λ_t:

$$\int_{\Lambda_t} \frac{d\sigma(\zeta)}{|w-\zeta|^{\alpha}} \cong \operatorname{dist}(w,\Lambda_t)^{-\alpha+1}, \qquad \text{if } \alpha>1 . \tag{1.12}$$

We conclude this section with an alternative description of chord-arc domains.

PROPOSITION 1.13. *Λ is an unbounded chord-arc curve if and only if it is the image of* $\mathbf{R}$ *under a global bi Lipschitzian mapping, that is there is a homeomorphism $S:\mathbf{C} \to \mathbf{C}$ satisfying*

$$S(\mathbf{R}) = \Lambda \quad \textit{and} \quad c_1|z-w| \leqq |S(z)-S(w)| \leqq c_2|z-w| .$$

PROOF. It is easy to check that if S is bi-Lipschitzian, $S(\mathbf{R})$ has the chord-arc property. Conversely, suppose that D is a chord-arc domain, $\partial D = \Lambda$. Let $S:\mathbf{R} \to \Lambda$ be the parametrization of Λ by arc-length in the same order as the conformal mapping $\Phi:\bar{\Sigma} \to \bar{D}$. Denote

$$x_k^j = k2^j;\ u_k^j = \Phi^{-1}(S(x_k^j));\ z_k^j = u_k^j + i(u_k^j - u_{k-1}^j) .$$

$$\eta(t) = \begin{cases} 2-t & 1 \leqq t \leqq 2 \\ 2t-1 & 1/2 \leqq t \leqq 1 \\ 0 & \text{otherwise} \end{cases} \qquad \psi(t) = \begin{cases} 1-s & 0 \leqq s \leqq 1 \\ s+1 & -1 \leqq s \leqq 0 \\ 0 & \text{otherwise} \end{cases}$$

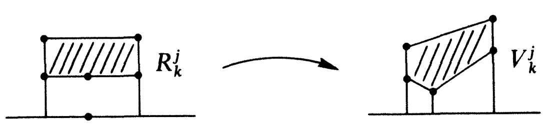
Define $S(x+it)$ for $y>0$ by

$$S(x+iy) = \Phi\Big(\sum_{j,k} \eta(2^{-j}y)\psi(2^{-j}x-k)z_k^j\Big),$$

where the sum is taken over all integers j and k. The sum has at most four nonzero terms, and S is the composition of Φ with a piecewise linear mapping from Σ to Σ that sends the five-sided figure R_k^j with vertices

$$\{x_k^{j+1} + i2^{j+1},\quad x_{k+1}^{j+1} + i2^{j+1},\quad x_{2k+2}^j + i2^j,\quad x_{2k+1}^j + i2^j,\quad x_{2k}^j + i2^j\}$$

to the figure V_k^j with vertices $\{z_k^{j+1}, z_{k+1}^{j+1}, z_{2k+2}^j, z_{2k+1}^j, z_{2k}^j\}$ (See Fig. 1.)



1.3 implies that successive differences Im $z_{2k+2}^j = u_{2k+2}^j - u_{2k+1}^j$ and Im $z_{2k+1}^j = u_{2k+1}^j - u_{2k}^j$ have comparable size. Therefore, the shape of V_k^j is not too far from the shape of R_k^j (although there may be a very large change of scale). The change of scale can easily be computed to be approximately $1/|\Phi'(z)|$ for $z \in V_k^j$. This follows because the arc length of Λ between $\Phi(u_k^j)$ and $\Phi(u_{(k+1)}^j)$ is 2^j, and therefore,

$$\int_{u_k^j}^{u_{k+1}^j} |\Phi'(s)|\, ds = 2^j .$$

By the chord-arc property of Λ and (1.6), the integral above is of the order of $(u_{k+1}^j - u_k^j)$. $|\Phi'(z)|$ for any $z \in V_k^j$. Therefore,

$$\frac{u_{k+1}^j - u_k^j}{2^j} \cong \frac{1}{|\Phi'(z)|} .$$

It is then easy to see that S is bi-Lipschitzian. Finally, define S from ${}^c\bar{\Sigma}$ to ${}^c\bar{D}$ in a similar way, using the conformal mapping from ${}^c\bar{\Sigma}$ to ${}^c\bar{D}$.

2. H^p, area integrals, and the Smirnov property.

Let D be a chord-arc domain. Another way to state Lavrentiev's theorem is that $|\Phi'(x)|\, dx$ is A_∞ equivalent to dx. This implies that the $L^p(\mathbf{R}, |\Phi'(x)|\, dx)$ norm of $N_\alpha(F)$ and $N_\beta(F)$ are comparable for any pair of positive real numbers α and β. Proposition 1.1 then implies

PROPOSITION 2.1. *Let F be holomorphic in a chord-arc domain D, then $F \in H^p(D, d\sigma)$ if and only if*

$$F \circ \Phi \in H^p(\Sigma, |\Phi'(x)|\, dx) \quad \text{and} \quad \|N_\alpha F\|_{L^p(\Lambda, d\sigma)} \cong \|N_\alpha(F \circ \Phi)\|_{L^p(\mathbf{R}, |\Phi'(x)|dx)} .$$

Our first task is to characterize H^p in terms of area integrals. We introduce several:

For $z \in \Lambda$, denote

$$g(F)(z)^2 = \int_{L(z)} |F'(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda) d\sigma(\zeta)$$

$$\tilde{g}(F)(z)^2 = \int_{L(z)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^2 d\sigma(\zeta)$$

$$S_\alpha(F)(z)^2 = \int_{\Gamma_\alpha(z)} |F'(\zeta)^2\, dV(\zeta)$$

$$\tilde{S}_\alpha(F)(z)^2 = \int_{\Gamma_\alpha(z)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^2\, dV(\zeta) ,$$

where $dV(\zeta)$ denotes the area element in the ζ-plane.

THEOREM 2.2. *Let D be a chord-arc domain and let F be holomorphic in D. Theorem following are equivalent*, $0 < p < \infty$,

(a) $N_\alpha F \in L^p(\Lambda, d\sigma)$.

(b) *For each* $z \in \Lambda$, $\lim F(\zeta) = 0$ *as* $\zeta \to \infty$, $\zeta \in L(z)$, *and either* $g(F) \in L^p(\Lambda, d\sigma)$ *or* $S_\alpha(F) \in L^p(\Lambda, d\sigma)$.

(c) *For each* $z \in \Lambda$, $\lim F(\zeta) = 0$ *and* $\lim F'(\zeta) = 0$ *as* $\zeta \to \infty$, $\zeta \in L(z)$, *and either* $\tilde{g}(F) \in L^p(\Lambda, d\sigma)$ *or* $\tilde{S}_\alpha(F) \in L^p(\Lambda, d\sigma)$.

Moreover, the L^p *norms in each case are comparable.*

PROOF. $(F \circ \Phi)' = (F' \circ \Phi)\Phi'$. From 1.6, $\zeta = \Phi(x+iy)$ implies $|\Phi'(x+iy)|y \cong \operatorname{dist}(\zeta, \Lambda)$. Therefore,

$$\int_{L(z)} |F'(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)\, d\sigma(\zeta) \cong \int_0^\infty |(F' \circ \Phi)(x+iy)|^2 |\Phi'(x+iy)|^2 y\, dy$$
$$= \int_0^\infty |(F \circ \Phi)'(x+iy)|^2 y\, dy\,,$$

the classical Littlewood-Paley g-function of $F \circ \Phi$. Thus $g(F) \in L^p(\Lambda, d\sigma)$ if and only if the classical g-function of $F \circ \Phi$ belongs to $L^p(\mathbf{R}, |\Phi'(x)|\, dx)$. Since $|\Phi'(x)|\, dx$ is A_∞ equivalent to dx it follows from a theorem of Gundy and Wheeden [9] that $g(F) \in L^p(\Lambda, d\sigma)$ if and only if $F \circ \Phi \in H^p(\Sigma, |\Phi'(x)|\, dx)$, which is equivalent to (a) by 2.1. A simpler proof works for $S_\alpha(F)$. Thus (a) is equivalent to (b).

