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BIBOUNDED OPERATORS ON W*-ALGEBRAS

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1. Introduction.

Gohberg–Krupnik [7] call a triple $(E, i, \mathscr{H})$ a Banach space with two norms, if E is a Banach space, $\mathscr{H}$ is a Hilbert space and i is a continuous injection from E into $\mathscr{H}$ with dense image. If we denote by E^* the dual space of E then the adjoint mapping $i^*\colon \mathscr{H} \to E^*$ and the map $j := i^* \circ i\colon E \to E^*$ are continuous injections, too. For our considerations the motivating examples of Banach spaces with two norms are the triple $(L^\infty(X, \Sigma, \mu), i, L^2(X, \Sigma, \mu))$ with (X, Σ, μ) a finite measure space and i the canonical embedding $L^\infty(X, \Sigma, \mu) \to L^2(X, \Sigma, \mu)$, and the triple $(\mathscr{T}(H), i, \mathscr{H}(H))$, where $\mathscr{T}(H)$ and $\mathscr{H}(H)$ denotes the trace class operators and Hilbert–Schmidt operators, respectively, on a Hilbert space H, i being the canonical injection $\mathscr{T}(H) \to \mathscr{H}(H)$.

Canonically associated to a Banach space with two norms $(E, i, \mathscr{H})$ is the class of bounded linear operators on E whose adjoints leave $j(E)$ in E^* invariant. It is well known that important properties of operators on $L^2(X, \Sigma, \mu)$ carry over to the class of bounded linear operators on $L^\infty(X, \Sigma, \mu)$ whose adjoints leave $j(L^\infty(X, \Sigma, \mu))$ in $L^\infty(X, \Sigma, \mu)^*$ invariant (Schaefer [17, V., § 8]). Furthermore, H. H. Schaefer [18] investigates spectral properties of operators in this class. In a recent paper E. Størmer [19] studies bounded linear operators T on the von Neumann algebra $\mathscr{L}(H)$ which are weak* continuous, hence possess a preadjoint $T_* \in \mathscr{L}(\mathscr{T}(H))$ such that $T(j(\mathscr{T}(H))) \subseteqq j(\mathscr{T}(H))$. For a special class of such mappings he obtains nice spectral properties and proves a kind of Bochner theorem. After these preliminaries we give a short outline of our paper.

In section two we endow a σ-finite W*-algebra $\mathfrak{A}$ with the structure of a Banach space with two norms. The Hilbert space is exhibited by the standard Hilbert space $\mathscr{H}$ associated with $\mathfrak{A}$ by the work of H. Araki [1] and A. Connes [3], whilst the injection depends on the choice of a faithful normal state φ on $\mathfrak{A}$. Using Tomita–Takesaki theory we establish, that the corresponding injections are well behaved with respect to order and topology. These results are in complete analogy to the commutative situation.

In the third section we study the class of bounded linear operators on $\mathfrak{A}$

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canonically associated to this structure. An operator in this class is called φ-*bibounded*, and one can define $T^+ := j^{-1} \circ T^* \circ j$ which is, by the closed graph theorem, a continuous linear operator on $\mathfrak{A}$ and again φ-bibounded. We list some elementary properties of φ-bibounded operators and show, what may be considered as a version of the M. Riesz convexity theorem, that such an operator has a continuous extension T_φ to $\mathscr{H}$. We furthermore establish the following examples for φ-bibounded operators. If $a, b \in \mathfrak{A}_\varphi$, $\mathfrak{A}_\varphi$ the set of all $x \in \mathfrak{A}$ with $\varphi(xy) = \varphi(yx)$ for every $y \in \mathfrak{A}$, we denote by L_a and R_b the mappings $x \mapsto ax$ and $x \mapsto xb$, respectively. Then L_a and R_b are φ-bibounded with $L_a^+ = L_{a^*}$ and $R_b^+ = R_{b^*}$. Given a locally compact group G and a weak* continuous representation U of G into Aut $(\mathfrak{A})$, for $\mu \in M^b(G)$ we denote by $U(\mu)$ the operator $U(\mu) = \int_G U(g)\, d\mu(g)$. If U leaves φ invariant, $U(\mu)$ is φ-bibounded for all $\mu \in M^b(G)$ with $U(\mu)^+ = U(\mu^*)$.

The last section is divided into two subsections: Applications to the spectral theory and applications to ergodic theory. For a φ-bibounded operator T we first discuss the relationship between the spectra of T, T^+ and T_φ. Next we give a sufficient condition for the pairwise identity of these spectra. As an application we get the following: If T and T^+ are compact, then T_φ is compact and the three spectra are identical. Another application yields that if the spectra of T and T^+ are contained in the unit circle, and if T is a positive operator on $\mathfrak{A}$ which commutes with T^+, then T is a Jordan *-automorphism. For a semi-group S consisting of φ-bibounded operators with $\|T\| \leqq 1$ and $\|T^+\| \leqq 1$ for $T \in S$, we then prove that S is weak* mean ergodic. We conclude this section with a non-commutative version of a theorem of Akcoglu–Sucheston.

2. A W*-algebra as a Banach space with two norms.

We consider a σ-finite W*-algebra $\mathfrak{A}$ with predual $\mathfrak{A}_*$ and dual $\mathfrak{A}^*$. Following the work of H. Araki [1] and A. Connes [3] we associate with $\mathfrak{A}$ a standard Hilbert space $(\mathscr{H}, \mathscr{P}, J)$ where the Hilbert space $\mathscr{H}$ is ordered by the self dual positive cone $\mathscr{P}$ and J is the modular involution. For a faithful normal state φ on $\mathfrak{A}$ there exists a unique vector ξ_φ in $\mathscr{P}$ and a faithful representation π_φ of $\mathfrak{A}$ on $\mathscr{H}$ such that the triple $(\pi_\varphi, \mathscr{H}, \xi_\varphi)$ may be identified (by unitary equivalence) with the corresponding triple arising from the GNS-construction for $\mathfrak{A}$ with respect to φ. The modular operator pertaining to φ will be denoted by Δ_φ. Since throughout the following we consider $\mathfrak{A}$ in the representation π_φ for φ fixed we omit the π_φ and denote the commutant of $\pi_\varphi(\mathfrak{A})$ simply by $\mathfrak{A}'$. We now define an injection as follows:

$$j_1 := (x \mapsto \Delta_\varphi^{\frac{1}{4}} x \xi_\varphi) : \mathfrak{A} \to \mathscr{H} .$$

If $\mathfrak{A}=L^\infty(X,\Sigma,\mu)$, then $\mathscr{H}=L^2(X,\Sigma,\mu)$, $\xi_\varphi=1_X$ and Δ_φ is the identity on $\mathscr{H}$, so j_1 reduces to the canonical injection as mentioned in the introduction.

2.1. PROPOSITION. 1. *$(\mathfrak{A}, j_1, \mathscr{H})$ is a Banach space with two norms and j_1 is a norm contraction.*

2. *j_1 possesses a preadjoint j_2: $\mathscr{H} \to \mathfrak{A}_*$ which is a norm contractive injection.* (Here we identify $\mathscr{H}$ with its dual space so that j_2 is conjugate linear.)

