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A STUDY OF GRADED EXTREMAL RINGS AND OF MONOMIAL RINGS

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If $R=(R,m,k)$ is a local Noetherian ring or if

$$R = \bigoplus_{i\geqq 0} R_i$$

is a Noetherian graded R_0-algebra ($R_0=k$ a field) we denote the Poincaré series for R,

$$P_R(Z) = \sum_{i\geqq 0} \dim_k(\mathrm{Tor}_i^R(k,k))Z^i .$$

There are now examples of local rings (even Gorenstein) and of graded rings (even with $R_3=0$) with $P_R(Z)$ non-rational (see [1], [2], [14], [4], [10], [11]). For monomial rings (rings of the type $k[X_1,\ldots,X_n]$ modulo an ideal generated by monomials) the question of rationality is still open. In this paper we first study two classes of "extremal" graded rings and show that they have rational Poincaré series. Then we make a closer study of some classes of extremal Gorenstein rings. For monomial rings we reduce the question of rationality of the Poincaré series to squarefree monomial rings, and we show by means of an example that the Poincaré series for a monomial ring may depend on the characteristic of the ground field.

Extremal rings.

Let k be an infinite field and R a commutative Noetherian graded k-algebra such that $R_0=k$ and $m=\bigoplus_{n\geqq 1} R_n$ is generated by R_1. R is then a factor of $A=k[X_1,\ldots,X_n]$, $R=A/I$ with I homogeneous. $H(n,R)=\dim_k R_n$ is called the Hilbert function of R. $H(n,R)$ is a polynomial for n large, the Hilbert–Samuel polynomial, we denote this polynomial by $h(n,R)$. The generating function for the Hilbert function, the Hilbert series

$$H_R(Z) = \sum_{n=0}^{\infty} H(n,R)Z^n ,$$

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is a rational function and has in lowest terms the form $p_R(Z)/(1-Z)^{\dim R}$, where $p_R(Z)$ is a polynomial with integer coefficients, with $p_R(0)=1$ and $p_R(1)\neq 0$. $p_R(1)=e(R)$ is called the multiplicity of R.

DEFINITION. If $H_R(Z)=p_1(Z)/p_2(Z)$ with p_i polynomials we define the index of irregularity for R as

$$i_1(R) = \deg p_1(Z)-\deg p_2(Z)+\dim R .$$

In some sense $i_1(R)$ measures the deviation for R from being regular, at least $i_1(R)\geqq 0$ with equality if and only if R is a polynomial ring. Schenzel defines the regularity index of R as

$$i(R) = \max\{n;\ H(n,R)\neq h(n,R)\}+1$$

in [15]. The following lemma explains the connection between $i_1(R)$ and $i(R)$, in fact we show that $i_1(R)=i(R)-1+\dim R$.

LEMMA 1. *Let $R=k[X_1,\ldots,X_n]/I$ where k is a field and I homogeneous, let $H_R(Z)=p_1(Z)/p_2(Z)$ be the Hilbert series for R (p_i polynomials) and let $H(n,R)$, respectively $h(n,R)$, denote the Hilbert function, respectively the Hilbert–Samuel polynomial for R. Then*

$$\deg p_1(Z)-\deg p_2(Z) = \max\{n;\ H(n,R)\neq h(n,R)\} \geqq -\dim R$$

with equality if and only if $R\cong k[X_1,\ldots,X_{\dim R}]$.

PROOF. Of course $\deg p_1(Z)-\deg p_2(Z)$ is independent of the representation of $H_R(Z)$ as quotient of polynomials, so we could use $H_R(Z)=p_R(Z)/(1-Z)^{\dim R}$ to see that

$$\deg p_1(Z)-\deg p_2(Z) = \deg p_R(Z)-\dim R \geqq -\dim R$$

with equality if and only if $p_R(Z)=1$, i.e. if and only if $H_R(Z)=1/(1-Z)^{\dim R}$, i.e. if and only if $R\cong k[X_1,\ldots,X_{\dim R}]$. Now let $p(Z)$ be any polynomial with integer coefficients. Then it is easy to see that the coefficient of Z^n in the power series expansion

$$p(Z)/(1-Z)^d = \sum_{i\geqq 0} h_i Z^i$$

is a polynomial $h(n)$ for n large, that $h_i=h(i)$ for $i>\deg p(Z)-d$ and that $h_{\deg p(Z)-d}\neq h(\deg p(Z)-d)$. This gives

$$\deg p_R(Z)-\dim R = \deg p_1(Z)-\deg p_2(Z) = \max\{n;\ H(n,R)\neq h(n,R)\} .$$