LEMMA 2.3. *If* $F(\zeta)$ *and* $F'(\zeta)$ *tend to zero as* $\zeta \to \infty$, $\zeta \in L(z)$, *then* $g(F)(z) \leqq C\tilde{g}(F)(z)$.

PROOF.

$$F'(z) = \int_{L(z)} F''(\zeta)\, d\zeta = \int_{L(z)} F''(\zeta) \operatorname{dist}(\zeta, \Lambda)^{\frac{1+\gamma}{2}} \operatorname{dist}(\zeta, \Lambda)^{-(\frac{1+\gamma}{2})} d\zeta\,.$$

Using 1.7,

$$|F'(z)|^2 \leqq C\left(\int_{L(z)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^{1+\gamma}\, d\sigma(\zeta)\right) \operatorname{dist}(z, \Lambda)^{-\gamma}\,.$$

Hence, using 1.8, for $z_0 \in \Lambda$,

$$\int_{L(z_0)} |F'(z)|^2 \operatorname{dist}(z, \Lambda)\, d\sigma(z)$$
$$\leqq C \int_{L(z_0)} \operatorname{dist}(z, \Lambda)^{1-\gamma} \int_{L(z)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^{1+\gamma}\, d\sigma(\zeta)\, d\sigma(z)$$

$$= C\int_{L(z_0)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^{1+\gamma}\left(\int_{L(z_0,\zeta)} \operatorname{dist}(z, \Lambda)^{1-\gamma}\, d\sigma(z)\right) d\sigma(\zeta)$$

$$\leqq C\int_{L(z_0)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^3\, d\sigma(\zeta)\,.$$

LEMMA 2.4. *With the hypothesis of* 2.2 (c), *there exists* α *such that* $\tilde{g}(F)(z) \leqq C\tilde{S}_\alpha(F)(z)$.

PROOF. Let $\zeta \in L(z)$. Denote $B(\zeta) = B(\zeta, c \operatorname{dist}(\zeta, \Lambda))$ for some small $c > 0$. By 1.1, for sufficiently small c and large α, $B(\zeta) \subset \Gamma_\alpha(z)$. $|F''|^2$ is subharmonic, so

$$|F''(\zeta)|^2 \leqq \frac{1}{|B(\zeta)|}\int_{B(\zeta)} |F''(w)|^2\, dV(w)\,.$$

Hence,

$$\tilde{g}(F)(z) = \int_{L(z)} |F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^3\, d\sigma(\zeta)$$

$$\leqq C\int_{L(z)} \operatorname{dist}(\zeta, \Lambda) \int_{B(\zeta)} |F''(w)|^2\, dV(w)\, d\sigma(\zeta)$$

$$\leqq C\int_{\Gamma_\alpha(z)} |F''(w)|^2 \left(\int_{L(z)\cap\{\zeta : w \in B(\zeta)\}} \operatorname{dist}(\zeta, \Lambda)\, d\sigma(\zeta)\right) dV(w)\,.$$

The lemma will be complete if we can prove

$$\int_{L(z)\cap\{\zeta' : w \in B(\zeta')\}} \operatorname{dist}(\zeta, \Lambda)\, d\sigma(\zeta) \leqq C \operatorname{dist}(w, \Lambda)^2\,.$$

This follows from three observations. First, by 1.1 the diameter of $L(z)\cap\{\zeta : w \in B(\zeta)\}$ is bounded by $C \operatorname{dist}(w, \Lambda)$. Consequently, the chord-arc property 1.5 implies $\sigma(L(z)\cap\{\zeta : w \in B(\zeta)\}) \leqq C \operatorname{dist}(w, \Lambda)$. Finally, by 1.1, if $\zeta \in L(z)\cap\{\zeta' : w \in B(\zeta')\}$, then $\operatorname{dist}(\zeta', \Lambda) \cong \operatorname{dist}(w, \Lambda)$.

The fact that the $L^p(\Lambda, d\sigma)$ norms of $\tilde{S}_\alpha(F)$ for different values of α are comparable is left to the reader. This concludes the proof that (c) implies (b). The converse follows from the well known estimate

$$|F''(\zeta)|^2 \operatorname{dist}(\zeta, \Lambda)^2 \leqq \frac{C}{|B(\zeta)|}\int_{B(\zeta)} |F'(w)|^2\, dV(w)\,.$$

The details are left to the reader.

It is easy to deduce from the geometry of chord-arc domains that

$$\|S_\alpha(F)\|^2_{L^2(\Lambda, d\sigma)} \cong \int_D |F'(w)|^2 \operatorname{dist}(w, \Lambda)\, dV(w)$$

$$\|\tilde{S}_\alpha(F)\|^2_{L^2(\Lambda, d\sigma)} \cong \int_D |F''(w)|^2 \operatorname{dist}(w, \Lambda)^3\, dV(w)\,.$$

In light of these equivalences, we obtain a consequence of Theorem 2.2 along with variants for higher derivatives that are proved in the same way.

COROLLARY 2.5. *Suppose that F is holomorphic in D and $F(\zeta), F'(\zeta), \ldots, F^{(m-1)}(\zeta)$ tend to zero as $\zeta \to \infty$ along "vertical" lines $L(z)$. Then*

$$\|F\|_{H^2(D, d\sigma)} \cong \int_D |F^{(m)}(w)|^2 \operatorname{dist}(w, \Lambda)^{2m-1} dV(w)\,.$$

THEOREM 2.6. *Let D be a chord-arc domain, then there exist constants m, $q>0$ such that*

(a) $\Phi'(z)/(i+z)^m \in H^1(\Sigma, dx)$

(b) $\Phi'(z)^{-q}/(i+z)^m \in H^1(\Sigma, dx)$.

PROOF. As a consequence of 1.4 and 1.6, there is M such that for any $t>0$,

$$C_t(1+|z|)^{-M} \leqq |\Phi'(z)| \leqq C_t'(1+|z|)^M \quad \text{for } \operatorname{Im} x > t\,.$$

It follows from the maximum principle that for $m>M$, $\Phi'(z)/(i+z)^m$ is represented by its Poisson integral on the domains $\operatorname{Im} z>t$, and similarly for $\Phi'(z)^{-q}/(i+z)^m$. Therefore, in order to prove (a) or (b) we need only evaluate the L^1 norm of the respective functions on level sets $\operatorname{Im} z=t$ for $0<t<\frac{1}{2}$. To prove this, we will apply some real variable lemmas concerning weights $w(x) = |\Phi'(x+it)|$. First, if $w(x)\,dx$ is A_∞ equivalent to dx, then there is $r>1$ depending only on the A_∞ constants of equivalence such that $w(x)$ belongs to Muckenhoupt's class $A_r(dx)$ defined by

$$\sup_I \left(\frac{1}{|I|}\int_I w(x)\,dx\right)\left(\frac{1}{|I|}\int_I w(x)^{-1/r-1}\,dx\right)^{r-1} < \infty\,.$$

(I denotes any finite interval of $\mathbf{R}$.)

LEMMA 2.7. *If $w \in A_r(dx)$ and $C^{-1} < \int_0^1 w(x)\,dx < C$, then there exists C' depending only on the constants involved such that*

$$\int_{\mathbf{R}} \frac{w(x)}{(1+x^2)^{m/2}}\,dx \leqq C'\,; \qquad \int_{\mathbf{R}} \frac{w(x)^{-q}}{(1+x^2)^{m/2}}\,dx \leqq C'\,,$$

with $q=(r-1)^{-1}$. (See [10]).