PROOF. 1. j_1 is an injection being the composition of the two injective maps $x \mapsto x\xi_\varphi$ and $x\xi_\varphi \mapsto \Delta_\varphi^{\frac{1}{4}}x\xi_\varphi$. Denoting by $\mathfrak{A}^+$ the positive part of $\mathfrak{A}$, we have $\overline{j_1(\mathfrak{A}^+)}=\mathscr{P}$ (Connes [3, 2.7]) and $\mathscr{H}=\mathscr{P}-\mathscr{P}+i(\mathscr{P}-\mathscr{P})$ (Connes [3, 4.1]), hence $\overline{j_1(\mathfrak{A})}=\mathscr{H}$. By the following computation j_1 is a contraction:

$$\begin{aligned}\|j_1(x)\|^2 &= (x\xi_\varphi \mid \Delta_\varphi^{\frac{1}{2}}x\xi_\varphi) = (x\xi_\varphi \mid Jx^*\xi_\varphi)\\ &\leqq (x\xi_\varphi \mid x\xi_\varphi)^{\frac{1}{2}}\cdot(Jx^*\xi_\varphi \mid Jx^*\xi_\varphi)^{\frac{1}{2}}\\ &\leqq \|x\|\cdot\|x^*\|\\ &= \|x\|^2, \qquad (x\in\mathfrak{A}).\end{aligned}$$

2. We only have to show the weak*-weak continuity of j_1, the other assertions follow from general properties of adjoints. If $(a_i)_{i\in I}\subseteqq\mathfrak{A}$ converges to $a_0\in\mathfrak{A}$ in the weak* topology, a_i^* converges to a_0^* in this topology. Since the weak operator topology on $\mathscr{L}(\mathscr{H})$ is weaker than the weak* topology, and since J is bounded on $\mathscr{H}$, $(a_i^*+Ja_i^*)\xi_\varphi$ converges to $(a_0^*+Ja_0^*)\xi_\varphi$ weakly on $\mathscr{H}$.

Δ_φ is a positive injective operator with dense image, hence $(\Delta_\varphi^{-\frac{1}{4}}+\Delta_\varphi^{\frac{1}{4}})^{-1}$ is bounded and may be considered as a continuous operator on $\mathscr{H}$. Therefore

$$\begin{aligned}j_1(a_i) &= \Delta_\varphi^{\frac{1}{4}}a_i\xi_\varphi = (\Delta_\varphi^{-\frac{1}{4}}+\Delta_\varphi^{\frac{1}{4}})^{-1}(\mathrm{id}+\Delta_\varphi^{\frac{1}{2}})(a_i\xi_\varphi)\\ &= (\Delta_\varphi^{-\frac{1}{4}}+\Delta_\varphi^{\frac{1}{4}})^{-1}(a_i+Ja_i^*)\xi_\varphi\end{aligned}$$

converges weakly to

$$(\Delta_\varphi^{-\frac{1}{4}}+\Delta_\varphi^{\frac{1}{4}})^{-1}(a_0+Ja_0^*)\xi_\varphi = j_1(a_0).$$

The next proposition and its corollary investigate the order theoretic behaviour of j_1 and j_2. For this we denote by $\mathfrak{A}^h$ ($\mathfrak{A}^+$) the set of all self adjoint (positive) elements of $\mathfrak{A}$, analogously we define $\mathfrak{A}_*^h$ and $\mathfrak{A}_*^+$ and we set

$$\mathscr{H}^J := \{\xi\in\mathscr{H}:\ J\xi=\xi\} = \mathscr{P}-\mathscr{P}.$$

$\mathfrak{A}^h$, $\mathscr{H}^J$, $\mathfrak{A}_*^h$ are real ordered Banach spaces with normal and generating positive cones $\mathfrak{A}^+$, $\mathscr{P}$, $\mathfrak{A}_*^+$ respectively. For the general theory of ordered Banach spaces we refer to Schaefer [16]. If E is any ordered Banach space and x,y are elements of E with $x\leqq y$ we define the order interval

$$[x,y] := \{z \in E:\ x \leqq z \leqq y\}\,.$$

If ψ is a faithful normal state on $\mathfrak{A}$, we set $\mathscr{F}^+(\psi) := \bigcup_{n\in\mathbb{N}}[0, n\cdot\psi]$,

$$\mathscr{F}^h(\psi) := \mathscr{F}^+(\psi) - \mathscr{F}^+(\psi) = \bigcup_{n\in\mathbb{N}} [-n\cdot\psi, n\cdot\psi]$$

and

$$\mathscr{F}(\psi) := \mathscr{F}^h(\psi) + i\cdot\mathscr{F}^h(\psi)\,.$$

Finally we put $j := j_2 \circ j_1$.

2.2. Proposition. *$j(x)$ is given by $j(x)(y) = (JxJy\xi_\varphi \mid \xi_\varphi)$ $(x, y \in \mathfrak{A})$ and j is an order isomorphism from $\mathfrak{A}^h$ onto $\mathscr{F}^h(\varphi)$ with $j([0,1]) = [0,\varphi]$.*

Proof. For $x, y \in \mathfrak{A}$ we have

$$\begin{aligned} j(x)(y) &= (j_1(y) | j_1(x)) = (y\xi_\varphi \mid \Delta_\varphi^{\frac{1}{2}} x\xi_\varphi) \\ &= (y\xi_\varphi \mid Jx^*J\xi_\varphi) = (JxJy\xi_\varphi \mid \xi_\varphi)\,. \end{aligned}$$

j_1 is positive by definition and by the results of 2.1, j is a positive injection. From the first part of this proposition we see $j(1) = \varphi$, hence $j([0,1]) \subseteqq [0,\varphi]$. By the commutant Radon–Nikodym theorem (see e.g. Stratila–Zsido [20, 5.19]) there exists for any $\psi \in \mathfrak{A}_*$ with $0 \leqq \psi \leqq \varphi$ a (unique) $x' \in \mathfrak{A}'$ with $0 \leqq x' \leqq 1$ such that $\psi(y) = (x'y\xi_\varphi \mid \xi_\varphi)$ for all $y \in \mathfrak{A}$. Since $z \to JzJ$ is an antiautomorphism from $\mathfrak{A}$ onto $\mathfrak{A}'$, there exists a unique $x \in \mathfrak{A}$ with $0 \leqq x \leqq 1$ such that

$$\psi(y) = (JxJy\xi_\varphi \mid \xi_\varphi) = j(x)(y) \quad \text{for all } y \in \mathfrak{A}\,.$$

Hence $j([0,1]) = [0,\varphi]$ and j is an order isomorphism from $\mathfrak{A}^h$ onto $\mathscr{F}^h(\varphi)$.

2.3. Corollary. 1. *j_1 is an order isomorphism of $\mathfrak{A}^h$ onto $\bigcup_{n\in\mathbb{N}} n\cdot[-\xi_\varphi, \xi_\varphi]$ with $j_1([0,1]) = [0,\xi_\varphi]$.*

2. *j_2 is an order isomorphism of $\bigcup_{n\in\mathbb{N}} n\cdot[-\xi_\varphi,\xi_\varphi]$ onto $\mathscr{F}^h(\varphi)$ with $j_2([0,\xi_\varphi]) = [0,\varphi]$.*

Proof. In view of 2.1 j_1 and j_2 are positive injections. We have $j_1(1) = \Delta_\varphi^{\frac{1}{4}}\xi_\varphi = \xi_\varphi$ and, by 2.2, $j_2(\xi_\varphi) = j_2(j_1(1)) = j(1) = \varphi$. Therefore $j_1([0,1]) \subseteqq [0,\xi_\varphi]$ and $j_2([0,\xi_\varphi]) \subseteqq [0,\varphi]$. If $\eta \in [0,\xi_\varphi]$, we set $\psi := j_2(\eta)$. By 2.2 there exists $x \in [0,1] \subseteqq \mathfrak{A}$ with $j(x) = \psi$. j_2 is injective and therefore $j_1(x) = \eta$ hence $j_1([0,1]) = [0,\xi_\varphi]$ and j_1 is an order isomorphism. By 2.2 we get

$$[0,\varphi] = j([0,1]) = j_2 \circ j_1([0,1]) \subseteqq j_2([0,\xi_\varphi])$$

so that $j_2([0,\xi_\varphi]) = [0,\varphi]$ and j_2 is an order isomorphism.

