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ON h-BASES FOR n II

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1. Introduction.

Given a sequence B: $b_0 < b_1 < \ldots < b_k$ of non-negative integers, we say that an integer M is *dependent* on B if there exist non-negative integers x_i such that

$$M = b_0 x_0 + b_1 x_1 + \ldots + b_k x_k .$$

If $\gcd B = 1$, it is well known that every sufficiently large integer is dependent on B. In this case we denote the largest integer *not* dependent on B, the *Frobenius number* of B, by $g(B)$.

For a positive integer h we write hB for the set of integers which can be written as the sum of h elements of B, allowing repetition of summands.

To B we also make correspond the sequence B^*: $b_0^* < b_1^* < \ldots < b_k^*$, where

$$b_i^* = b_k - b_{k-i}, \quad i = 0, 1, \ldots, k .$$

Note that $b_0^* = 0$, and if $b_0 = 0$, then $b_k^* = b_k$ and $\gcd B^* = \gcd B$.

An integer sequence

$$(1.1) \qquad A_k:\ a_0 = 0 < 1 = a_1 < a_2 < \ldots < a_k$$

is called an *h-basis* for a non-negative integer n if all the integers $0, 1, 2, \ldots, n$ belong to hA_k (Rohrbach [8]). The *h-range* $n(h, A_k)$ of A_k is the largest n for which A_k is an h-basis.

The following important result connecting the h-range and the Frobenius number was obtained by Meures [4]: Given A_k, if h is sufficiently large, then

$$(1.2) \qquad n(h, A_k) = a_k h - g(A_k^*) - 1 .$$

Let $h_0 = h_0(A_k)$ be the smallest h for which $a_k \leqq n(h, A_k)$, and let $h_1 = h_1(A_k)$ be the smallest $h \geqq h_0 - 1$ for which (1.2) is true. Then (1.2) is valid for all $h \geqq h_1$. (See [7].)

In [7] we gave some general upper bounds for h_1. A combination of Lemma 1 in [7] with ideas from [5] also led us to a new proof of the known result $h_1(A_3) \leqq h_0(A_3)$.

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In this paper we combine ideas introduced in [7] with ideas from [6] to determine the h_1 of the integer sequence

$$A_k:\ 0 < 1 < d < 2d < \ldots < (k-2)d < a_k, \quad k \geqq 3\,. \tag{1.3}$$

We also study the sequence

$$A_k:\ 0 < 1 < 2 < \ldots < k-2 < a_{k-1} < a_k, \quad k \geqq 3\,. \tag{1.4}$$

However, the h_1 of this sequence seems to behave more irregularly than that of (1.3), and we settle for, though sharp, less precise results.

The common feature of these two sequences is that the sequence A_k^* forms an *almost arithmetic sequence*, i.e. $k-1$ of the non-zero elements of A_k^* form an ordinary arithmetic sequence. The Frobenius number of an almost arithmetic sequence was determined in [6].

There is also a third case where A_k^* forms an almost arithmetic sequence; namely

$$A_k:\ 0 < 1 < 1+d < 1+2d < \ldots < 1+(k-2)d < a_k, \quad d \geqq 2,\ k \geqq 3\,.$$

However, for this sequence our present technique does not enable us to improve the general bounds for h_1 given in [7], unless we impose rather heavy additional conditions upon A_k.

2. Preliminaries.

In the following we write $[x]$ for the integral part of a real number x, and we use $\langle x \rangle$ to denote the smallest integer greater than or equal to x.

Given the sequence (1.1), k fixed, we write A_i for the sequence

$$A_i:\ a_0 = 0 < 1 = a_1 < a_2 < \ldots < a_i, \quad 1 \leqq i \leqq k\,.$$

For a positive integer M we write $\Lambda_i(M)$ for the least number of elements of A_i with sum M. Also put $\Lambda_i(0)=0$. Then $M \in hA_i$ if and only if $\Lambda_i(M) \leqq h$.

For the notion of a "pleasant" sequence we refer the reader to [7, § 1], and for the fact that pleasantness implies $h_1 = h_0 - 1$, to [7, § 2].

Given integers $s_{-1} > s_0 > 0$, we use the Euclidean algorithm in the form

$$\begin{aligned}
s_{-1} &= q_1 s_0 - s_1, & 0 &\leqq s_1 < s_0\\
s_0 &= q_2 s_1 - s_2, & 0 &\leqq s_2 < s_1\\
s_1 &= q_3 s_2 - s_3, & 0 &\leqq s_3 < s_2\\
&\ldots\\
s_{m-2} &= q_m s_{m-1} - s_m, & 0 &\leqq s_m < s_{m-1}\\
s_{m-1} &= q_{m+1} s_m, & 0 &= s_{m+1} < s_m\,.
\end{aligned}$$

We also recursively define integers P_i, Q_i for $i=-1,0,\ldots,m+1$, by

$$\begin{cases} P_{i+1} = q_{i+1}P_i - P_{i-1}, & P_0=1,\ P_{-1}=0 \\ Q_{i+1} = q_{i+1}Q_i - Q_{i-1}, & Q_0=0,\ Q_{-1}=-1 . \end{cases} \tag{2.1}$$

Now,

$$P_iQ_{i+1} - P_{i+1}Q_i = 1 \tag{2.2}$$

$$s_{-1}Q_i = s_0P_i - s_i \tag{2.3}$$

$$s_iQ_{i+1} - s_{i+1}Q_i = s_0 , \tag{2.4}$$

and, since $q_i \geqq 2$, we also have $P_i < P_{i+1}$, $Q_i < Q_{i+1}$.