Using Corollary 1.10, we see that all that it remains to prove is the estimate $C^{-1} < \int_0^1 w(x)\,dx < C$ for $w(x) = |\Phi'(x+it)|$, $0 < t < \frac{1}{2}$. In fact, the chord-arc property for level sets Λ_t, 1.3 and 1.6 simply that

$$\int_0^1 w(x)\,dx = \sigma(\Phi\{x+it : 0 < x < 1\}) \cong \operatorname{dist}(\Phi(i), \Lambda) .$$

As a consequence of Theorem 2.6, we obtain a version for unbounded domains of a theorem of Laurentiev [14].

COROLLARY 2.8. *Let D be a chord-arc domain, then* $\log|\Phi'(z)|$ is represented *by its Poisson integral*

$$\log|\Phi'(x+iy)| = \frac{1}{\pi}\int_{\mathbf{R}} \frac{y}{(x-t)^2+y^2}\log|\Phi'(t)|\,dt .$$

In fact, $\log|\Phi'(x+iy)| \in \mathrm{BMO}\,(dx)$ uniformly as $y-0$. (See [10]).

REMARKS (2.9). A domain with the property that $\log|\Phi'(z)|$ is represented by its Poisson integral is called a *Smirnov domain.* These are usually studied in the context of bounded domains. The conformal mapping Φ is then the mapping from the disk to the domain. Let D be a bounded Jordan domain with rectifiable boundary. Denote by $E^p(D)$ the class of holomorphic functions F in D such that there exists a sequence of Jordan curves C_n that eventually surround every compact subset of D for which

$$\sup_n \int_{C_n} |F(\zeta)^p\,d\sigma(\zeta) < \infty .$$

D is a Smirnov domain if and only if the space of polynomials in z is dense in $E^p(D)$ in $L^p(\partial D)$ norm. See [7].

(2.10). A bounded domain made up of a finite union of arcs of Smirnov domains in a Smirnov domain. ([19]).

One can regard the function $\Phi'(z)$ as an analytic extension of the measure $|\Phi'(x)|\,dx$. In general, for a measure ν that is A_∞-equivalent to dx, Kenig [12] has constructed an analytic function $G(z)$ such that $|G(x)|\,dx \cong d\nu\,(x)$. $G(z)$ is used to study the theory of H^p with measures on ∂D other than arc length. We would like to correct an error in the definition of the class $AE(\nu)$ of functions $G(z)$ extending ν given in [12] (Definition 2.10). In addition to the properties stated there, the definition should include a Smirnov type condition: $\log|G(z)|$

is represented by the Poisson integral of its boundary values. This is necessary to avoid pathologies such as those described in Section 4 (see [8]).

THEOREM 2.11. *Let D be a chord-arc domain. Then $F \in H^p(D, d\sigma)$ if and only either*

$$\text{(a)} \quad (F\circ\Phi)(\Phi')^{1/p} \in H^p(\Sigma, dx), \qquad or$$

$$\text{(b)} \quad \sup_t \int_{\Lambda_t} |F(\zeta)|^p \, d\sigma(\zeta) < \infty$$

and the corresponding norms are equivalent $(0<p<\infty)$.

PROOF. The equivalence of (a) and (b) is just the classical theorem that $H^p(\Sigma, dx)$ (as defined with the non-tangential maximal function) equals the space of holomorphic functions f on Σ such that $\sup_y \int_{\mathbf{R}} |f(x+iy)|^p \, dx < \infty$.

Suppose that $F \in H^p(D, d\sigma)$. Consider an increasing sequence of real numbers $\{x_k\}_{k=-\infty}^{\infty}$. Denote

$$I_k = \{\Phi(x+it) : x_k < x < x_{k+1}\} .$$

We can choose x_k and α so that

$$\sigma(I_k) \cong \operatorname{dist}(I_k, \Lambda) \cong \int_{x_k}^{x_{k+1}} |\Phi'(x)| \, dx$$

and $I_k \subset \Gamma_\alpha(\Phi(x))$ for all x, $x_k \leqq x \leqq x_{k+1}$. Therefore,

$$\begin{aligned} \int_{\Lambda_t} |F(\zeta)|^p \, d\sigma(\zeta) &= \sum_{k=-\infty}^{\infty} \int_{I_k} |F(\zeta)|^p \, d\sigma(\zeta) \\ &\leqq C \sum_{k=-\infty}^{\infty} \int_{x_k}^{x_{k+1}} N_\alpha(F\circ\Phi(x))^p |\Phi'(x)| \, dx \\ &\leqq C \|N_\alpha(F\circ\Phi)\|_{L^p(\mathbf{R}, |\Phi'(x)|\,dx)}^p \\ &\leqq C \|F\|_{H^p(D, d\sigma)}^p \qquad \text{by 2.1} . \end{aligned}$$

Hence (b) is proved.

Conversely, we want to show that (a) implies $F\circ\Phi \in H^p(\Sigma, |\Phi'(x)|\, dx)$. This is [12, 2.17]. The proof will be repeated for completeness. Because $(F\circ\Phi)(\Phi')^{1/p} \in H^p(\Sigma, dx)$, $F\circ\Phi$ has non-tangential limit $F\circ\Phi(x)$ and $\int_{\mathbf{R}} |F\circ\Phi(x)|^p |\Phi'(x)| \, dx < \infty$. Choose r so that $|\Phi'(x)| \in A_r(dx)$. (See Lemma 2.7).

CASE 1. $p \geqq 1+q^{-1}=r$.

By [10], $F\circ\Phi(x) \in L^p(|\Phi'(x)|\,dx)$ and $p \geqq r$ implies that the Poisson integral of $F\circ\Phi$,

$$R(x+it) = \frac{1}{\pi}\int \frac{t}{(x-u)^2+t^2} F\circ\Phi\,(u)\,du$$

is absolutely convergent. Also, it is well known that $N_\alpha(R)(x) \leqq CM(F\circ\Phi)(x)$, where M denotes the Hardy Littlewood maximal function. Since $p \geqq r$, Muckenhoupt's theorem [16] implies that $N_\alpha(R) \in L^p(\mathrm{R}, |\Phi'(x)|(dx)$. Thus $R \in H^p(\Sigma, |\Phi'(x)|\,dx)$ and it suffices to prove that $R = F\circ\Phi$.

Note that $p \geqq 1+q^{-1}$ implies $p'/p \leqq q$, where $1/p+1/p'=1$. Let s be defined by $sp'=m$, where m is defined in Theorem 2.6.

Then for any $\varepsilon>0$,

$$\begin{aligned}
&\int_{\mathrm{R}} \frac{|F\circ\Phi(x+it)|}{|i+\varepsilon(x+it)|^s}\,dx \\
&\leqq \left(\int_{\mathrm{R}} |F\circ\Phi(x+it)|^p |\Phi'(x+it)|\,dx\right)^{1/p} \\
&\qquad\qquad \left(\int_{\mathrm{R}} |\Phi'(x+it)|^{-p'/p} \frac{dx}{|i+\varepsilon(x+it)|^{sp'}}\right)^{1/p'} \\
&\leqq C_\varepsilon\,.
\end{aligned}$$

Thus, $F\circ\Phi(z)/(i+\varepsilon z)^s \in H^1(\Sigma, dx)$. Consequently,

$$\frac{F\circ\Phi(x+iy)}{(i+\varepsilon(x+iy))^s} = \frac{1}{\pi}\int_{\mathrm{R}} \frac{y}{(x-u)^2+y^2}\frac{F\circ\Phi(u)}{(i+\varepsilon u)^s}\,du\,.$$

Let ε tend to zero to obtain $F\circ\Phi = R$ by dominated convergence.

CASE 2. $p<r$.