Now suppose that $R=k[X_1,\ldots,X_n]/I$, k an infinite field, I a homogeneous ideal $\neq (0)$ and let t be the least degree of a generator for I. Schenzel shows in [15] that, in our terminology, $i_1(R)\geqq t-1$ if R is Cohen–Macaulay and that $i_1(R)\geqq 2t-2$ if R is Gorenstein. A Cohen–Macaulay (respectively Gorenstein) ring is called extremal Cohen–Macaulay (respectively extremal Gorenstein) if there is equality. Note that an extremal Gorenstein ring is not an extremal Cohen–Macaulay ring with these definitions unless it is isomorphic to a polynomial ring over k. We disgress a little to show that extremal rings are extremal with respect to Hilbert functions also. If R is graded and x is a non-zerodivisor in R of degree one, $H_R(Z)=(1-Z)H_{R/(x)}(Z)$ so R is extremal if and only if $R/(x)$ is extremal ($\dim R/(x)=\dim R-1$). If R is a zero-dimensional extremal Cohen–Macaulay ring $k[X_1,\ldots,X_n]/I$, $I\subseteqq (X_1,\ldots,X_n)^2$,

$$H(i,R) = H(i,k[X_1,\ldots,X_n] \text{ if } i<t \quad \text{and} \quad H(i,R) = 0 \text{ if } i\geqq t$$

(so $R\cong k[X_1,\ldots,X_n]/(X_1,\ldots,X_n)^t$). If R is a zero-dimensional Gorenstein ring (graded) it is well-known that $H(i,R)=H(s-i,R)$, $s=\deg\operatorname{soc} R$. Hence, if $R=k[X_1,\ldots,X_n]/I$, $I\subseteqq (X_1,\ldots,X_n)^2$, is a zero-dimensional extremal Gorenstein ring then

$$H(i,R) = H(2t-2-i,R) = H(i,k[X_1,\ldots,X_n] \quad \text{for } i<t\,,$$

so also in this case the Hilbert function is "as large as it can be". We will now calculate the Poincaré series for extremal Cohen–Macaulay and Gorenstein rings.

If R is graded, $\operatorname{Tor}^R(k,k)$ is bigraded. We call

$$P_R(X,Y) = \sum_{i,j} \dim_k(\operatorname{Tor}^R_{i,j}(k,k))X^iY^j$$

the double Poincaré series for R (first degree is homological). $P_R(Z,1)$ is the Poincaré series for R. A graded ring R is called a graded trivial Golod ring if one could choose representing cycles $\{z_i\}$ for a basis of $H(K(R))$ (the homology of the Koszul complex $K(R;\, dT_i=\bar{X}_i)$) such that $z_iz_j=0$ for all (i,j). Graded trivial Golod rings have rational (even double) Poincaré series, see e.g. [5]. We will show that zero-dimensional extremal Cohen–Macaulay rings are graded trivial Golod rings and conclude that they have rational (double) Poincaré series. If R is an extremal zero-dimensional Gorenstein ring we show that $R/(0\colon m)$ is a graded trivial Golod ring. Furthermore, there is a rational correspondence between the (double) Poincaré series for R and $R/(0\colon m)$ if R is a graded zero-dimensional Gorenstein ring. As one can easily reduce to the zero-dimensional case, we can conclude that extremal Gorenstein rings have rational (double) Poincaré series.

THEOREM 2. *Extremal Cohen–Macaulay and Gorenstein rings have rational (double) Poincaré series. In fact, an extremal Cohen–Macaulay ring* $R=k[X_1,\dots,X]/I$, $I\subseteqq(X_1,\dots,X_n)^2$, *of codimension* k $(=n-\dim R)$ *and least degree of* $I=t$ *has the series*

$$P_R(X,Y) = (1+XY)^n\Big/\left(1-\sum_{i=1}^{k}\binom{t-1+k}{t-1+i}\binom{t+i-2}{i-1}X^{i+1}Y^{t+i-1}\right)$$

hence

$$P_R(Z) = (1+Z)^n\Big/\left(1-\sum_{i=1}^{k}\binom{t-1+k}{t-1+i}\binom{t+i-2}{i-1}Z^{i+1}\right).$$