For $-1 \leqq i \leqq m$, we define sets X_i, Y_i of lattice points by

$$X_i = \{(x,y) \mid 0 \leqq x < s_i - s_{i+1},\ 0 \leqq y < P_{i+1}\}$$
$$Y_i = \{(x,y) \mid 0 \leqq x < s_i,\ 0 \leqq y < P_{i+1} - P_i\} .$$

We say that two lattice points (x,y) and (x',y') are *congruent* if

$$x + s_0y \equiv x' + s_0y' \pmod{s_{-1}} .$$

It was shown in [7] that for each $i=0,1,\ldots,m$, there is a *bijection*

$$\varphi:\ X_{i-1} \cup Y_{i-1} \to X_i \cup Y_i$$

given by

$$\varphi(x,y) = \left(x - s_i\left[\frac{x}{s_i}\right],\ y + P_i\left[\frac{x}{s_i}\right]\right) . \tag{2.5}$$

φ also has the property that if $(x,y) \in X_{i-1} \cup Y_{i-1}$, then the lattice points (x,y) and $\varphi(x,y)$ are congruent.

It follows that the set

$$\{x + s_0y \mid (x,y) \in X_i \cup Y_i\}$$

forms a complete residue system modulo s_{-1} for each $i=-1,0,\ldots,m$.

Now fix r, $0 \leqq r < s_{-1}$. Let (x_i, y_i) be the unique lattice point in $X_i \cup Y_i$ which is congruent to $(r,0)$, $i=-1,0,\ldots,m$. Then

$$x_i + s_0y_i = x_{i-1} + s_0y_{i-1} + s_{-1}Q_i\left[\frac{x_{i-1}}{s_i}\right], \quad i \geqq 0 , \tag{2.6}$$

and

$$r = x_{-1} + s_0y_{-1} = x_0 + s_0y_0 \leqq x_1 + s_0y_1 \leqq \cdots \leqq x_m + s_0y_m . \tag{2.7}$$

3. The sequence (1.3).

We now consider the sequence

$$A_k: \ a_0 = 0 < a_1 = 1 < a_2 = d < a_3 = 2d < \ldots < a_{k-1} = \varkappa d < a_k ,$$

where $\varkappa = k-2 \geqq 1$.

Put

$$s_{-1} = a_k, \quad s_0 = d ,$$

and

$$R_i = \frac{1}{a_k}((a_k-1)\varkappa s_i - (a_k - \varkappa d)P_i) .$$

Since

$$R_{-1} = (a_k-1)\varkappa, \quad R_0 = \varkappa d - 1$$

$$R_{i+1} = q_{i+1}R_i - R_{i-1} ,$$

all the R_i are integers. Further

$$-\frac{a_k - \varkappa d}{s_m} = R_{m+1} < R_m < \ldots < R_0 = \varkappa d - 1 .$$

Hence there is a unique integer $v = v(A_k)$, $0 \leqq v \leqq m$, satisfying

$$R_{v+1} \leqq 0 < R_v .$$

We also have

(3.1) $$R_i = \varkappa s_i - P_i + \varkappa Q_i .$$

It is easily seen that the sequence A_{k-1} is pleasant, and

(3.2) $$A_{k-1}(M) = r + \left\langle \frac{N}{\varkappa} \right\rangle \quad \text{if } M = r + dN, \ 0 \leqq r < d$$

(3.3) $$n(h, A_{k-1}) = \varkappa d(h+1-d) + 2d - 2, \quad h \geqq h_0(A_{k-1}) - 1 ,$$

where

$$h_0(A_{k-1}) = \begin{cases} d-1 & \text{if } \varkappa = 1 \\ d & \text{if } \varkappa \geqq 2 . \end{cases}$$

(Alternatively, see Djawadi [2, Satz 1] and Hofmeister [3, Satz 1].)