The results 2.2 and 2.3 show that j, j_1, and j_2 are well behaved with respect to the order structures of $\mathfrak{A}$, $\mathscr{H}$, and $\mathfrak{A}_*$. This singles out j_1 among other possible injections of $\mathfrak{A}$ into $\mathscr{H}$.

In addition these injections have, in complete analogy to the commutative situation (see Schaefer [17, Lemma V.8.3]), nice topological properties which will be basic for the last section. In the following we denote by $\mathfrak{A}_1$ the unit ball of $\mathfrak{A}$.

2.4. PROPOSITION. *The restriction of j_2 to $j_1(\mathfrak{A}_1)$ is a homeomorphism for the weak (norm) topologies on $\mathscr{H}$ and $\mathfrak{A}_*$.*

PROOF. By 2.1.2 $j_1(\mathfrak{A}_1)$ is weakly compact and the restriction of the weakly continuous injection j_2 to $j_1(\mathfrak{A}_1)$ is a homeomorphism.

To prove the assertion for the norm topologies we consider any sequence $(\psi_n)_{n\in\mathbf{N}}$ in $j_2(j_1(\mathfrak{A}_1))=j(\mathfrak{A}_1)$ with $\lim_n \psi_n=\psi_0 \in j(\mathfrak{A}_1)$. Putting $x_n:=j^{-1}(\psi_n-\psi_0)$ and $\xi_n:=j_2^{-1}(\psi_n-\psi_0)$ we have $j_1(x_n)=\xi_n$ and $x_n \in 2\cdot\mathfrak{A}_1$ (compare 2.2 and 2.3). Therefore

$$\begin{aligned} 2\cdot\|\psi_n-\psi_0\| &\geqq |(\psi_n-\psi_0)(x_n)| = |j(x_n)(x_n)| \\ &= |(j_1(x_n)\mid j_1(x_n))| = |(\xi_n\mid\xi_n)| \\ &= \|\xi_n\|^2 \quad \text{for all } n\geqq 1\,. \end{aligned}$$

3. Bibounded operators.

In this section we endow the W*-algebra $\mathfrak{A}$ with the structure of a Banach space with two norms corresponding to the faithful normal state φ as described in the preceding section. We introduce and investigate the class of operators on $\mathfrak{A}$ whose adjoints leave $\mathscr{F}(\varphi)$ invariant. If T is a bounded linear operator on $\mathfrak{A}$, we denote by T^* its adjoint and in the case of weak* continuity, by T_* the preadjoint of T on $\mathfrak{A}_*$.

3.1. DEFINITON. Let $\mathscr{L}(\mathfrak{A})$ be the Banach space of all bounded linear operators on $\mathfrak{A}$. We call $T\in\mathscr{L}(\mathfrak{A})$ *φ-bibounded* if $T^*(\mathscr{F}(\varphi))\subseteqq\mathscr{F}(\varphi)$. Since $\mathscr{F}(\varphi)=j(\mathfrak{A})$ we denote in this case by T^+ the (well defined) linear operator $T^+:=j^{-1}\circ T^*\circ j$ on $\mathfrak{A}$.

3.2. REMARKS. 1. It follows immediately that a φ-bibounded operator is weak* continuous, hence possesses a preadjoint T_*.

2. An application of the closed graph theorem shows that T^+ is continuous on $\mathfrak{A}$.

3. An operator $T \in \mathscr{L}(\mathfrak{A})$ is φ-bibounded iff there exists $S \in \mathscr{L}(\mathfrak{A})$ with $j(Sx)(y) = j(x)(Ty)$ for every $x, y \in \mathfrak{A}$. In that case $S = T^+$.

Before presenting examples of φ-bibounded operators we list some elementary properties of T^+.

3.3. PROPOSITION. *Let $\mathfrak{A}$ be a W^*-algebra with a faithful normal state φ and let $T \in \mathscr{L}(\mathfrak{A})$ be φ-bibounded. Then the following holds:*

1. *Let $A \subseteqq \mathfrak{A}$. Then $T^+(A) \subseteqq A$ iff $T_*(j(A)) \subseteqq j(A)$ and $T(A) \subseteqq A$ iff $(T^+)^*(j(A)) \subseteqq j(A)$.*
2. *T is ψ-bibounded for all states ψ with $\mathscr{F}(\psi) = \mathscr{F}(\varphi)$ and $(T^+)^+ = T$.*
3. *T^+ is n-positive iff T is n-positive ($n \geqq 1$).*
4. *For φ-bibounded operators $T_1, T_2 \in \mathscr{L}(\mathfrak{A})$ we have $(T_1 \circ T_2)^+ = T_2^+ \circ T_1^+$ and $(\alpha T_1 + \beta T_2)^+ = \bar{\alpha} T_1^+ + \bar{\beta} T_2^+$ for all $\alpha, \beta \in \mathbf{C}$.*
5. *Endowed with the norm*

$$T \mapsto \|T\|_{\mathrm{bi}} := \max\{\|T\|, \|T^+\|\}$$

the algebra $\mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$ of all φ-bibounded operators on $\mathfrak{A}$ is a Banach algebra with involution $T \mapsto T^+$, and for $T \in \mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$ invertible in $\mathscr{L}(\mathfrak{A})$ we have $T^{-1} \in \mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$ iff $(T^+)^{-1}$ exists. In this case $(T^+)^{-1} = (T^{-1})^+$.

PROOF. 1. By definition we have $T^+(A) \subseteqq A$ iff $T_*(j(A)) \subseteqq j(A)$. It is easy to see that $(T^+)^*$ on $j(A)$ is given by the operator $j \circ T \circ j^{-1}$. Thus for all $A \subseteqq \mathfrak{A}$ we have

$$\begin{aligned} T(A) \subseteqq A &\Leftrightarrow (j \circ T \circ j^{-1})(j(A)) \subseteqq j(A) \\ &\Leftrightarrow (T^+)^*(j(A)) \subseteqq j(A). \end{aligned}$$

2. This follows immediately from part 1.

3. Recall that an operator $S \in \mathscr{L}(\mathfrak{A})$ is called *n-positive* ($n \geqq 1$) iff $S \otimes \mathrm{id}_n$ is a positive linear operator on $\mathfrak{A} \otimes M_n$, M_n the algebra of all $n \times n$-matrices over $\mathbf{C}$ (see e.g. Takesaki [21, IV.3.3]). Given a faithful state τ on M_n, $\varphi \otimes \tau$ is a faithful state on $\mathfrak{A} \otimes M_n$ and

$$(T \otimes \mathrm{id}_n)_*(\varphi \otimes \tau) \leqq (\lambda \cdot \varphi) \otimes \tau = \lambda(\varphi \otimes \tau)$$

for some $0 \leqq \lambda \in \mathbf{R}$. If $T \otimes \mathrm{id}_n$ is positive we obtain

$$(T \otimes \mathrm{id}_n)_*[0, \varphi \otimes \tau] \subseteqq \lambda[0, \varphi \otimes \tau]$$

and

$$(T \otimes \mathrm{id}_n)_* \mathscr{F}(\varphi \otimes \tau) \subseteqq \mathscr{F}(\varphi \otimes \tau).$$

Thus by part 1, $T \otimes \mathrm{id}_n$ is $\varphi \otimes \tau$-bibounded with $(T \otimes \mathrm{id}_n)^+ = T^+ \otimes \mathrm{id}_n$.