An extremal Gorenstein ring $R=k[X_1,\dots,X_n]/I$, $I\subseteqq(X_1,\dots,X_n)^2$, *of codimension* k *and least degree of* $I=t$, *has the series*

$$P_R(X,Y) = (1+XY)^n\Big/\left(1-\sum_{i=1}^{k-1}\left(\binom{t-1+k}{t-1+i}\binom{t+i-2}{i-1}+\right.\right.$$
$$\left.\left.+\binom{t-1+k}{i}\binom{t+k-i-2}{t-1}-\binom{k}{i}\binom{t+k-2}{t-1}\right)X^{i+1}Y^{t+i-1}+X^{2+k}Y^{2t-2+k}\right)$$

hence

$$P_R(Z) = (1+Z)^n\Big/\left(1-\sum_{i=1}^{k-1}\left(\binom{t-1+k}{t-1+i}\binom{t+i-2}{i-1}+\right.\right.$$
$$\left.\left.+\binom{t-1+k}{i}\binom{t+k-i-2}{t-1}-\binom{k}{i}\binom{t+k-2}{t-1}\right)Z^{i+1}+Z^{2+k}\right).$$

a). EXTREMAL COHEN–MACAULAY RINGS. Schenzel shows that an extremal Cohen–Macaulay ring R has a free minimal resolution

$$E\colon 0\to A^{b_k}\xrightarrow{f_k}\dots\to A^{b_1}\xrightarrow{f_1}A\to R\to 0$$

with f_i homogeneous of degree 1 for $1<i\leqq k$ and of degree t for $i=1$. This means that

$$H_{i,j}(E\otimes k) = \operatorname{Tor}^A_{i,j}(R,k) \neq 0$$

only if

$$(i,j)\in\{(0,0),(1,t),(2,t+1),(3,t+2),\dots,(k,t+k-1)\},$$

where the first degree is homological and the second (total degree) comes from the grading of R. So

$$H_{i,r}(K(R)) = \operatorname{Tor}^A_{i,i+r}(R,k) \neq 0$$

only if

$$(i,r) \in \{(0,0),(1,t-1),(2,t-1),\dots (k,t-1)\}$$

(we call r ring degree). Now if we factor out by a maximal R-sequence consisting of elements in R_1 (which is possible since k is infinite), $\bar{R}=R/(y_1,\dots,y_d)$, it follows from Theorem A in [15] that $\bar{R}_t=0$ (in fact $\bar{R}\cong A/(X_1,\dots,X_n)^t$). Since $H_{i,r}(K(R))=H_{i,r}(K(\bar{R}))$ it follows that $\bar{R}$ is a graded trivial Golod ring (choose homogeneous z_i, i.e. ring degree $(z_i)=t-1$ for all i, then ring degree $(z_iz_j)=2t-2$, so $z_iz_j=0$ since every element of ring degree $\geqq t$ is zero). Since $P_R(X,Y)$ differs from $P_{\bar{R}}(X,Y)$ only by a factor $(1+XY)^d$, $P_R(X,Y)$ is rational. The formula follows from [5] and [15].

Schenzel also gives some examples of extremal Cohen–Macaulay rings:

1. Let $X=(X_{ij})$, $1\leqq i\leqq n$, $1\leqq j\leqq m$ be a matrix of indeterminates, I the ideal generated by all maximal minors. Then $k[X]/I$ is an extremal Cohen–Macaulay ring.

2. Let $X=(X_{ij})$, $1\leqq i,j\leqq n$, be a symmetric matrix of indeterminates, I the ideal generated by the submaximal minors. Then $k[X]/I$ is an extremal Cohen–Macaulay ring.

3. Let (S,n) be a d-dimensional local Cohen–Macaulay ring of embedding dimension $e(S)+d-1$ (the maximal possible). Then $\mathrm{Gr}_n(S)$ is an extremal Cohen–Macaulay ring. ($e(S)$ denotes the multiplicity of $\mathrm{Gr}_n(S)$.)

For references to these last statements, see [15].

b) *Extremal Gorenstein rings.* In this case R has a free minimal resolution

$$E\colon 0 \to A^{b_k} \to \dots \to A^{b_1} \xrightarrow{f_1} A \to R \to 0$$

with f_i homogeneous of degree 1 for $1<i<k$ and homogeneous of degree t for $i=1$ and for $i=k$ [15]. A similar calculation as above shows that

$$H_{i,r}(K(R)) \neq 0$$

only if

$$(i,r) \in \{(0,0),(1,t-1),(2,t-1),\dots,(k-1,t-1),(k,2t-2)\}\,.$$

As before we factor out a maximal R-sequence consisting of element in R_1, $\bar{R}=R/(y_1,\dots,y_d)$, and this does not change the homology of the Koszul complex. Then Theorem 2 in [9], which gives the correspondence

$$P_{S/(0:n)}(Z) = P_S(Z)/(1-Z^2P_S(Z))$$

for a zero-dimensional local Gorenstein ring (S,n) is easily extended to the graded case which gives

$$P_{\bar{R}/(0:\bar{m})}(X,Y) = P_{\bar{R}}(X,Y)/(1-X^2Y^{2t-2}P_{\bar{R}}(X,Y))\,.$$