It follows that $h_0 = h_0(A_k)$ is given by

$$h_0 = d + \left[\frac{a_k - 2d}{\varkappa d} \right],$$

and in particular that

(3.4) $$h_0 = d-1+\left\langle\frac{q_1-2}{\varkappa}\right\rangle \quad \text{if } v \geqq 1 .$$

Putting

(3.5) $$h' = s_v - s_{v+1} - 2 + \left\langle\frac{P_{v+1}+R_v-1}{\varkappa}\right\rangle ,$$

we are now in the position to state

THEOREM 1. *For the integer sequence*

$$A_k:\ 0 < 1 < d < 2d < \ldots < \varkappa d < a_k ,$$

where $\varkappa = k-2 \geqq 1$, *we have*

$$h_1 = \begin{cases} h_0 - 1 & \text{if } v=0 \\ h_0 & \text{if } v \geqq 1 \text{ and } R_v \geqq \varkappa \\ \max\{h_0, h'\} & \text{if } v \geqq 1 \text{ and } R_v < \varkappa . \end{cases}$$

We prove this theorem by going through the following steps:

(3.6) $$v = 0 \Rightarrow h_1 = h_0 - 1$$

(3.7) $$v \geqq 1 \Rightarrow h_1 \geqq h_0$$

(3.8) $$v \geqq 1 \quad \text{and} \quad R_v \geqq \varkappa \Rightarrow h_1 \leqq h_0$$

(3.9) $$v \geqq 1 \quad \text{and} \quad R_v < \varkappa \Rightarrow h_1 \leqq \max\{h_0, h'\}$$

(3.10) $$v \geqq 1 \quad \text{and} \quad R_v < \varkappa \Rightarrow h_1 \geqq h' .$$

PROOF OF (3.6). Let

$$q_1 = \left\langle\frac{q_1}{\varkappa}\right\rangle \varkappa - t .$$

Then

$$a_k = \left\langle\frac{q_1}{\varkappa}\right\rangle \varkappa d - (td + s_1) ,$$

where

$$0 \leqq td + s_1 < \varkappa d .$$

According to Satz 1 of Djawadi [2] we now have that A_k is pleasant if and only if

$$\left\langle \frac{q_1}{\varkappa} \right\rangle > \left\langle \frac{t}{\varkappa} \right\rangle + s_1 ;$$

that is, if and only if $v=0$.

Thus, if $v=0$, then A_k is pleasant, whence $h_1 = h_0 - 1$.

PROOF OF (3.7). Suppose that $h_1 = h_0 - 1$. Then

$$n(h_0-1, A_{k-1}) = n(h_0-1, A_k) = a_k(h_0-1) - g(A_k^*) - 1 ,$$

and, by (3.3),

$$g(A_k^*) = a_k(h_0+1-d+s_1) - d(\varkappa(h_0-d)+1+q_1) + a_{k-1}^*(d-1-s_1) .$$

Now, if $v \geqq 1$, then by (3.4) and (3.1),

$$0 \leqq \varkappa(h_0-d)+1+q_1 \leqq \varkappa(h_0+1-d+s_1) ,$$

and by Lemma 1 in [6], $g(A_k^*)$ is dependent on A_k^*; a contradiction.

PROOF OF (3.8). Let t_l^* be the smallest integer dependent on A_k^* and $\equiv l \pmod{a_k}$. By Lemma 1 in [6], we then have

$$t_l^* = (a_k-1)x + a_k \left\langle \frac{y}{\varkappa} \right\rangle - dy , \tag{3.11}$$

and the same technique as used in [6, § 4] shows that we can take

$$(x, y) \in X_v \cup Y_v . \tag{3.12}$$

We now want to express t_l^* on the form

$$t_l^* = (a_k-1)x + \sum_{i=1}^{\varkappa} a_{k-(i+1)}^* x_i^{(l)}, \quad x_i^{(l)} \geqq 0 ,$$

and the proof of Lemma 1 in [6] tells us how to do this: If $y=0$, put $x_i^{(l)}=0$, $i=1,2,\ldots,\varkappa$.

If $y>0$, let

$$y = q\varkappa - s, \ 0 \leqq s < \varkappa; \ s = \sigma q + \varrho, \ 0 \leqq \varrho < q;$$

$$j = \varkappa - \sigma - 1, \ x_j^{(l)} = \varrho, \ x_{j+1}^{(l)} = q - \varrho, \ x_i^{(l)} = 0 \text{ otherwise} .$$

(Here we only have $0 \leqq j < \varkappa$. However, if $j=0$, then $x_j^{(l)}=0$.) Then it follows that

$$x + \sum_{i=1}^{\varkappa} a_{i+1} x_i^{(l)} = x + dy .$$

Hence, by (3.12) and Lemma 1 in [7], if for each r, $0 \leqq r < a_k$, all the integers

(3.13) $$r < r+a_k < r+2a_k < \dots < x_v+dy_v-a_k$$

belong to hA_k for some $h \geqq h_0-1$, then $h \geqq h_1$.

If $v=0$, then the set (3.13) is empty, and again we have (3.6).

Suppose that $v \geqq 1$, and let M be an arbitrary integer in the sequence (3.13). By (2.6) and (2.7), we then have

(3.14) $$M = x_{i-1}+dy_{i-1}+a_k z,\ 0 \leqq z < Q_i\left[\frac{x_{i-1}}{s_i}\right],$$

for some i, $1 \leqq i \leqq v$.