The usual factorization of $H^p(\Sigma, dx)$ says that $F\circ\Phi(z)\,(\Phi'(z))^{1/p} = b(z)g(z)$, where $|b(z)| \leqq 1$, $g(z) \neq 0$ for $z \in \Sigma$ and $g \in H^p(\Sigma, dx)$. Let $h(z) = g(z)(\Phi'(z))^{-1/p}$. $h(z)$ is never zero, so we can consider $h(z)^{1/n}$ for any integer n. Choose n sufficiently large that $np>r$. Then

$$h(z)^{1/n}(\Phi'(z))^{1/np} \in H^{np}(\Sigma, dx)\,.$$

Applying Case 1 with np playing the role of p in Case 1, we obtain $h(z)^{1/n} \in H^{np}(\Sigma, |\Phi'(x)|\,dx)$. Thus $h(z) \in H^p(\Sigma, |\Phi'(x)|\,dx)$, and since $|F\circ\Phi| \leqq |h|$, $F\circ\Phi \in H^p(\Sigma, |\Phi'(x)|\,dx)$.

Our final characterization of H^p is in terms of level sets for the function measuring distance to Λ. Denote

$$Y_t = \{S(x+it) : x \in \mathbf{R}\},$$

where S is the bi-Lipschitzian mapping constructed in Proposition 1.13.

THEOREM 2.12. *Let D be a chord-arc domain. $F \in H^p(D, d\sigma)$ if and only if*

$$\sup_{t>0} \int_{Y_t} |F(\zeta)|^p \, d\sigma(\zeta) < \infty$$

and the corresponding norms are equivalent.

PROOF. The proof that $F \in H^p(D, d\sigma)$ implies the integrals on level sets Y_t are bounded is similar to the proof of 2.11 (b) and will be omitted. Conversely, suppose that the supremum above is finite. Denote

$$G_\delta = \{S(x+iy) : y > \delta\}.$$

We show first that $F(z)$ is bounded in G_δ. Indeed, since $|F(z)|^p$ is subharmonic,

$$\begin{aligned} |F(z)|^p &\leqq \frac{1}{|B(z,\delta/2)|} \int_{B(z,\delta/2)} |F(\zeta)|^p \, dV(\zeta) \\ &\leqq \frac{1}{|B(z,\delta/2)|} \int_{c_1\delta}^{c_2\delta} \int_{Y_t} |F(\zeta)|^p \, d\sigma(\zeta)\, dt \leqq C_{\delta,F} . \end{aligned}$$

Next, note that G_δ is a chord-arc domain with constants independent of δ. Let $\Phi_n \colon \Sigma \to G_{1/n}$ denote a conformal mapping with $\Phi_n(\infty) = \infty$, $\Phi_n(i) = w_0$ for some fixed $w_0 \in G_1$. $\Phi_n \to \Phi$ uniformly on compact subsets of D and $\Phi_n'(z)/(i+\varepsilon z)^m \in H^1(\Sigma, dx)$. Let

$$J_n(z) = (F \circ \Phi_n)(\Phi_n'/(i+\varepsilon z)^m)^{1/p} .$$

Since F is bounded on $G_{1/n}$, $J_n \in H^p(\Sigma, dx)$. By the usual H^p theory, its H^p norm is given by its L^p norm on $\mathbf{R}$

$$\|J_n\|_{H^p(\Sigma,dx)} \cong \int_{\mathbf{R}} |F \circ \Phi_n(x)|^p |\Phi_n'(x)| \, dx = \int_{Y_{1/n}} |F(\zeta)|^p \, d\sigma(\zeta) \leqq C .$$

The convergence as $n \to \infty$ and $\varepsilon \to 0$ is uniform on compact subsets, so that $(F \circ \Phi)(\Phi')^{1/p} \in H^p(\Sigma, dx)$. Hence, by 2.11, $F \in H^p(D, d\sigma)$.

3. Singular integrals on curves.

We will study Calderón-Zygmund type kernels on chord-arc curves. But first we must construct non-trivial examples of such kernels other than the Cauchy kernel $K(w, \zeta) = (z-\zeta)^{-1}$.

To define a branch of $\log(w-\zeta)$ in D, choose $\zeta_0 \in \partial D = \Lambda$, $w_0 \in D$ and let a_0 be defined by $0 \leqq \operatorname{Im} a_0 < 2\pi$, $\exp a_0 = w_0 - \zeta_0$. Let $\beta(w_0, w)$ be a path in $\bar{D}$ from w_0 to w and for $\zeta \in \mathbf{C} \setminus L(w)$, let $\beta(\zeta_0, \zeta)$ be a path in $\mathbf{C} \setminus L(w)$ from ζ_0 to ζ. For $w \in \bar{D}$ and $\zeta \in \mathbf{C} \setminus L(w)$, denote

$$k(w,\zeta) = \int_{\beta(w_0,w)} \frac{dz}{(z-\zeta_0)} + \int_{\beta(\zeta_0,\zeta)} \frac{dz}{z-w} + a_0 . \tag{3.1}$$

$k(w,\zeta)$ is well-defined and it is easy to see that $\exp k(w,\zeta) = (w-\zeta)$. Hence, $\operatorname{Re} k(w,\zeta) = \log|w-\zeta|$. Denote

$$K_s(w,\zeta) = \exp[(-1+is)k(w,\zeta)]; \qquad s \in \mathbf{R} .$$

Since $(\partial/\partial w)k = -(\partial/\partial \zeta)k$, we have

$$\frac{\partial}{\partial w} K_s(w,\zeta) = -\frac{\partial}{\partial \zeta} K_s(w,\zeta) . \tag{3.2}$$

Unfortunately, the estimate $|K_s(w,\zeta)| \leqq C_s |w-\zeta|^{-1}$ can fail in a chord-arc domain. The problem is that a chord-arc can have an infinite spiral, so that $\operatorname{Im} k(w,\zeta)$ is unbounded.

Definition. The curve Λ has *finite rotation* if

$$\sup\left\{\left|\operatorname{Im} \int_{(z_1,z_2)} \frac{dz}{z-z_0}\right| : z_0, z_1, z_2 \text{ are consecutive points of } \Lambda\right\} < \infty .$$

Remark. A curve has finite rotation if it does not spiral unboundedly often around any of its points. For instance, this is true if Λ is given as a graph. In particular, if $\varphi = \mathbf{R} \to \mathbf{R}$ and $\varphi' \in \mathrm{BMO}$, then $\Lambda = \{x + i\varphi(x) : x \in \mathbf{R}\}$ is a chord-arc curve with finite rotation. The asymmetry of the definition is not essential. As Lemma 3.5 shows, finite rotation with one linear ordering of Λ implies finite rotation with the reverse ordering.

Proposition 3.3. *If Λ is a chord-arc curve with finite rotation, then* $|\operatorname{Im} k(w,\zeta)|$ defined in 3.1 is bounded for $w \in \bar{D}$, $\zeta \in \mathbf{C} \setminus L(w)$.

Corollary 3.4. *If Λ is as in* 3.3, *then* $|K_s(w,\zeta)| \leqq C_s|w-\zeta|^{-1}$ *and*

$$\left|\frac{\partial^j}{\partial w^j} K_s(w,\zeta)\right| \leqq C_{s,j}|w-\zeta|^{-1-j} .$$

The main step in the proof of 3.3 is

LEMMA 3.5. *Let $\gamma: \mathbf{R} \to \mathbf{C}$ be an unbounded locally rectifiable Jordan curve. and suppose that*

$$\sup\left\{\left|\operatorname{Im} \int_{t_1}^{t_2} \frac{\gamma'(t)\,dt}{\gamma(t)-\gamma(0)}\right| : 0 < t_1 < t_2\right\} \leqq C .$$

Denote $L_0 = \{\gamma(t): 0 \leqq t < \infty\}$. Then for any finite path $L \subset \mathbf{C} \setminus L_0$,

$$\left|\operatorname{Im} \int_L \frac{dz}{z-\gamma(0)}\right| \leqq 2C + 6\pi .$$

PROOF: We will connect the endpoints of L by a path homotopic to L in $\mathbf{C} \setminus \{\gamma(0)\}$ made up of at most five parts. Three will be arcs of circles with center $\gamma(0)$, which contribute at most 2π each to the integral. The other two parts will be arcs of $L_0 \setminus \{\gamma(0)\}$, which each contribute at most C by hypothesis.