4. This follows easily from the definition and the fact that j is conjugate linear.

5. By parts 2 and 4, the map $T \mapsto T^+$ is an involution. Since $\|T\|_{\mathrm{bi}}$ is an algebra norm one only has to show the completeness of $\mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$. If $(T_n)_{n \geq 1}$ is a Cauchy sequence in $\mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$, then so are $(T_n)_{n \geq 1}$ and $(T_n^+)_{n \geq 1}$ in $\mathscr{L}(\mathfrak{A})$. Therefore there exist $T, S \in \mathscr{L}(\mathfrak{A})$ with $\lim_n T_n = T$ and $\lim_n T_n^+ = S$. Now for all x and y in $\mathfrak{A}$:

$$j(Sx)(y) = \lim_n j(T_n^+ x)(y) = \lim_n j(x)(T_n y) = j(x)(Ty)\,.$$

This shows $T^+ = S$ and $\lim_n T_n = T$ in $\mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$.

3.4. EXAMPLES. 1. For the commutative case we refer to the book of Schaefer [17, p. 343, Examples].

2. Let $\mathfrak{A}_\varphi := \{x \in \mathfrak{A} : \varphi(xy) = \varphi(yx) \text{ for all } y \in \mathfrak{A}\}$ be the centralizer of φ in $\mathfrak{A}$. We note that $\mathfrak{A}_\varphi$ is a W*-subalgebra of $\mathfrak{A}$ and can also be characterized as the set of elements in $\mathfrak{A}$ that commute on $\mathscr{H}$ with Δ_φ (see, e.g. Stratila–Zsido [20, p. 238]). For $a, b \in \mathfrak{A}_\varphi$ we define the operators L_a and R_b on $\mathfrak{A}$ as follows:

$$L_a(x) := ax, \qquad R_b(x) := xb \qquad (x \in \mathfrak{A})\,.$$

The following computation shows the φ-biboundedness of L_a and R_b with $L_a^+ = L_{a^*}$ and $R_b^+ = R_{b^*}$. Indeed, for $x, y \in \mathfrak{A}$ we have:

$$\begin{aligned} j(y)(L_a x) &= (\Delta_\varphi^{\frac14} a x \xi_\varphi \mid \Delta_\varphi^{\frac14} y \xi_\varphi) \\ &= (\Delta_\varphi^{\frac14} x \xi_\varphi \mid a^*(\Delta_\varphi^{\frac14} y \xi_\varphi)) \\ &= (\Delta_\varphi^{\frac14} x \xi_\varphi \mid \Delta_\varphi^{\frac14} a^* y \xi_\varphi) \\ &= j(L_{a^*} y)(x) \end{aligned}$$

and

$$\begin{aligned} j(y)(R_b x) &= (\Delta_\varphi^{\frac14} x b \xi_\varphi \mid \Delta_\varphi^{\frac14} y \xi_\varphi) \\ &= (\Delta_\varphi^{\frac14} J \Delta_\varphi^{\frac12} b^* x^* \xi_\varphi \mid \Delta_\varphi^{\frac14} y \xi_\varphi) \\ &= (J \Delta_\varphi^{-\frac14} \Delta_\varphi^{\frac12} b^* x^* \xi_\varphi \mid \Delta_\varphi^{\frac14} y \xi_\varphi) \\ &= (J \Delta_\varphi^{\frac14} y \xi_\varphi \mid b^* \Delta_\varphi^{\frac14} x^* \xi_\varphi) \\ &= (bJ \Delta_\varphi^{\frac14} J \Delta_\varphi^{\frac12} y^* \xi_\varphi \mid \Delta_\varphi^{\frac14} J \Delta_\varphi^{\frac12} x \xi_\varphi) \\ &= (\Delta_\varphi^{\frac14} (yb^*)^* \xi_\varphi \mid J \Delta_\varphi^{\frac14} x \xi_\varphi) \\ &= (J \Delta_\varphi^{\frac14} y b^* \xi_\varphi \mid J \Delta_\varphi^{\frac14} x \xi_\varphi) \\ &= (\Delta_\varphi^{\frac14} x \xi_\varphi \mid \Delta_\varphi^{\frac14} y b^* \xi_\varphi) \\ &= j(R_{b^*} y)(x)\,. \end{aligned}$$

Now the assertion follows from 3.2.3. Moreover, the linear operator $T_{a,b}(x) = axb$ $(x \in \mathfrak{A})$ is φ-bibounded with $T^{+}_{a,b} = T_{a^*,b^*}$ since $T_{a,b} = L_a \circ R_b$.

3. Consider a locally compact group G and a representation U of G into the automorphism group Aut $(\mathfrak{A})$ of $\mathfrak{A}$ which is continuous for the weak* operator topology on Aut $(\mathfrak{A})$. For $\mu \in M^b(G)$ the space of bounded Radon measures on G, we put $U(\mu)x := \int_G U(g)x\, d\mu(g)$ $(x \in \mathfrak{A})$, thus defining a bounded linear operator on $\mathfrak{A}$ (Arveson [2, 1.4]). If we assume $\varphi \circ U(g) = \varphi$ for all $g \in G$, then $U(\mu)$ is φ-bibounded for every $\mu \in M^b(G)$. Since the involution $\mu \to \mu^*$ on $M^b(G)$ is defined as $\mu^*(f) = (\int_G \bar{f}(g^{-1})\, d\mu(g))^-$ $(f \in C^b(G))$ (the bar denotes the complex conjugation on C), we have for all $x, y \in \mathfrak{A}$, noting that the operator $U(g)$ has a unitary extension $U(g)_\varphi$ to $\mathscr{H}$ which commutes with J (Araki [1, p. 347]):

$$
\begin{aligned}
j(U(\mu^*)y)(x) &= (JU(\mu^*)yJx\xi_\varphi \mid \xi_\varphi) \\
&= \left(\int_G (JU(g^{-1})yJx\xi_\varphi \mid \xi_\varphi)^-\, d\mu(g)\right)^- \\
&= \left(\int_G (JU(g^{-1})_\varphi (yJx)\xi_\varphi \mid \xi_\varphi)^-\, d\mu(g)\right)^- \\
&= \left(\int_G ((U(g)x)^*\xi_\varphi \mid JyJ\xi_\varphi)\, d\mu(g)\right)^- \\
&= \left(\int_G (\xi_\varphi \mid JyJU(g)x\xi_\varphi)\, d\mu(g)\right)^- \\
&= (\xi_\varphi \mid JyJU(\mu)x\xi_\varphi)^- = (JyJU(\mu)x\xi_\varphi \mid \xi_\varphi) \\
&= j(y)(U(\mu)x);
\end{aligned}
$$

hence $U(\mu)^+ = U(\mu^*)$. In particular we have for abelian G and $*$ the convolution on $M^b(G)$:

$$
\begin{aligned}
U(\mu)^+ \circ U(\mu) &= U(\mu^*) \circ U(\mu) = U(\mu^* * \mu) = U(\mu * \mu^*) \\
&= U(\mu) \circ U(\mu^*) = U(\mu) \circ U(\mu)^+ .
\end{aligned}
$$

A remarkable fact of φ-bibounded operators is the following extension property which is a C*-version of the M. Riesz convexity theorem (see, e.g., Dunford–Schwartz [6, VI. 10.12]). For the proof compare Gohberg–Krupnik [7].