As in [7, § 4] we have, by (2.3),

$$M = x'+dy'+a_k z',$$

where

$$x' = x_{i-1}-s_i\left[\frac{z}{Q_i}\right] \geqq 0, \quad y' = y_{i-1}+P_i\left[\frac{z}{Q_i}\right], \quad z' = z-Q_i\left[\frac{z}{Q_i}\right],$$

so that, by (3.2),

$$\Lambda_k(M) \leqq x'+\left\langle\frac{y'}{\varkappa}\right\rangle+z'.$$

We have

$$\varkappa x'+y'+\varkappa z' \leqq \varkappa x_{i-1}+y_{i-1}+\varkappa(Q_i-1)+(P_i-\varkappa s_i)\left[\frac{z}{Q_i}\right],$$

so that

$$x'+\left\langle\frac{y'}{\varkappa}\right\rangle+z' \leqq x_{i-1}+\left\langle\frac{y_{i-1}}{\varkappa}\right\rangle+Q_i-1 \quad \text{if } P_i \leqq \varkappa s_i,$$

and, by (2.5) and (3.1),

$$x'+\left\langle\frac{y'}{\varkappa}\right\rangle+z' \leqq x_i+\left\langle\frac{y_i+R_i}{\varkappa}\right\rangle-1 \quad \text{if } P_i > \varkappa s_i.$$

If $R_v \geqq \varkappa$, then $R_i \geqq \varkappa$ for $i=1,2,\dots,v$. Hence, by Lemma 1 below, we have $M \in h_0 A_k$, so that $h_1 \leqq h_0$.

Thus the proof of (3.8) is complete as soon as we have proved the following

LEMMA 1. *If* $i \geqq 1$, *then*

(3.15) $$x_{i-1}+\left\langle\frac{y_{i-1}}{\varkappa}\right\rangle+Q_i-1 \leqq h_0 \quad \textit{if } P_i \leqq \varkappa s_i$$

(3.16) $$x_i+\left\langle\frac{y_i+R_i}{\varkappa}\right\rangle-1 \leqq h_0 \quad \text{if } P_i > \varkappa s_i \text{ and } R_i \geqq \varkappa .$$

PROOF. Put

$$\gamma_i = \max_{(x,y)\in X_i}\{\varkappa x+y\} = \varkappa(s_i-s_{i+1}-1)+P_{i+1}-1$$

$$\delta_i = \max_{(x,y)\in Y_i}\{\varkappa x+y\} = \varkappa(s_i-1)+P_{i+1}-P_i-1 .$$

Suppose that $P_i \leqq \varkappa s_i$. Then $\gamma_{i-1}<\delta_{i-1}$, and we prove (3.15) by showing that $\Delta \geqq 0$, where (cf. (3.4))

$$\Delta = \varkappa d+q_1-\varkappa-2-\delta_{i-1}-\varkappa(Q_i-1) .$$

By (2.4), we have

$$\Delta = (Q_i-1)\varkappa s_{i-1}-Q_{i-1}\varkappa s_i-\varkappa Q_i-P_i+P_{i-1}+\varkappa+q_1-1 ,$$

and, since $s_{i-1}\geqq s_i+1$,

$$\Delta \geqq (Q_i-Q_{i-1}-1)\varkappa s_i-P_i+P_{i-1}+q_1-1 .$$

Using the assumption $P_i \leqq \varkappa s_i$, we further have

(3.17) $$\Delta \geqq (Q_i-Q_{i-1}-2)P_i+P_{i-1}+q_1-1 .$$

If $Q_i-Q_{i-1}-2\geqq 0$, then $i\geqq 2$ and

$$\Delta \geqq P_1+q_1-1 \geqq 3 .$$

If $Q_i-Q_{i-1}-2\leqq -1$, then we have as in the proof of Lemma 5 in [7], that $i=1$ or $q_2=\ldots=q_i=2$, whence

(3.18) $$Q_j = j, \quad P_j = (q_1-1)j+1$$

for $0\leqq j\leqq i$, and the right hand side of (3.17) equals 0. This completes the proof of (3.15).

Next, suppose that $P_i>\varkappa s_i$ and $R_i\geqq\varkappa$. Then $\gamma_i>\delta_i$, and we prove (3.16) by showing that $\Gamma\geqq 0$, where

$$\Gamma = \varkappa d+q_1-2-\gamma_i-R_i .$$

By (2.4) and (3.1), we have

$$\Gamma = (Q_{i+1}-2)\varkappa s_i-(Q_i-1)\varkappa s_{i+1}-P_{i+1}+P_i-\varkappa Q_i+q_1+\varkappa-1 ,$$

and, since $s_{i+1}\leqq s_i-1$,

$$\Gamma \geqq (Q_{i+1}-Q_i-1)\varkappa s_i-P_{i+1}+P_i+q_1-1 .$$

Since $R_i \geqq \varkappa$, we have by (3.1), that $\varkappa s_i \geqq \varkappa + P_i - \varkappa Q_i$, and using (2.1), we further get

$$(3.19)\qquad \Gamma \geqq (Q_{i+1} - Q_i - q_{i+1})(P_i - \varkappa Q_i) + P_{i-1} - \varkappa Q_{i-1} + q_1 - \varkappa - 1 .$$