Let λ: $[a, b] \to \mathbf{C}$ be a curve such that $\lambda(t) \notin L_0$ for $a < t < b$, $\lambda(a)$ and $\lambda(b)$ belong to $L_0 \setminus \{\gamma(0)\}$. Let D denote the Jordan domain to the left of γ. We will say that λ has *type* $(+, +)$ if there is $\varepsilon > 0$ such that $\lambda(t) \in D$ for $a < t < a + \varepsilon$ and $b - \varepsilon < t < b$. λ has *type* $(+, -)$ if there is $\varepsilon > 0$ such that $\lambda(t) \in D$, $a < t < a + \varepsilon$ and $\lambda(t) \in {}^cD$ for $b - \varepsilon < t < b$. We define types $(-, +)$ and $(-, -)$ similarly. The endpoints $\lambda(a), \lambda(b)$ will also be referred to by their signs $+$ or $-$.

Choose w_1 and w_2 in $\mathbf{C} \setminus L_0$. The intersection of the circle $\{\gamma(0) + re^{i\theta}: \theta \in \mathbf{R}\}$ with $\mathbf{C} \setminus L_0$ is a union of subarcs of types $(+, +)$, $(+, -)$ etc. Denote by λ_j the subarc of $\{\gamma(0) + r_j e^{i\theta}: \theta \in \mathbf{R})$ to which w_j belongs $(r_j = |w_j - \gamma(0)|)$. Suppose that both λ_1 and λ_2 have a $+$ endpoint. We can join w_1 to w_2 as follows. Join w_1 to a $+$ endpoint of λ_1 by following λ_1. Then join the endpoint of λ_1 to a $+$ endpoint of λ_2 by following L_0. Finally, join the endpoint to w_2 by following λ_2. The integral $|\operatorname{Im} \int dz/z - \gamma(0)|$ along the path just described is at most $C + 4\pi$ because the two circular arcs contribute at most 2π. The path is homotopic to L in $\mathbf{C} \setminus \{\gamma(0)\}$ because it never crosses L_0 from $\bar{D}$ into ${}^c\bar{D}$ or from ${}^cD$ into D.

The only remaining case not taken care of by symmetry is the one where λ_1 has type $(+, +)$ and λ_2 has type $(-, -)$. Since we showed above how to connect either of these two arcs to an arc of type $(+, -)$, it suffices to show that an arc of type $(+, -)$ exists. Indeed, let $\Psi: \mathbf{C} \to \mathbf{C}$ be a homeomorphism such that $\Psi(\Sigma) = D$, $\Psi(\{t: t \geqq 0\}) = \gamma(\{t: t \geqq 0\}$, $\Psi(0) = \gamma(0)$. (This is possible because D is to the left of Ψ.) Denote $\lambda(\theta) = \Psi^{-1}(\gamma(0) + e^{i\theta})$. Choose θ_0 such that $\lambda(\theta_0)$ is real and positive. We can specify a branch of $\arg \lambda(\theta)$ by putting $\arg \lambda(\theta_0) = 0$. Since the circle $\gamma(0) + e^{i\theta}$, $\theta_0 \leqq \theta < \theta_0 + 2\pi$ has winding number $+1$, $\arg \lambda(\theta_0 + 2\pi) = 2\pi$. Define

$$\theta_1 = \sup\{\theta : \theta < \theta_0 + 2\pi \quad \text{and} \quad \arg \lambda(\theta) = 0\}$$

$$\theta_2 = \inf\{\theta : \theta_1 < \theta \quad \text{and} \quad \arg \lambda(\theta) = 2\pi\} .$$

The arc $\Psi(\lambda(\theta))$ $\theta_1 \leqq \theta \leqq \theta_2$ has type $(+, -)$. This concludes the proof of 3.5.

Lemma 3.5 shows that the first term in formula 3.1 for $\operatorname{Im} k(w, \zeta)$, namely

$$\operatorname{Im} \int_{\beta(w_0, w)} \frac{dz}{(z-\zeta_0)}$$

is bounded. To prove that

$$\operatorname{Im} \int_{\beta(\zeta_0, \zeta)} \frac{dz}{(z-w)}$$

is bounded, we would like to show that the hypothesis of Lemma 3.5 is satisfied on a "vertical" line $L(w)$. Let $w = \Phi(x+it_0)$; $w_1 = \Phi(x+it_1)$; $w_2 = \Phi(x+it_2)$; $t_0 < t_1 < t_2$. We must show that

$$\left|\operatorname{Im} \int_{t_1}^{t_2} \frac{\Phi'(x+it)i\,dt}{\Phi(x+it)-\Phi(x+it_0)}\right| = \left|\operatorname{Im} \int_{L(w_1, w_2)} \frac{dz}{z-w}\right| \leqq C' . \tag{3.6}$$

When $t_0 = 0$, $(w \in \Lambda)$, 3.6 follows from Lemma 3.5 applied to the curve Λ with $\gamma(0) = w$. When $t_0 > 0$, it suffices to check two cases.

(i) $t_0 < t_1 \leqq t_2 \leqq (1+\varepsilon)t_0$,

(ii) $(1+\varepsilon)t_0 \leqq t_1 \leqq t_2$.

The general case is a sum of an integral of type (i) and one of type (ii). Let $r = \operatorname{dist}(w, \Lambda)$. From 1.3 and 1.6 we deduce that for $|z-(x+it_0)| < \frac{1}{2}t_0$, $|\Phi'(z)| \cong r/t_0$ and $|\Phi''(z)| \leqq Cr/t_0^2$. Therefore,

$$\Phi(x+it) - \Phi(x+it_0) = (it - it_0)\Phi'(x+it_0) + O(t-t_0)^2 r/t_0^2 .$$

$$\Phi'(x \pm it) = \Phi'(x+it_0) + O(t-t_0)r/t_0^2 .$$

For ε sufficiently small, $t_0 < t < (1+\varepsilon)t_0$ implies that

$$\operatorname{Im} \frac{i\Phi'(x+it)}{\Phi(x+it)-\Phi(x+it_0)} = \operatorname{Im} \frac{i\Phi'(x+it_0)}{i(t-t_0)\Phi'(x+it_0)} + O(t_0^{-1})$$
$$= 0 + O(t_0^{-1}) = O(t_0^{-1}) .$$

Hence part (i) is dominated by

$$\int_{t_0}^{(1+\varepsilon)t_0} \left|\operatorname{Im} \frac{i\Phi'(x+it)}{\Phi(x+it)-\Phi(x+it_0)}\right| dt \leqq C \int_{t_0}^{(1+\varepsilon)t_0} t_0^{-1} \leqq C .$$

For part (ii), notice that when $t_1 \geqq (1+\varepsilon)t_0$, 1.3 implies that for $z \in L(w_1, w_2)$, $|z-w| \cong |z-\Phi(x)| \cong \operatorname{dist}(z, \Lambda)$ and $|w-\Phi(x)| \cong r = \operatorname{dist}(w, \Lambda)$.

Therefore

$$\left|\operatorname{Im}\int_{L(w_1,w_2)}\frac{dz}{z-w}-\operatorname{Im}\int_{L(w_1,w_2)}\frac{dz}{z-\Phi(x)}\right|$$

$$=\left|\operatorname{Im}\int_{L(w_1,w_2)}\frac{(w-\Phi(x))\,dz}{(z-w)(z-\Phi(x))}\right|\leqq\int_{L(w_1,w_2)}\frac{|w-\Phi(x)|\,d\sigma(z)}{|z-w|\,|z-\Phi(x)|}$$

$$\leqq C\int_{L(w_1,w_2)}r\operatorname{dist}(z,\Lambda)^{-2}\,d\sigma(z)\leqq C^1\quad\text{by 1.7.}$$

We have already shown that

$$\operatorname{Im}\int_{L(w_1,w_2)}\frac{dz}{z-\Phi(x)}$$

is bounded – that was the case $t_0=0$. Thus

$$\operatorname{Im}\int_{L(w_1,w_2)}\frac{dz}{z-w}$$

is bounded and 3.6 holds.