3.5. THEOREM. *Let $\mathfrak{A}$ be a W*-algebra with a faithful normal state φ, and let $T \in \mathscr{L}(\mathfrak{A})$ be φ-bibounded. Then the linear operators T_φ and T^+_φ defined on $j_1(\mathfrak{A})$ by*

$$T_\varphi := j_1 \circ T \circ j_1^{-1}$$
$$T_\varphi^+ := j_1 \circ T^+ \circ j_1^{-1}$$

have continuous extensions on $\mathscr{H}$, again denoted by T_φ and T_φ^+, with the following properties:

1. $T_\varphi^* = T_\varphi^+$.
2. $\|T_\varphi\| \leqq \|T^+T\|^{\frac{1}{2}}$, $\|T_\varphi^+\| \leqq \|TT^+\|^{\frac{1}{2}}$.
3. $\|T_\varphi\| \leqq \|T\|_{\text{bi}}$.

PROOF. By injectivity of j_1 the operators T_φ and T_φ^+ are well defined on $j_1(\mathfrak{A})$. An easy computation shows that

$$(T_\varphi j_1(x)|j_1(y)) = (j_1(x)|T_\varphi^+ j_1(y)) \qquad (x, y \in \mathfrak{A}) .$$

To prove the continuity of T_φ and T_φ^+ we first consider the special case $T = T^+$. Then $T_\varphi = T_\varphi^*$ on $j_1(\mathfrak{A})$ and we may assume $\|T\| \leqq 1$.

For given $0 \neq x \in \mathfrak{A}$ with $\|j_1(x)\| \leqq 1$ we define

$$r_n := (T_\varphi^n j_1(x)|T_\varphi^n j_1(x)) \qquad (n \geqq 0) .$$

Then for all $\lambda \in \mathsf{R}$ and $n \geqq 1$ we have

$$(T_\varphi^{n-1} j_1(x) + \lambda T_\varphi^{n+1} j_1(x)|T_\varphi^{n-1} j_1(x) + \lambda T_\varphi^{n+1} j_1(x)) = r_{n-1} + 2\lambda r_n + \lambda^2 r_{n+1} \geqq 0 ,$$

which implies

$$r_n^2 \leqq r_{n-1} \cdot r_{n+1} \qquad (n \geqq 1) .$$

Since $r_0 \neq 0$, we have for $r_1 \neq 0$:

$$r_n > 0 \quad \text{and} \quad r_1 \leqq \frac{r_2}{r_1} \leqq \frac{r_3}{r_2} \leqq \ldots .$$

It follows that $r_1^n \leqq r_n$, hence

$$r_1 \leqq \liminf_{n\to\infty} (r_n)^{1/n} \qquad \text{for all } n \in \mathsf{N} .$$

Since j_1 is a contraction, we have $r_n \leqq \|x\|^2$ and therefore

$$\limsup_{n\to\infty} (r_n)^{1/n} \leqq 1 .$$

This implies $r_1 \leqq 1$ and so T_φ has a contractive extension to $\mathscr{H}$.

For the general case we consider the operator $T^+ \circ T$. Since $(T^+ \circ T)^+ = T^+ \circ T$ by the results above this operator has a continuous extension on $\mathscr{H}$ with $\|(T^+ \circ T)_\varphi\| \leqq \|T^+ \circ T\|$. For all $x \in \mathfrak{A}$ we have

$$\begin{aligned}\|T_\varphi j_1(x)\|^2 &= (T_\varphi j_1(x) \mid T_\varphi j_1(x)) \\ &= ((T_\varphi^+ \circ T_\varphi) j_1(x) \mid j_1(x)) \\ &\leqq \|T^+ T\| \, \|j_1(x)\|^2 .\end{aligned}$$

Hence $T_\varphi \in \mathscr{L}(\mathscr{H})$,

$$\|T_\varphi\| \leqq \|T^+ T\|^{\frac{1}{2}}$$

and

$$\|T_\varphi\| \leqq \max\{\|T\|, \|T^+\|\} = \|T\|_{\mathrm{bi}} .$$

By the same method we see $T_\varphi^+ \in \mathscr{L}(\mathscr{H})$ with $\|T_\varphi^+\| \leqq \|T \circ T^+\|^{\frac{1}{2}}$.

3.6. Proposition. *Let φ and ψ be faithful normal states on $\mathfrak{A}$ with $\mathscr{F}(\varphi) = \mathscr{F}(\psi)$ and let $T \in \mathscr{L}(\mathfrak{A})$ be φ-bibounded. Then T is ψ-bibounded and there exists a bijection $V \in \mathscr{L}(\mathscr{H})$ with*

$$V \circ T_\psi = T_\varphi \circ V .$$

Proof. By 3.3.2, T is ψ-bibounded and by 3.5, T_φ and T_ψ are bounded linear operators on $\mathscr{H}$. Furthermore $\lambda^{-1}\psi \leqq \varphi \leqq \lambda\psi$ for some $0 < \lambda \in \mathsf{R}$. Then it follows from Araki [1, Theorem 12, (9.4)]

$$\lambda^{-\frac{1}{2}}\xi_\psi \leqq \xi_\varphi \leqq \lambda^{\frac{1}{2}}\xi_\psi .$$

Using $xJxJ\ (\mathscr{P}) \subseteqq \mathscr{P}$ for all $x \in \mathfrak{A}$ we estimate:

$$\begin{aligned}\lambda^{-1}\|\Delta_\psi^{\frac{1}{4}} x\xi_\psi\|^2 &= \lambda^{-1}(xJxJ\xi_\psi \mid \xi_\psi) \\ &= \lambda^{-\frac{1}{2}}(xJxJ\xi_\psi \mid \lambda^{-\frac{1}{2}}\xi_\psi) \\ &\leqq (xJxJ\xi_\varphi \mid \xi_\varphi) \\ &\leqq (xJxJ\xi_\varphi \mid \xi_\varphi) \\ &= \|\Delta_\varphi^{\frac{1}{4}} x\xi_\varphi\|^2 \\ &\leqq \lambda^{\frac{1}{2}}(xJxJ\xi_\varphi \mid \xi_\psi) \\ &\leqq \lambda(xJxJ\xi_\psi \mid \xi_\psi) \\ &= \|\Delta_\psi^{\frac{1}{4}} x\xi_\psi\|^2 .\end{aligned}$$

Therefore the operators

$$V\colon \Delta_\psi^{\frac{1}{4}} x\xi_\psi \mapsto \Delta_\varphi^{\frac{1}{4}} x\xi_\varphi \quad \text{and} \quad W\colon \Delta_\varphi^{\frac{1}{4}} x\xi_\varphi \mapsto \Delta_\psi^{\frac{1}{4}} x\xi_\psi$$

have continuous extensions $\bar{V}, \bar{W}$ on the Hilbert space $\mathscr{H}$ with $\bar{V} \circ \bar{W} = \mathrm{id}_{\mathscr{H}}$, $\bar{W} \circ \bar{V} = \mathrm{id}_{\mathscr{H}}$, and $\bar{V} \circ T_\psi = T_\varphi \circ \bar{V}$.

3.8. COROLLARY. *Let φ and ψ be faithful normal states on $\mathfrak{A}$ with $\mathcal{F}(\varphi)=\mathcal{F}(\psi)$ and let $T\in\mathcal{L}(\mathfrak{A})$ be φ-bibounded. Then the spectra of T_ψ and T_φ in $\mathcal{L}(\mathcal{H})$ coincide.*

4. Applications.

A. *Spectral theory.*

In this section we study the relationship between the spectra of T, T^+, and T_φ for T φ-bibounded. We give a sufficient condition for the pairwise identity of the three spectra, extending a recent result of Schaefer [18] to the non-commutative setting.