By (2.2), we have

$$Q_{j+1}(P_j - \varkappa Q_j) = 1 + Q_j(P_{j+1} - \varkappa Q_{j+1}) ,$$

and, since $P_1 - \varkappa Q_1 = q_1 - \varkappa \geqq 1 = P_0 - \varkappa Q_0$, it follows that

$$(3.20)\qquad P_j - \varkappa Q_j \leqq P_{j+1} - \varkappa Q_{j+1}, \quad j = 0, 1, \ldots, m .$$

Since $P_i > \varkappa s_i$ and $R_i \geqq \varkappa$, we have $i \geqq 2$. If $Q_{i+1} - Q_i - q_{i+1} \geqq 0$, we thus get by (3.19) and (3.20) that

$$\Gamma \geqq P_1 - \varkappa Q_1 + q_1 - \varkappa - 1 \geqq 1 .$$

If $Q_{i+1} - Q_i - q_{i+1} \leqq -1$, we have as in the proof of Lemma 5 in [7], that $q_2 = \ldots = q_{i+1} = 2$, and (3.18) holds for $0 \leqq j \leqq i+1$. Then the right hand side of (3.19) equals 0. This completes the proof of Lemma 1.

PROOF OF (3.9). Now suppose that $v \geqq 1$ and $R_v < \varkappa$. Again we consider the M given by (3.14).

By (3.1) and (3.20), it follows that

$$R_j - R_{j+1} \geqq \varkappa, \quad j = 0, 1, \ldots, m ,$$

so that $R_i > \varkappa$ if $1 \leqq i < v$. Thus we have, by Lemma 1, that if $1 \leqq i < v$, then $M \in h_0 A_k$.

Moreover, $P_v > \varkappa s_v$, and

$$A_k(M) \leqq \left\langle \frac{\gamma_v + R_v - \varkappa}{\varkappa} \right\rangle' = h' \quad \text{if } i = v .$$

Thus $M \in hA_k$, where $h \leqq \max\{h_0, h'\}$.

PROOF OF (3.10). Putting

$$\gamma = \alpha\varkappa - \beta, \quad 0 \leqq \beta < \varkappa ,$$

we have, by (3.11),

$$t_i^* = (a_k - 1)x + (a_k - \varkappa d)\alpha + d\beta ,$$

and, by (3.12), we see that

$$\max_{(x,y)\in X_v} t_l^* = S$$

$$\max_{(x,y)\in Y_v} t_l^* = \begin{cases}(a_k-1)(s_v-1) & \text{if } P_{v+1}-P_v-1=0\\ T \text{ otherwise}, & \end{cases}$$

where

$$S = (a_k-1)(s_v-s_{v+1}-1)+(a_k-\varkappa d)\left\langle\frac{P_{v+1}-1}{\varkappa}\right\rangle+d(\varkappa-1) \tag{3.21}$$

$$T = (a_k-1)(s_v-1)+(a_k-\varkappa d)\left\langle\frac{P_{v+1}-P_v-1}{\varkappa}\right\rangle+d(\varkappa-1)\,.$$

Since $R_v<\varkappa$, we have by (3.1) and (2.3),

$$T = -a_k(Q_v+1)-s_v+1+\varkappa d(s_v+Q_v)+(a_k-\varkappa d)\left\langle\frac{P_{v+1}+R_v-1}{\varkappa}\right\rangle+d(\varkappa-1)$$

$$\leqq dR_v-\varkappa d+1+(a_k-\varkappa d)\left\langle\frac{P_{v+1}-1}{\varkappa}\right\rangle+d(\varkappa-1) \leqq S\,.$$

Thus

$$\max t_l^* = S\,. \tag{3.22}$$

Next, consider

$$M = 2s_v-s_{v+1}-1+d\left(\varkappa\left\langle\frac{P_{v+1}-P_v-1}{\varkappa}\right\rangle-\varkappa+1\right)+a_k(Q_v-1)\,. \tag{3.23}$$

Then, by (3.1) and (2.3), we also have

$$M = s_v-s_{v+1}-1+d\left(\varkappa\left\langle\frac{P_{v+1}+R_v-1}{\varkappa}\right\rangle-R_v-\varkappa+1\right)-a_k\,. \tag{3.24}$$

Suppose that $h_1<h'$. By (1.2), (3.21), (3.22), (3.24), and the formula

$$g(A_k^*) = -a_k+\max t_l^* \tag{3.25}$$

of Brauer and Shockley [1], we then have

$$n(h'-1,A_k) = a_kh'-S-1$$

$$= M+(a_k-\varkappa d)\left(\left\langle\frac{P_{v+1}+R_v-1}{\varkappa}\right\rangle-\left\langle\frac{P_{v+1}-1}{\varkappa}\right\rangle\right)+dR_v-1\,.$$

Since $v\geqq 1$, we also get, using (3.23),

$$M \geqq 2s_v-s_{v+1}-1+d(P_{v+1}-P_v-\varkappa)+a_k(Q_v-1)$$

$$\geqq s_v+d(q_1-1-\varkappa) \geqq s_v\,.$$

Thus we have

$$0 \leqq M \leqq n(h'-1, A_k) ,$$

and in particular,

$$A_k(M) \leqq h'-1 .$$

We continue to show that this leads to a contradiction, thus proving (3.10).