For $f\in L^2(\Lambda,d\sigma)$ and $w\in D$, denote $T_sf(w)=\int_\Lambda K_s(w,\zeta)f(\zeta)\,d\zeta$. The integral converges by Schwarz' inequality, provided Λ is a chord-arc curve. Coifman and Meyer [5] have proved that if Λ is a chord-arc domain with sufficiently small chord-arc constant and $f\in L^2(\Lambda,d\sigma)$, then there exist $f_+\in H^2(D,d\sigma)$ and $f_-\in H^2({}^c\bar{D},d\sigma)$ such that $f=f_+|_\Lambda-f_-|_\Lambda$. (Here and in what follows $F|_\Lambda$ refers to the non-tangential limit on Λ of a holomorphic function in D or ${}^c\bar{D}$.) In fact, f_+ and f_- are given by

$$f_+(w)=\frac{1}{2\pi i}T_0f(w)=\frac{1}{2\pi i}\int_\Lambda\frac{f(\zeta)}{w-\zeta}d\zeta;\qquad w\in D$$

$$f_-(w)=\frac{1}{2\pi i}\int_\Lambda\frac{f(\zeta)}{w-\zeta}d\zeta;\qquad w\in{}^cD-\ .$$

The theorem of Coifman and Meyer is equivalent to the following L^2 estimate on the Cauchy integral T_0:

$$\|T_0f|_\Lambda\|_{L^2(\Lambda,d\sigma)}=\|f_+\|_{H^2(D,d\sigma)}\leqq C\|f\|_{L^2(\Lambda,d\sigma)}\tag{3.7}$$

THEOREM 3.8. *Let Λ be a chord-arc curve with finite rotation and suppose that 3.7 holds. Then for*

$$f\in L^2(\Lambda,d\sigma),\qquad T_sf\in H^2(D,d\sigma)\ \textit{and}$$

$$\|T_sf|_\Lambda\|_{L^2(\Lambda,d\sigma)}\leqq C\|f\|_{L^2(\Lambda,d\sigma)}\,.$$

PROOF. We will prove the estimate on $T_s f$ for continuous functions f on Λ with compact support, with a constant C depending only on the finite rotation and chord-arc constants of Λ, and the constant in 3.7. The passage to general f in $L^2(\Lambda, d\sigma)$ is then easy.

LEMMA 3.9. *If $G \in H^2(D, d\sigma)$, then for $w \in D$,*

$$\lim_{\delta \to 0^+} \int_{\Lambda_\delta} K_s(w,\zeta)G(\zeta)\,d\zeta = \int_\Lambda K_s(w,\zeta)G(\zeta)\,d\zeta .$$

PROOF. The lemma can be rephrased as

$$\lim_{\delta\to 0^+} \int_{\mathbf{R}} K_s(w, \Phi(x+i\delta))G(\Phi(x+i\delta))\Phi'(x+i\delta)\,dx$$
$$= \int_{\mathbf{R}} K_s(w, \Phi(x))G(\Phi(x))\Phi'(x)\,dx ,$$

where the second integrand is the non-tangential limit of the first. Bacause the integrand is holomorphic it is enough to prove that the $L^1(dx)$ norm is bounded uniformly as $\delta \to 0^+$. Indeed,

$$\int_{\mathbf{R}} |K_s(w, \Phi(x+i\delta))G(\Phi(x+i\delta))\Phi'(x+i\delta)|\,dx = \int_{\Lambda_\delta} |K_s(w,\zeta)G(\zeta)|\,d\sigma(\zeta)$$
$$\leqq \left(\int_{\Lambda_\delta} |K_s(w,\zeta)|^2\,d\sigma(\zeta)\right)^{1/2}\left(\int_{\Lambda_\delta} |G(\zeta)|^2\,d\sigma(\zeta)\right)^{1/2} \leqq C_w \|G\|_{H^2(D,d\sigma)} ,$$

by Theorem 2.11, the chord-arc property for Λ_δ and 3.4.

Next, we prove

$$T_s f_+(w) = T_s f(w) \quad \text{for} \quad w \in D . \tag{3.10}$$

To verify 3.10 we must show that $T_s f_-(w) = 0$ for $w \in D$. Let $\Lambda_{-\delta} = \{\Psi(x - i\delta) : x \in \mathbf{R}\}$, where Ψ is a conformal mapping from the lower half space $\{x+iy\colon y<0\}$ to ${}^c\bar{D}$. By a slight variant of Lemma 3.9,

$$T_s f_-(w) = \lim_{\delta\to 0^+} \int_{\Lambda_{-\delta}} K_s(w,\zeta)f_-(\zeta)\,d\zeta .$$

The integrand is holomorphic in all of ${}^c\bar{D}$. Thus a contour integration and the following simple estimates at infinity yield $T_s f_-(w) = 0$ for $w \in D$:

$$|K_s(w,\zeta)| \leqq C_w|\zeta|^{-1} \quad \text{and} \quad |f_-(\zeta)| \leqq C_{f,\delta}|\zeta|^{-1}$$

for ζ such that $|\zeta| > C_f$ and such that ζ is "below" $\Lambda_{-\delta}$.

Denote $F(w) = (T_s f_+)(w)$, $w \in D$. To prove the theorem it suffices to show that

$$\|F\|_{H^2(D,d\sigma)} \leqq C \|f_+\|_{H^2(D,d\sigma)} .$$

It is easy to check (since f is continuous with compact support) that $F(\zeta)$, $F'(\zeta)$, $F''(\zeta)$, etc. tend to zero as ζ tends to infinity along curves $L(w)$. Thus, by Corollary 2.5 the desired estimate can be written.

$$(3.11) \quad \int_D |F^{(m)}(w)|^2 \operatorname{dist}(w,\Lambda)^{2m-1}\, dV(w) \leqq C \int_D |f'_+(w)|^2 \operatorname{dist}(w,\Lambda)\, dV(w) .$$

Let $w = \Phi(u+2it)$. A contour integration shows that $\int_{\Lambda_\delta} K_s(w,\zeta) f_+(\zeta)\, d\zeta$ is constant for $2t > \delta > 0$. Hence, by Lemma 3.9,

$$\begin{aligned}
F(w) &= \int_{\Lambda_t} K_s(w,\zeta) f_+(\zeta)\, d\zeta . \\
F^{(m)}(w) &= \int_{\Lambda_t} \frac{\partial^m}{\partial w^m} K_s(w,\zeta) f_+(\zeta)\, d\zeta \\
&= \int_{\Lambda_t} -\frac{\partial^{m-1}}{\partial w^{m-1}} \frac{\partial}{\partial \zeta} K_s(w,\zeta) f_+(\zeta)\, d\zeta \\
&= \int_{\Lambda_t} \frac{\partial^{m-1}}{\partial w^{m-1}} K_s(w,\zeta) f'_+(\zeta)\, d\zeta
\end{aligned}$$

by 3.2 and an integration by parts. Recall from 3.4 that

$$\left| \frac{\partial^{m-1}}{\partial w^{m-1}} K_s(w,\zeta) \right| \leqq C |w-\zeta|^{-m} .$$

Therefore,

$$\begin{aligned}
|F^{(m)}(w)| &\leqq C \left(\int_{\Lambda_t} \frac{d\sigma(\zeta)}{|w-\zeta|^m} \right)^{\frac{1}{2}} \left(\int_{\Lambda_t} \frac{|f_+(\zeta)|^2}{|w-\zeta|^m}\, d\sigma(\zeta) \right)^{\frac{1}{2}} \\
&\leqq C \operatorname{dist}(w,\Lambda)^{\frac{1}{2}-\frac{m}{2}} \left(\int_{\Lambda_t} \frac{|f'_+(\zeta)|^2}{|w-\zeta|^m}\, d\sigma(\zeta) \right)^{\frac{1}{2}} .
\end{aligned}$$