4.1. PROPOSITION. *If $T\in\mathcal{L}(\mathfrak{A})$ is φ-bibounded and if T and T^+ are invertible, then so are T_φ and T_φ^+.*

In particular,

$$\sigma(T_\varphi)\subseteqq\sigma(T)\cup\overline{\sigma(T^+)}=\sigma_\varphi^{\mathrm{bi}}(T),$$

where $\sigma_\varphi^{\mathrm{bi}}(T)$ denotes the spectrum of T in the Banach algebra $\mathcal{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$ and the bar is the complex conjugation in $\mathbf{C}$.

PROOF. The first assertion follows from 3.3.5 and 3.5. If $\lambda\in\sigma(T_\varphi)$, then $(\lambda-T)$ or $(\lambda-T)^+=(\bar\lambda-T^+)$ are not invertible by the first consideration. Hence $\lambda\in\sigma(T)$ or $\bar\lambda\in\sigma(T^+)$. The last equality follows via 3.3.5. from the fact that $(\lambda-T)$ is not invertible in $\mathcal{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$ iff $(\lambda-T)$ or $(\lambda-T)^+$ is not invertible in $\mathcal{L}(\mathfrak{A})$.

4.2. EXAMPLE. Let $T\in\mathcal{L}(\mathfrak{A})$ be φ-bibounded, $\|T\|=1$, $T(\mathbf{1})=\mathbf{1}$ and let the spectra of T and T^+ be contained in the unit circle. If T and T^+ commute, then T is a Jordan *-isomorphism.

Indeed, by 4.1 the spectrum of T_φ is contained in the unit circle and T_φ is normal on $\mathcal{H}$ since T and T^+ commute. But this implies T_φ unitary in $\mathcal{L}(\mathcal{H})$. Thus

$$j_1(T^+x)=T_\varphi^*j_1(x)=T_\varphi^{-1}j_1(x)=j_1(T^{-1}x)$$

for every $x\in\mathfrak{A}$, hence $T^+=T^{-1}$. Since $\|T\|=1$ and $T(\mathbf{1})=\mathbf{1}$, T is a positive operator on $\mathfrak{A}$, and therefore by 3.3.3, T^{-1} is positive. Thus T is bipositive with $T(\mathbf{1})=\mathbf{1}$ which implies the assertion by Stratila–Zsido [20, Theorem 3, p. 135].

Recall that a (non-empty) topological Hausdorff space is called 0-*dimensional*, if its topology has a base of open-and-closed sets. For example every finite or countable subset in $\mathbf{C}$ as well as the Cantor discontinuum are 0-dimensional.

4.3. THEOREM. *Let $\mathfrak{A}$ be a* W*-*algebra with a faithful normal state φ and let $T \in \mathscr{L}(\mathfrak{A})$ be φ-bibounded. If the spectra $\sigma(T)$ and $\sigma(T^+)$ are both* 0-*dimensional, then we have $\sigma(T) = \overline{\sigma(T^+)} = \sigma(T_\varphi)$.*

PROOF. Let $p(t) = \sum_{i=0}^n \alpha_i t^i$ be a polynomial over C and suppose $p(T) = 0$. Then we obtain for all x and y in $\mathfrak{A}$:

$$\begin{aligned} j(\bar{p}(T^+)x)(y) &= j\left(\sum_{i=0}^n \bar{\alpha}_i (T^+)^i x\right)(y) \\ &= \sum_{i=0}^n \alpha_i j((T^+)^i x)(y) \\ &= \sum_{i=0}^n \alpha_i j(x)(T^i y) \\ &= j(x)(p(T)y) = 0 , \end{aligned}$$

thus $\bar{p}(T^+) = 0$. In that case it follows $\sigma(T) = \overline{\sigma(T^+)}$ from Dunford–Schwartz [6, VII.3]. Therefore without loss of generality we may assume in the following that T, hence T^+, satisfies no polynomial identity $p(T) = 0$ for some $p \in \mathrm{C}[t]$, $\mathrm{C}[t]$ the ring of polynomials over C in one variable.

Since T is φ-bibounded $p(T) \in \mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$ for every $p \in \mathrm{C}[t]$ and we denote by $\mathfrak{B}_1$ (respectively $\mathfrak{B}_2$) the completion of the algebra $\mathrm{C}[T] := \{p(T) : p \in \mathrm{C}[t]\}$ with respect to the norm $\|p(T)\|_1 := \|p(T)\|$ (respectively $\|p(T)\|_2 := \|p(T^+)\|$). Then the spectrum of T in $\mathfrak{B}_1$ (respectively $\mathfrak{B}_2$) is equal to $\sigma(T)$ (respectively $\overline{\sigma(T^+)}$), since in view of the hypothesis these spectra have connected complement.

Since the complement of $\sigma(T) \cup \overline{\sigma(T^+)}$ in C is connected, it follows from Schaefer [18, Proposition 2] that $\sigma_{\mathfrak{B}}(T) = \sigma(T) \cup \overline{\sigma(T^+)}$, $\sigma_{\mathfrak{B}}(T)$ denoting the spectrum of T in the Banach algebra $\mathfrak{B}$ and $\mathfrak{B}$ the closure of $\mathrm{C}[T]$ in $\mathscr{L}_\varphi^{\mathrm{bi}}(\mathfrak{A})$. Hence $\sigma_{\mathfrak{B}}(T)$ is 0-dimensional. Now if K is an open-and-closed neighborhood of $\lambda \in \sigma_{\mathfrak{B}}(T)$, the spectral projection

$$p(K) := \frac{1}{2\pi i} \int_\Gamma (\gamma - T)^{-1} \, d\gamma$$

is $\neq 0$, where Γ is any simple, closed, rectifiable and oriented curve lying entirely in $\mathrm{C} - \sigma_{\mathfrak{B}}(T)$ and separating K from $\sigma_{\mathfrak{B}}(T) - K$. Since $\Gamma \subseteqq \mathrm{C} - \sigma(T)$, respectively $\Gamma \subseteqq \mathrm{C} - \overline{\sigma(T^+)}$, $p(K)$ is a non-zero spectral projection in $\mathfrak{B}_1$ and $\mathfrak{B}_2$, since these algebras are in a canonical manner continuously embedded in $\mathfrak{B}$. But this implies $\sigma(T) \cap K \neq \varnothing$ and $\overline{\sigma(T^+)} \cap K \neq \varnothing$, hence $\lambda \in \sigma(T) \cap \overline{\sigma(T^+)}$ by the closedness of these sets.

Next for $p(T) \in \mathbf{C}[T]$ we define

$$\|p(T)\|_1 := \|p(T)\|, \qquad \|p(T)\|_2 := \|p(T_\varphi)\| ,$$

$$|||p(T)||| := \max\{\|p(T)\|_1, \|p(T)\|_2\}$$

and denote by $\mathfrak{B}_1$, $\mathfrak{B}_2$, $\mathfrak{B}$, respectively, the completion of $\mathbf{C}[T]$ with respect to these norms. Then since $\sigma(T_\varphi)$ is 0-dimensional, in view of the hypothesis and 4.1 we get with the same arguments as above $\sigma(T_\varphi)=\sigma_{\mathfrak{B}}(T)=\sigma(T)$, hence $\sigma(T) =\sigma(T_\varphi)=\overline{\sigma(T^+)}$ as desired.