By (3.2), there are three-tuples (x, y, z) of non-negative integers such that

$$M = x + dy + a_k z \tag{3.26}$$

$$A_k(M) = x + \left\langle \frac{y}{\varkappa} \right\rangle + z ,$$

and where $\lambda = \varkappa x + y + \varkappa z$ is minimal. Among these three-tuples choose the one where y is minimal. Then

$$\left\langle \frac{\lambda}{\varkappa} \right\rangle = A_k(\mathrm{M}) \leqq \mathrm{h}'-1 ,$$

so that

$$\lambda \leqq \varkappa(h'-1) . \tag{3.27}$$

There is a unique lattice point $(x_v, y_v) \in X_v \cup Y_v$ such that

$$x_v + dy_v \equiv M \pmod{a_k} .$$

Now,

$$0 \leqq \varkappa \left\langle \frac{P_{v+1}+R_v-1}{\varkappa} \right\rangle - R_v - \varkappa + 1 < P_{v+1} ,$$

so that, by (3.24),

$$M = x_v + dy_v - a_k .$$

Hence, by (3.26),

$$(x, y) \notin X_v \cup Y_v . \tag{3.28}$$

By (2.3), we have

$$M = (x - s_v) + d(y + P_v) + a_k(z - Q_v) ,$$

where, by (3.1),

$$\varkappa(x - s_v) + (y + P_v) + \varkappa(z - Q_v) = \lambda - R_v .$$

Hence, by the minimality of λ, we have

(3.29) $$x < s_v \quad \text{or} \quad z < Q_v .$$

Further

$$M = (x+s_{v+1})+d(y-P_{v+1})+a_k(z+Q_{v+1}) ,$$

where

$$\varkappa(x+s_{v+1})+(y-P_{v+1})+\varkappa(z+Q_{v+1}) = \lambda+R_{v+1} .$$

If $R_{v+1}<0$, we thus have

(3.30) $$y < P_{v+1} .$$

Because of the minimality of y, (3.30) also holds in the case of $R_{v+1}=0$.

We also have

$$M = (x-s_v+s_{v+1})+d(y+P_v-P_{v+1})+a_k(z-Q_v+Q_{v+1}) ,$$

where

$$\varkappa(x-s_v+s_{v+1})+(y+P_v-P_{v+1})+\varkappa(z-Q_v+Q_{v+1}) = \lambda-(R_v-R_{v+1}) ,$$

so that

(3.31) $$x < s_v-s_{v+1} \quad \text{or} \quad y < P_{v+1}-P_v .$$

Now, if $y \geqq P_{v+1}-P_v$, then (3.31) and (3.30) imply $(x,y) \in X_v$, which contradicts (3.28). Therefore

(3.32) $$y < P_{v+1}-P_v .$$

If $z \geqq Q_v$, then (3.29) and (3.32) imply $(x,y) \in Y_v$, which also contradicts (3.28). Hence

(3.33) $$z < Q_v .$$

Now, by (3.27), (3.5), and (3.1), we have

$$\varkappa x+y+\varkappa z \leqq \varkappa\left(2s_v-s_{v+1}-3+\left\langle\frac{P_{v+1}-P_v-1}{\varkappa}\right\rangle+Q_v\right).$$

Using (3.26) and (3.23) to eliminate x from this inequality, we further get

$$1 \leqq (\varkappa d-1)\left(y-\varkappa\left\langle\frac{P_{v+1}-P_v-1}{\varkappa}\right\rangle+\varkappa-1\right)+(a_k-1)\varkappa(z-Q_v+1) ,$$

so that, by (3.32),

$$1 \leqq (\varkappa d-1)(\varkappa-1)+(a_k-1)\varkappa(z-Q_v+1) .$$

It follows that $z \geqq Q_v-1$; hence, by (3.33), we have

$$z = Q_v - 1 .$$

Clearly $x < d$, and since

$$2s_v - s_{v+1} - 1 \leqq s_1 + (s_v - s_{v+1}) - 1 \leqq s_1 + (s_0 - s_1) - 1 < d ,$$

it follows from (3.26) and (3.23) that we also have

$$x = 2s_v - s_{v+1} - 1, \quad y = \varkappa\left\langle\frac{P_{v+1} - P_v - 1}{\varkappa}\right\rangle - \varkappa + 1 .$$

Hence, by (3.5),

$$\lambda = \varkappa(h' - 1) + 1 ,$$

which contradicts (3.27).