(See 1.12). Next,

$$\begin{aligned}
&\int_D |F^{(m)}(w)|^2 \operatorname{dist}(w,\Lambda)^{2m-1}\, dV(w) \\
&\qquad \leqq C \int_D \int_{\Lambda_t} \frac{|f'_+(\zeta)|^2}{|w-\zeta|^m}\, d\sigma(\zeta) \operatorname{dist}(w,\Lambda)^m\, dV(w) .
\end{aligned}$$

Since $dV(w)=\frac{1}{4}|\Phi'(u+2it)|^2\,du\,dt$, the second integral can be written as a constant times

$$\iiint \frac{|f'_+(\Phi(x+it))|^2}{|\Phi(u+2it)-\Phi(x+it)|^m}$$
$$|\Phi'(x+it)|\,dx\,\mathrm{dist}\,(\Phi(u+2it),\Lambda)^m\,|\Phi'(u+2it)|\,du dt$$
$$= \iint |f'_+(\Phi(x+it))|^2 R(x+it)|\Phi'(x+it)|\,dx\,dt$$

where

$$R(x+it) = \int \frac{\mathrm{dist}\,(\Phi(u+2it),\Lambda)^m}{|\Phi(u+2it)-\Phi(x+it)|^m}|\Phi'(u+2it)|^2\,du\ .$$

If we can show that $R(x+it)\leqq C\,\mathrm{dist}\,(\Phi(x+it),\Lambda)|\Phi'(x+it)|$, then 3.11 follows. To do so, use 1.6 to rewrite the estimate on $R(x+it)$ as

$$\int_{\mathbf{R}} \frac{\mathrm{disy}\,(\Phi(u+2it),\Lambda)^{m+1}}{|\Phi(u+2it)-\Phi(x+it)|^m}|\Phi'(u+2it)|\,du \ \leqq\ \mathrm{dist}\,(\Phi(x+it),\Lambda)^2\ .$$

This is the same as

$$\int_{\Lambda_{2t}} \frac{\mathrm{dist}\,(w,\Lambda)^{m+1}}{|w-\zeta|^m}\,d\sigma(w)\ \leqq\ C\,\mathrm{dist}\,(\zeta,\Lambda)^2 \qquad \text{for } \zeta\in\Lambda_t\ .$$

Notice that $\mathrm{dist}\,(\zeta,\Lambda)\cong\mathrm{dist}\,(\zeta,\Lambda_{2t})$. Using variants of 1.11 and 1.12,

$$\begin{aligned}\int_{\Lambda_{2t}} \frac{\mathrm{dist}\,(w,\Lambda)^{m+1}}{|w-\zeta|^m}\,d\sigma(w) &\leqq C\int_{\Lambda_{2t}}\left(\frac{\mathrm{dist}\,(\zeta,\Lambda_{2t})}{|w-\zeta|}\right)^{(m+1)\varepsilon}|w-\zeta|\,d\sigma(w)\\ &\cong \mathrm{dist}\,(\zeta,\Lambda)^{(m+1)\varepsilon}\int_{\Lambda_{2t}}\frac{d\sigma(w)}{|w-\zeta|^{(m+1)\varepsilon-1}}\\ &\cong \mathrm{dist}\,(\zeta,\Lambda)^{(m+1)\varepsilon}\,\mathrm{dist}\,(\zeta,\Lambda)^{-(m+1)\varepsilon+2}\\ &\cong \mathrm{dist}\,(\zeta,\Lambda)^2 \qquad \text{provided } (m+1)\varepsilon>2\ .\end{aligned}$$

This concludes the proof.

4. Characterization of chord-arc domains in terms of harmonic measure.

THEOREM 4.1. *Suppose that D is a quasicircle and that (in the notation of* 1.10) *σ_t and ω_t are A_∞-equivalent with constants independent of t as $t\to 0$. Then D is a chord-arc domain.*

THEOREM 4.2. *Suppose that D is a quasicircle satisfying the Smirnov property 2.9 and σ is A_∞equivalent to ω. Then D is a chord-arc domain.*

Recall that we have already proved the converse of these theorems (see 1.10 and 2.8). Theorem 4.2 is sharp in the sense that no single hypothesis can be omitted without making the theorem false. For example, the Smirnov property cannot be omitted because there exist non-Smirnov domains that are rectifiable, quasicircles such that $|\Phi'(x)|=1$ a.e. (see [8]). Thus the domain cannot satisfy the chord-arc property because all chord-arc domains have the Smirnov property. The quasicircle condition cannot be omitted either, as is easily seen by looking at a domain exterior to a spike. Finally, the A_∞ condition is necessary as can be seen by construction of a function φ such that $\varphi' \in L^1(dx)$, $|\varphi(x+t)+\varphi(x-t)-2\varphi(x)| \leqq |t|$, but $\varphi' \notin$ BMO. The domain $D=\{x+iy\colon y>\varphi(x)\}$ is a rectifiable quasicircle. It is a Smirnov domain because it is given by a graph. However, one can check that it does not satisfy the chord-arc property.

To prove 4.2, consider an interval $I \subset \mathbf{R}$ and restrict Φ to the square Q with base I and side length $|I|$. Since D is a quasicircle, we know that if $r=\operatorname{diam}\Phi(Q)$ and $z=$center of Q, then there exists c, such that $B(\Phi(z), c\cdot r) \subset \Phi(Q)$. It suffices to show that $\int_I |\Phi'(x)|\,dx \leqq Cr$. Multiplying Φ by r^{-1} we may as well assume $r=1$. We can also change variables by the factor $|I|^{-1}$ and assume that $|I|=1$. Φ maps a square of size 1 to a region of diameter 1 such that $B(\Phi(z), c_1) \subset \Phi(Q)$. By Schwarz' lemma, applied to Φ and Φ^{-1}, $|\log|\Phi'(x)||<C$. By the quasicircle property, $\Phi(\partial Q \pm I)$ is a chord-arc curve. Hence, by 2.10, $\Phi(Q)$ satisfies the Smirnov property. Since the unit square has the Smirnov property, $\log|\Phi'(z)|$ is represented by its Poisson integral:

$$\left| \int_{\partial Q} P_z(\zeta) \log|\Phi'(\zeta)|\, d\sigma(\zeta) \right| = |\log|\Phi'(z)|| \leqq C ,$$

where $P_z(\zeta)$ is the Poisson integral for the square. By Hölder continuity 1.4 and 1.6, we see that

$$c_1 (\operatorname{Im} z)^{1/\alpha-1} \leqq |\Phi'(z)| \leqq c_2 (\operatorname{Im} z)^{\alpha-1} \qquad \text{for some } \alpha<1,\ z \in Q .$$

(The normalizations $\operatorname{diam}\Phi(Q)=1$ and $|I|=1$ imply that the constants c_1 and c_2 depend only on the dilatation constant K). Therefore,

$$\left| \int_{\partial Q \setminus I} P_z(\zeta) \log|\Phi'(\zeta)|\, d\sigma(\zeta) \right| \leqq C ,$$

and we conclude that

$$\left| \int_I P_z(\zeta) \log|\Phi'(\zeta)|\, d\sigma(\zeta) \right| \leqq C .$$

Finally, we have reduced matters to a real variable lemma.

LEMMA 4.3. *Let* $h(x)=(1-x)^{1/2}x^{1/2}$, $0\leqq x\leqq 1$. *If* $w(x)\,dx$ *is* A_∞*-equivalent to* dx *on the unit interval and*

$$\left|\int_0^1 \log w(x)h(x)\,dx\right| < C\,,$$

then

$$\int_0^1 w(x)\,dx < C'\,,$$

where C' *depends only on the* A_∞ *constants and* C.