4.4. EXAMPLES. 1. Let $T\in \mathscr{L}(\mathfrak{A})$ be φ-bibounded with T and T^+ compact. Since the spectra of compact operators are countable we get $\sigma(T)=\sigma(T_\varphi) =\overline{\sigma(T^+)}$.

2. Let $T\in \mathscr{L}(\mathfrak{A})$ be φ-bibounded and compact. Then we assert $\sigma(T)=\sigma(T_\varphi)$, and T_φ is compact. To see this, by the second half of the proof of 4.3 we only have to show that T_φ is compact. But it is easy to see that for $S=T\circ T^+$, the linear operator S_φ is compact on $j_1(\mathfrak{A})$, hence its extension $T_\varphi \circ T_\varphi^*$ is compact on $\mathscr{H}$. But this implies the assertion.

3. Let $T\in \mathscr{L}(\mathfrak{A})$ be φ-bibounded with $\sigma(T)=\sigma(T^+)=\{1\}$ and let T commute with T^+. Then $T=T^+=\mathrm{id}_{\mathfrak{A}}$: By 4.3 we have $\sigma(T_\varphi)=\{1\}$. But T_φ is normal on $\mathscr{H}$, which implies $T_\varphi=\mathrm{id}_{\mathscr{H}}$. From this the assertion follows.

B. *Bicontractions and ergodic theorems.*

If (X,Σ,μ) is a finite measure space and T: $L^1(X,\Sigma,\mu)\to L^1(X,\Sigma,\mu)$ is a linear operator leaving $L^\infty(X,\Sigma,\mu)$ invariant with $\|T\|_1\leqq 1$ and $\|T\|_\infty \leqq 1$, then T induces a contraction on $L^2(X,\Sigma,\mu)$ by the M. Riesz convexity theorem (see e.g. Schaefer [17, Proposition V.8.2]). With the help of ergodic theorems for operators on Hilbert space one can prove ergodic theorems for these operators (Schaefer [17, V.8]). In this section we consider operators in $\mathscr{L}(\mathfrak{A})$ which fulfil a non-commutative version of the above condition.

4.5. DEFINITION. Let $\mathfrak{A}$ be a W*-algebra with unit ball $\mathfrak{A}_1$ and let φ be a faithful normal state on $\mathfrak{A}$. We call an operator $T\in \mathscr{L}(\mathfrak{A})$ a *φ-bicontraction*, if $\|T\|\leqq 1$ and T fulfils any of the following two equivalent conditions:

1. $T^*(j(\mathfrak{A}_1))\subseteqq j(\mathfrak{A}_1)$,
2. T is φ-bibounded with $\|T^+\|\leqq 1$.

It is obvious that, in contrast to remark 3.4, the notion of a "φ-bicontraction" depends on the state φ.

4.6. DEFINITION. Let E be a Banach space. A semi-group $S\subseteqq \mathscr{L}(E)$ is called *mean ergodic* if there exists a projection $P \in \overline{\text{co}}\, S$ with $P \circ T = T \circ P = P$ for all $T \in S$, $\overline{\text{co}}\, S$ being the convex closure of S in the strong operator topology.

4.7. REMARK. P is uniquely determined and is a projection onto the fixed space of S. We call it the *mean ergodic projection corresponding to* S. By definition we have $Px \in \overline{\text{co}}\{Tx : T \in S\}$ for all $x \in E$.

4.8. DEFINITION. A semi-group $S \subseteq \mathscr{L}(\mathfrak{A})$ is called *weak* mean ergodic if* S consists of weak* continuous operators and if the preadjoint semigroup $S_* := \{T_* : T \in S\}$ is mean ergodic in $\mathscr{L}(\mathfrak{A}_*)$. In this case we denote the corresponding mean ergodic projection in $\mathscr{L}(\mathfrak{A}_*)$ by P_*.

For the general theory of mean ergodic semi-groups we refer to Schaefer [17, III.7] and Nagel [11]. More information on weak* mean ergodic semi-groups on W*-algebras may be found in Kümmerer–Nagel [10].

Theorem 3.5 serves as our analogue of the M. Riesz convexity theorem, from which we obtain:

If $T \in \mathscr{L}(\mathfrak{A})$ *is a* φ*-bicontraction then* $\|T_\varphi\| \leqq 1$.

The Alaoglu–Birkhoff theorem states that every contraction semi-group on a Hilbert space is mean ergodic (see, e.g. Schaefer [17, Theorem III.7.11]). It appears to be the only mean ergodic theorem placing no restriction on the algebraic structure (such as commutativity or amenability) of the semigroup S considered. It is, therefore, remarkable that an arbitrary φ-bicontractive semi-group on a W*-algebra is weak* mean ergodic.

4.9. THEOREM. *Let* $\mathfrak{A}$ *be a* W**-algebra with faithful normal state* φ. *Every semi-group* $S \subseteq \mathscr{L}(\mathfrak{A})$ *consisting of* φ*-bicontractions is weak* mean ergodic.*

PROOF. By the mean ergodic theorem of Alaoglu–Birkhoff, the contraction semi-group

$$S_\varphi^+ := \{T_\varphi^+ : \ T \in S\}$$

on $\mathscr{H}$ is mean ergodic. We denote by P_φ^+ the corresponding mean ergodic projection.

$j_1(\mathfrak{A}_1)$ is invariant under S_φ^+ and therefore

$$P_\varphi^+(\xi) \in \overline{\text{co}}\, S_\varphi^+ \xi \subseteq j_1(\mathfrak{A}_1)$$

for every $\xi \in j_1(\mathfrak{A}_1)$. If we set

$$j_2(\xi) =: \psi \in j(\mathfrak{A}_1)$$

we get by definition $j_2(T_\varphi^+\xi) = T_*(\psi)$ ($T \in \boldsymbol{S}$), and from 2.4 we conclude

$$j_2(P_\varphi^+\xi) \in \overline{\text{co}}\, \boldsymbol{S}_*\psi\ .$$

Moreover, we get

$$\|j_2(P_\varphi^+\xi)\| \leqq \|P_\varphi^+\xi\| \leqq \|\xi\| \leqq c\|j_2(\xi)\| = c\|\psi\|\ ,$$

where $c > 0$ is a constant independent of ξ. Hence if we define a mapping P_0 on $\mathscr{F}(\varphi)$ by

$$P_0 := (\psi \mapsto (j_2 \circ P_\varphi^+ \circ j_2^{-1})\psi) : \mathscr{F}(\varphi) \to \mathscr{F}(\varphi)\ ,$$

P_0 has a norm continuous extension $P_*\colon \mathfrak{A}_* \to \mathfrak{A}_*$. Clearly, for $\psi \in \mathscr{F}(\varphi)$ we have

$$(P_*T_*)(\psi) = (T_*P_*)(\psi) = P_*(\psi) = P_*^2(\psi)\ .$$

Hence by continuity this relation holds for $\psi \in \mathfrak{A}_*$. If $\psi \in \mathscr{F}(\varphi)$, then $P_*\psi \in \overline{\text{co}}\, \boldsymbol{S}_*\psi$ and $\boldsymbol{S}_*$ is mean ergodic by Nagel [11, 1.2]. Therefore, $\boldsymbol{S}$ is weak* mean ergodic with mean ergodic projection $P := (P_*)^*$.

4.10. Remark. In theorem 4.9 we need some norm condition. If instead we assume that $\boldsymbol{S}$ is an amenable semi-group (see Day [4]) of φ-bibounded operators such that $\boldsymbol{S}$ and $\boldsymbol{S}^+$ are equicontinuous we can prove once more that $\boldsymbol{S}$ is weak* mean ergodic (compare the results in Nagel [11]).