REMARK 1. For the sequence 0, 1, 6, 12, 20 we have $v = 2$, $R_v = 1 < \varkappa = 2$, $h_0 = 6 > h' = 5$, so that

$$\max\{h_0, h'\} = h_0 .$$

For the sequence 0, 1, 4, 8, 11 we have $v = 1$, $R_v = 1 < \varkappa = 2$, $h_0 = 4 < h' = 5$, so that

$$\max\{h_0, h'\} = h' .$$

REMARK 2. The value of $\max t_l^*$ is given at the beginning of the "proof of (3.10)". Hence, by (3.25) and (1.2), we know the value of $n(h, A_k)$ for all $h \geqq h_1$.

We also note that

$$\max t_l^* = \max\{S, T\} \quad \text{if } \varkappa \geqq 2 .$$

For if $P_{v+1} - P_v - 1 = 0$, then $q_1 = \ldots = q_{v+1} = 2$, and $s_{v+1} = (v+2)d - (v+1)a_k$. Since $v \geqq 0$ and $a_k > \varkappa d$, we then have $s_{v+1} < 0$ if $\varkappa \geqq 2$, which is impossible. Hence $P_{v+1} - P_v - 1 > 0$ if $\varkappa \geqq 2$.

4. The sequence (1.4).

We now consider the sequence

$$A_k:\ a_0 = 0 < a_1 = 1 < a_2 = 2 < \ldots < a_\varkappa = \varkappa < a_{k-1} < a_k ,$$

where $\varkappa = k - 2 \geqq 1$.

Put

$$s_{-1} = a_k, \quad s_0 = a_{k-1} ,$$

and

$$R_i = \frac{1}{a_k}((a_k - \varkappa)s_i - (a_k - a_{k-1})\varkappa P_i) .$$

Then

$$-\frac{\varkappa}{s_m}(a_k - a_{k-1}) = R_{m+1} < R_m < \ldots < R_0 = a_{k-1} - \varkappa ,$$

and there is a unique integer $v = v(A_k)$, $0 \leqq v \leqq m$, satisfying

$$R_{v+1} \leqq 0 < R_v .$$

We also have

$$R_i = s_i - \varkappa P_i + \varkappa Q_i$$

$$R_{i+1} = q_{i+1} R_i - R_{i-1} .$$

The sequence A_{k-1} is pleasant, and

$$A_{k-1}(M) = \left\langle \frac{r}{\varkappa} \right\rangle + N \quad \text{if } M = r + a_{k-1}N,\ 0 \leqq r < a_{k-1}$$

$$n(h, A_{k-1}) = a_{k-1}h - (a_{k-1} - \varkappa)\left[\frac{a_{k-1}-2}{\varkappa}\right], \quad h \geqq \left[\frac{a_{k-1}-2}{\varkappa}\right].$$

This gives us

$$h_0 = h_0(A_k) = \left[\frac{a_k - 2 - \varkappa\left[\frac{a_{k-1}-2}{\varkappa}\right]}{a_{k-1}}\right] + \left[\frac{a_{k-1}-2}{\varkappa}\right] + 1 ,$$

and in particular that

$$h_0 = q_1 + \left[\frac{a_{k-1}-2}{\varkappa}\right] - 1 \quad \text{if } v \geqq 1 .$$

Let t_l^* be the smallest integer dependent on A_k^* and $\equiv l (\mathrm{mod}\, a_k)$. Then, by [6, § 4],

$$t_l^* = a_k\left\langle \frac{x}{\varkappa} \right\rangle - x + (a_k - a_{k-1})y, \quad (x, y) \in X_v \cup Y_v .$$

Thus

$$t_l^* = \sum_{i=1}^{\varkappa} a_{k-i}^* x_i^{(l)} + a_1^* y ,$$

where $x_i^{(l)}=0$ if $x=0$, and if $x>0$, then

$$x_j^{(l)} = \varrho, \quad x_{j+1}^{(l)} = q-\varrho, \quad x_i^{(l)} = 0 \quad \text{otherwise},$$

where

$$x = q\varkappa - s,\ 0 \leqq s < \varkappa; \quad s = \sigma q + \varrho,\ 0 \leqq \varrho < q; \quad j = \varkappa - \sigma - 1 .$$

(Here we only have $0 \leqq j < \varkappa$. However, if $j=0$, then $x_j^{(l)}=0$.)

Now

$$\sum_{i=1}^{\varkappa} a_i x_i^{(l)} + a_{k-1} y = x + a_{k-1} y .$$

Hence, by [7, Lemma 1], if for each r, $0 \leqq r < a_k$, all the integers

$$(4.1) \qquad r < r + a_k < r + 2a_k < \ldots < x_v + a_{k-1} y_v - a_k$$

belong to hA_k for some $h \geqq h_0 - 1$, then $h \geqq h_1$.