This lemma is exactly what we need to apply to $w(x)=|\Phi'(x)|$ because $P_z(\zeta)$ restricted to I is comparable to the weight $h(x)$.

PROOF: Since $w(x)\,dx$ and $h(x)\,dx$ are A_∞-equivalent to dx, they are A_∞-equivalent to each other. Thus,

$$\log\frac{w(x)}{h(x)} \in \mathrm{BMO}\,(h(x)\,dx) \qquad (\text{see [10]})\,.$$

Also,

$$\left|\int_0^1 \left(\log\frac{w(x)}{h(x)}\right)h(x)\,dx\right|$$
$$\leqq \left|\int_0^1 (\log w(x))h(x)\,dx\right| + \left|\int_0^1 \left(\log\frac{1}{h(x)}\right)h(x)\,dx\right| \leqq \text{const.}$$

Hence, by the John-Nirenberg inequality for weighted BMO, there exists $p>0$ such that

$$\int_0^1 \left(\frac{w(x)}{h(x)}\right)^p h(x)\,dx < C_1\,.$$

Recall that for some $r<\infty$, $(w(x)/h(x))\in A_r(h(x)\,dx)$. It follows that for all $q\leqq 1$,

$$\left(\frac{w(x)}{h(x)}\right)^q \in A_r(h(x)\,dx)$$

with the same constants. Thus [3], for all $q\leqq 1$,

$$\int_0^1 \left(\frac{w(x)}{h(x)}\right)^{q(1+\delta)} h(x)\,dx \leqq C\left(\int_0^1 \left(\frac{w(x)}{h(x)}\right)^q h(x)\,dx\right)^{1+\delta}.$$

Now if we apply this estimate many times there exist M depending only on δ and p such that

$$\int_0^1 \frac{w(x)}{h(x)} h(x)\,dx \leqq C_2\left(\int_0^1 \left(\frac{w(x)}{h(x)}\right)^p h(x)\,dx\right)^M \leqq C_2 C_1^M .$$

∎

Last of all, to prove 4.1, apply the same reasoning to each D_δ. Λ_δ is smooth, so the image of a square under the conformal mapping to D_δ will satisfy the Smirnov property. The result is an estimate on the chord-arc constant of Λ_δ independent of δ. Taking the limit $\delta \to 0$ we see that Λ is a chord-arc curve.

5. Further remarks and open questions.

1. All of the theorems above have analogues in bounded domains. The proofs are the same or simpler.

2. Theorem 4.2 and Proposition 1.13 combine to say that a quasicircle satisfying the Smirnov property is the biLipschitzian image of Σ (or the disk in the bounded case) if and only if harmonic measure is A_∞-equivalent to arc length.

3. The quasicircle condition can be characterized completely in terms of harmonic measure. D is a quasicircle if and only if harmonic measure for D and ${}^c\bar{D}$ satisfy a doubling condition (see Jerison and Kenig [11]).

4. Theorem 2.11 can be used to obtain a solution and estimates in the Neumann problem in bounded chord-arc domains. This was carried out in Lipschitz domains by Kenig [13]. The same proof yields.

THEOREM. *Let D be a bounded chord-arc domain. If $p>1$ is sufficiently small that arc length on ∂D belongs to the weight class $A_{p'}$ with respect to harmonic measure* $(1/p+1/p'=1)$, *then for any $f \in L^p(d\sigma)$ with $\int_{\partial D} g(\zeta)\,d\sigma(\zeta)=0$, there exists a harmonic function u in D such that $N_\alpha(\nabla u) \in L^p(d\sigma)$ and $n_\zeta \cdot \nabla u(z)$ tends to $g(\zeta)$ as $z \to \zeta$ non-tangentially* (a.e. $\zeta\, d\sigma$; *n_ζ denotes the normal ∂D*).

There is a similar theorem for the Dirichlet problem with boundary data in $L_1^p(d\sigma)$.

5. A different proof of Theorem 3.8 can be given based on the level sets Y_t. It is exactly analogous to the one given by Coifman and Meyer [4] in Lipschitz domains and only requires the first and second area integrals.

6. It is worthwhile to compare our converse to Laurentiev's theorem (Section 4.) to a theorem of Pommerenke. Pommerenke [18] proved that a holomorphic function f in the disk has boundary values in BMO if and only if there exists $b \in \mathbf{C}$, $f(z)=b \log \Phi'(z)$ for some conformal mapping Φ from the

disk to a (bounded) chord-arc domain. Recall that if $w(\theta)\,d\theta$ and $d\theta$ are A_∞-equivalent, then $\log w(\theta) \in \mathrm{BMO}\,(d\theta)$, but the converse is not quite true. If $\log w \in \mathrm{BMO}$, then the best we can say is that there exists $\varepsilon>0$ such that $w(\theta)^\varepsilon\, d\theta$ is A_∞-equivalent to $d\theta$. Thus one direction of Pommerenke's theorem follows from Laurentiev's theorem. The gap in the relation between A_∞ and BMO accounts for the constant b in Pommerenke's converse.

7. Proposition 1.1 shows that the space $H^p(D, d\omega)$ is invariant under conformal mapping if D is a quasicircle.

8. Using results on $H^p(\Sigma, w(x)dx)$, one can deduce from the theorems above about $H^p(D, d\sigma)$, atomic decompositions, duality results, and theorems on interpolation of analytic families of operators as in Kenig [13].

Open questions. 1. Does the chord-arc condition imply that $|\Phi'(x)|$ belongs not only to $A_r(dx)$ for some $r<\infty$, but to $A_2(dx)$? This is true for a logarithmic spiral and for Lipschitz domains.

2. Let D be a Jordan domain and define as usual $\Phi\colon \Sigma \to D$, $\Psi\colon {}^c\Sigma \to {}^cD$, conformal mappings ($\Phi(\infty)=\infty$, $\Psi(\infty)=\infty$). Consider $f\colon \mathbf{R} \to \mathbf{R}$ defined by $f=\Phi^{-1}\circ\Psi$. It is well known (see [15]) that D is a quasicircle if and only if f satisfies a kind of doubling condition. (This is related to 3 above.) Does the stronger condition $f'(x)\,dx$ is A_∞-equivalent to dx imply that D is a chord-arc domain? This assumption is equivalent to the statement that harmonic measure for D and ${}^cD$ are A_∞-equivalent.

3. Let us proceed to higher dimensions. Let $D \subset \mathbf{R}^n$, $n>2$, be a bounded domain. Let σ denote surface measure on ∂D and ω denote harmonic measure. In light of Proposition 1.13 we can ask if σ and ω are A_∞-equivalent in the case where D is the biLipschitzian image of a ball. This was proved by Dahlberg [6] for Lipschitz domains and by Jerison and Kenig [11] for domains given locally by graphs of functions whose gradient is in BMO.

4. There remains the problem of identifying biLipschitzian images of the ball. Suppose that $D \subset \mathbf{R}^n$, $n>2$, and that D is the image of a ball under quasiconformal mapping. Suppose also that $\sigma(B_r \cap \partial D) \leqq Cr^{n-1}$, where B_r is any ball in $\mathbf{R}^n$ of radius r. (This is an analogue in $\mathbf{R}^n$ of the chord-arc property.) Is it true that D is the image of a ball under bi-Lipschitzian mapping? One might also wish to replace the "quasi-sphere" hypothesis with a more geometric one (see [11]).

5. Caffarelli et al [1] have proved that there are uniformly elliptic divergence class operators $\Sigma\partial_i a_{ij}\partial_j$ with bounded, measurable coefficients a_{ij} such that the associated "harmonic" measure for the ball is singular with respect to surface measure. These operators are constructed by transforming the ordinary Laplacian by a quasiconformal change of variable. Despite this, especially if 3. is true, thre is a large collection of divergence operators for which "harmonic"

measure on the ball is A_∞-equivalent to surface measure. The final problem we would like to pose is to characterize that class.

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