The following result roots in the investigation of mixing properties of measure preserving transformations and may be viewed as a non-commutative version of a theorem of Akcoglu–Sucheston (see e.g. Schaefer [17, V.8]). The proof is based on the analogous theorem for operators on a Hilbert space for which we refer to Schaefer [17, V.8.5]. Here we consider a φ-bicontraction $T \in \mathscr{L}(\mathfrak{A})$. By 4.9 the semi-group $\{T_*^n : n \in \mathbf{N}\}$ is mean ergodic with projection P_*. General ergodic theory gives the explicit expression

$$P_* = \lim_n \frac{1}{n} \sum_{i=0}^{n-1} T_*^i$$

(see, e.g., Nagel [12]).

4.11. Theorem. *Let $\mathfrak{A}$ be a W*-algebra with faithful normal state φ and let $T \in \mathscr{L}(\mathfrak{A})$ be a φ-bicontraction. Then the following are equivalent.*

1. $\lim_n T_*^n = P_*$ *in the weak operator topology on* $\mathscr{L}(\mathfrak{A}_*)$.
2. $\lim_N N^{-1} \sum_{i=1}^N T_*^{n_i} = P_*$ *in the strong operator topology for each infinite subsequence* $(n_i)_{i \in \mathbf{N}}$ *of* $\mathbf{N}$.

PROOF. 1. $\Rightarrow$ 2.: Since T is a φ-bicontraction we have $T_*(j(\mathfrak{A}_1)) \subseteqq j(\mathfrak{A}_1)$. For $\psi \in j(\mathfrak{A}_1)$ there exists a $\xi \in j_1(\mathfrak{A}_1) \subseteqq \mathscr{H}$ with $\psi = j_2(\xi)$ and $T_*(\psi) = j_2(T_\varphi^+ \xi)$. From proposition 2.4 it follows that $((T_\varphi^+)^n \xi)_{n \in \mathsf{N}}$ converges to $P_\varphi^+ \xi = P_\varphi \xi$ in the weak topology of $\mathscr{H}$. Since $\|T_\varphi^+\| \leqq 1$, the Alaoglu–Birkhoff theorem implies that the semi-geoup $((T_\varphi^+)^n)_{n \in \mathsf{N}}$ is mean ergodic in $\mathscr{L}(\mathscr{H})$ with mean ergodic projection P_φ. Hence by the Hilbert space version of this theorem, for all infinite sequences $(n_i)_{i \in \mathsf{N}}$ of N, the averages

$$\frac{1}{N} \sum_{i=1}^{N} (T_\varphi^+)^{n_i} \xi$$

converge to $P_\varphi \xi$ in the norm. Since $j_1(\mathfrak{A}_1)$ j_2 is also a homeomorphism for the norm topology, the averages

$$\frac{1}{N} \sum_{i=1}^{N} T_*^{n_i} \psi$$

converge to $P_* \psi$ in $\mathscr{L}(\mathfrak{A}_*)$. An application of (Schaefer [16, III.4.5]) yields the assertion.

2. $\Rightarrow$ 1.: The assertion is trivial for the restriction of T_* to its fixed space $P_*(\mathfrak{A}_*)$, and it suffices to consider $\psi \in P_*^{-1}(0)$. For all infinite sequences $(n_i)_{i \in \mathsf{N}}$ in N and for all $x \in \mathfrak{A}$ one obtains

$$\lim_N \frac{1}{N} \sum_{i=1}^{N} \langle T_*^{n_i} \psi, x \rangle = 0 .$$

Since the sequence $(\langle T^{n_i} \psi, x \rangle)_{i \in \mathsf{N}}$ is bounded, we have $\lim_n \langle T_*^n \psi, x \rangle = 0$ and therefore $\lim_n T_*^n = P_*$ in the weak operator topology of $\mathscr{L}(\mathfrak{A}_*)$.

BIBLIOGRAPHY

1. H. Araki, *Some properties of modular conjugation operator of von Neumann algebras and a non-commutative Radon–Nikodym theorem with a chain rule*, Pacific J. Math. 50 (1974), 309–354.
2. W. Arveson, *On groups of automorphisms of operator algebras*, J. Funct. Anal. 15 (1974), 217–243.
3. A. Connes, *Caractérisation des espaces vectoriels ordonnés sous-jacents aux algèbres de von Neumann*, Ann. Inst. Fourier (Grenoble), 24 (1974), 121–155.
4. M. H. Day, *Semigroups and amenability*, in *Semigroups* ed. K. W. Folley (Proceedings of a Symposium on Semigroups held at Wayne State University, Detroit, Michigan, 1968), pp. 5–53, Academic Press, New York - London, 1969.
5. J. Dieudonné, *Treatise in Modern Analysis*, Part I, Academic Press, New York - London, 1960.
6. N. Dunford and J. T. Schwartz, *Linear Operators*, Part I: *General Theory* (Pure and Applied Mathematics 7), Interscience Publishers Inc., New York, 1958.

7. I. Gohberg und N. Krupnik, *Einführung in die Theorie der eindimensionalen singulären Integraloperatoren* (Mathematische Reihe 63) Birkhäuser Verlag, Basel - Bonn - Stuttgart, 1979.
8. H. Halpern, *Spectra of operator equations and automorphism groups of von Neumann algebras*, Math. Scand. 42 (1978), 135–149.
9. H. Halpern, *Spectral representation for abelian automorphism groups of von Neumann algebras*, J. Funct. Anal. 36 (1980), 313–342.
10. B. Kümmerer and R. Nagel, *Mean ergodic semigroups on* W*-*algebras*, Acta Sci. Math. (Szeged) 41 (1979), 151–159.
11. R. J. Nagel, *Mittelergodische Halbgruppen linearer Operatoren*, Ann. Inst. Fourier (Grenoble), 23 (1973), 75–87.
12. R. J. Nagel, *Ergodic and mixing properties of linear operators*, Proc. Roy. Irish Acad. Sect. A 74 (1974), 245–261.
13. D. Olesen, *On spectral subspaces and their applications to automorphism groups*, Symposia Math. Vol. 20, pp. 253–296, Academic Press, London - New York, 1976.
14. G. K. Pedersen and M. Takesaki, *The Radon–Nikodym theorem for von Neumann algebras*, Acta Math. 130 (1973), 53–87.
15. W. Rudin, *Fourier Analysis on Groups* (Interscience Tracts in Pure and Applied Mathematics 12), Interscience Publishers, London, 1967.
16. H. H. Schaefer, *Topological Vector Spaces* (Graduate Texts in Mathematics 3), 4th corr. printing, Springer-Verlag, Berlin - Heidelberg - New York, 1980.
17. H. H. Schaefer, *Banach Lattices and Positive Operators* (Die Grundlehren der Mathematischen Wissenschaften 215), Springer-Verlag, Berlin - Heidelberg - New York, 1974.
18. H. H. Schaefer, *Interpolation of spectra*, Integral Equations Operator Theory 3 (1980), 463–469.
19. E. Størmer, *Regular abelian algebras of linear maps of operator algebras*, J. Funct. Anal. 37 (1980), 331–373.
20. S. Stratila and L. Zsido, *Lectures on von Neumann Algebras*, Abacus Press, Tunbridge Wells, Kent, 1979.
21. M. Takesaki, *Theory of Operator Algebras* I, Springer-Verlag, Berlin - Heidelberg - New York, 1979.

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