If $v=0$, then the set (4.1) is empty, and $h_1 = h_0 - 1$. Therefore suppose that $v \geqq 1$, and consider an arbitrary integer M in the set (4.1). By (2.6) and (2.7), we then have

$$M = x_{i-1} + a_{k-1} y_{i-1} + a_k z, \quad 0 \leqq z < Q_i \left[\frac{x_{i-1}}{s_i}\right].$$

As in § 3 and in [7, § 4], we write

$$M = x' + a_{k-1} y' + a_k z' ,$$

where

$$x' = x_{i-1} - s_i \left[\frac{z}{Q_i}\right] \geqq 0, \quad y' = y_{i-1} + P_i \left[\frac{z}{Q_i}\right], \quad z' = z - Q_i \left[\frac{z}{Q_i}\right].$$

Thus

$$\Lambda_k(M) \leqq \left\langle \frac{x'}{\varkappa} \right\rangle + y' + z' .$$

We further have

$$\left\langle \frac{x'}{\varkappa} \right\rangle + y' + z' \leqq \begin{cases} \left\langle \dfrac{x_{i-1}}{\varkappa} \right\rangle + y_{i-1} + Q_i - 1 & \text{if } \varkappa P_i \leqq s_i \\ \left\langle \dfrac{x_i + R_i}{\varkappa} \right\rangle + y_i - 1 & \text{if } \varkappa P_i > s_i . \end{cases}$$

The following lemma is quite similar to Lemma 1, and therefore we do not include a proof.

LEMMA 2. *If* $1 \leqq i \leqq v$, *then*

$$\left\langle \frac{x_{i-1}}{\varkappa} \right\rangle + y_{i-1} + Q_i - 1 \leqq a_{k-1} - q_1(\varkappa - 2) - 3 \qquad \text{if } \varkappa P_i \leqq s_i$$

$$\left\langle \frac{x_i + R_i}{\varkappa} \right\rangle + y_i - 1 \leqq a_{k-1} - q_1(\varkappa - 2) + \varkappa - 4 \qquad \text{if } \varkappa P_i > s_i .$$

It follows that

$$A_k(M) \leqq a_{k-1} - 2 - (q_1 - 1)(\varkappa - 2) .$$

Since the right hand side is $\geqq h_0$ (for $v \geqq 1$), we thus have

THEOREM 2. *For the sequence*

$$A_k:\ 0 < 1 < 2 < \dots < \varkappa < a_{k-1} < a_k ,$$

where $\varkappa = k - 2 \geqq 1$, *we have* $h_1 = h_0 - 1$ *if* $v = 0$, *and*

$$h_1 \leqq a_{k-1} - 2 - (q_1 - 1)(\varkappa - 2) \qquad \text{if } v \geqq 1 . \tag{4.2}$$

REMARK 3. It follows from [6, Theorem 1′], that

$$g(A_k^*) = -1 + (a_k - a_{k-1})(P_{v+1} - 1) + \\ + \max\left\{ (a_k - \varkappa)\left[\frac{s_v - s_{v+1} - 2}{\varkappa}\right],\quad (a_k - \varkappa)\left[\frac{s_v - 2}{\varkappa}\right] - (a_k - a_{k-1})P_v \right\} .$$

Hence, by (1.2), we know the value of $n(h, A_k)$ for all $h \geqq h_1$.

EXAMPLE 1. Take $a_{k-1} = a_k - 1 \geqq k$. Then $v \geqq 1$, and Theorem 2 gives us $h_1 \leqq a_{k-1} - k + 2$. On the other hand, since $h_0 \geqq 2$, we have by [7, formula (2.9)] that $h_1 \geqq a_{k-1} - k + 2$. Thus (4.2) is "sharp".

EXAMPLE 2. For the sequence A_4: 0, 1, 2, 7, 15 we have $v = 1$, $h_1 = h_0 - 1 = 3$. Theorem 2, however, gives us only $h_1 \leqq 5$.

Also for the sequence A_k considered in this section, we have that A_k is pleasant if and only if $v = 0$ (Djawadi [2, Satz 1]). Thus it follows that if $v = 0$, then $h_1 = h_0 - 1$.

The reversed implication, however, is *not* true in this case, as shown by Example 2.

Suppose that $v \geqq 1$ and $h_1 = h_0 - 1$. Then

$$\begin{aligned} g(A_k^*) &= a_k(h_0-1)-n(h_0-1, A_{k-1})-1 \\ &= a_1^*(2q_1-2)+a_k(h_0+2-2q_1)-(\varkappa(h_0+1-q_1)+1-s_1) . \end{aligned}$$

Since $v \geqq 1$, we have

$$\varkappa(h_0+1-q_1)+1-s_1 \leqq \varkappa(h_0+2-2q_1) .$$

If also

$$0 \leqq \varkappa(h_0+1-q_1)+1-s_1 , \tag{4.3}$$

then, by [6, Lemma 1], we have a contradiction.

Now, (4.3) can be written as

$$s_1-1 \leqq \varkappa\left[\frac{a_{k-1}-2}{\varkappa}\right], \tag{4.4}$$

and we have that *if* $v \geqq 1$ *and* (4.4) *holds, then* $h_1 \geqq h_0$.

As shown by Example 2, the "extra" condition (4.4) cannot be removed.

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