Furthermore, Theorem 1 in [9] gives a correspondance between $H(K(S))$ and

$H(K(S/0{:}n))$ for a zero-dimensional local Gorenstein ring (S, n). A similar extension of this theorem to the graded case gives that

$$H_{i,r}(K(\bar{R}/(0{:}m))) \neq 0$$

only if

$$(i, r) \in$$
$$\{(0,0), (1, t-1), (2, t-1), \ldots, (k-1, t-1), (1, 2t-3), (2, 2t-3), \ldots, (k, 2t-3)\}\ .$$

Since $\bar{R}_{2t-1} = 0$, [15], we have $(\bar{R}/(0{:}m))_{2t-2} = 0$ and it follows that $\bar{R}/(0{:}m)$ is graded trivial Golod ring. Putting these pieces together, we get that $P_R(X, Y)$ is rational if R is an extremal Gorenstein ring. The formula follows from [5], [9] and [15].

Schenzel also gives examples of extremal Gorenstein rings:

4. Let $X = (X_{ij})$ be a skew-symmetric $(2n+1)\times(2n+1)$-matrix of indeterminates, I the ideal generated by the $2n \times 2n$-Pfaffians. Then $k[X]/I$ is an extremal Gorenstein ring.

5. Let $X = (X_{ij})$, $1 \leqq i, j \leqq n$, be a matrix of indeterminates, I the ideal generated by all submaximal minors. Then $k[X]/I$ is an extremal Gorenstein ring.

6. Let (S, n) be a d-dimensional local Gorenstein ring of embedding dimension $e(S)+d-2$. Then $\mathrm{Gr}_n(S)$ is an extremal Gorenstein ring.

7. Let Δ be the simplicial polytope defined by n different points on $(a, a^2, \ldots, a^d)$, and $k[\Delta]$ the associated squarefree monomial ring. Then $k[\Delta]$ is an extremal Gorenstein ring.

For references, see [15].

We could use the construction in [6] of $[d/2]$-neighbourly polytopes which are not combinatorially equivalent to cyclic polytopes to construct another sequence of squarefree monomial rings which are extremal Gorenstein, the smallest with 8 variables and 50 relations (squarefree monomials) of degree 3 and with Hilbert series $(1+4Z+10Z^2+4Z^3+Z^4)/(1-Z)^4$.

Pfaffians, minors and cyclic polytopes.

Let M be the moment curve in R^d, defined parametrically by $x(t) = (t, t^2, \ldots, t^d)$, $-\infty < t < \infty$. A *cyclic polytope* $C(n, d)$ is the convex hull of any $n \geqq d+1$ distinct points on M. $C(n, d)$ is a *simplicial polytope*, i.e. each face of dimension $< d$ is a simplex [12]. The boundary of $C(n, d)$ thus gives a simplicial complex which is a triangulation of the sphere S^{d-1}. To each simplicial complex σ on n points we associate its *Stanley–Reisner ring* $R_\sigma = \mathsf{R}[X_1,\ldots,X_n]/I_\sigma$, where I_σ is generated by the monomials $X_{i_1}X_{i_2}\ldots X_{i_k}$, $1 \leqq i_1 < i_2 < \ldots < i_k \leqq n$, for which $\{i_1,\ldots,i_k\}$ is not a face in σ. Every triangulation of a sphere gives a Gorenstein Stanley–Reisner ring [8]. $R_{\partial C(n,2d)}$ is an extremal Gorenstein ring of embedding dimension n and dimension $2d$.

Let A_{2d+3} be the $(2d+3)\times(2d+3)$-matrix (a_{ij}) with $a_{ij}=X_{ij}$ if $i<j$, $a_{ii}=0$ and $a_{ij}=-X_{ij}$ if $i>j$, where X_{ij} are variables. The minors $M_{i,i}^{2d+3}$ of A_{2d+3} (row i and column i is deleted) are $\pm$ squares, $M_{i,i}^{2d+3} = \pm(\mathrm{Pf}_i^{2d+3})^2$, the polynomials Pf_i^{2d+3} are called Pfaffians. The ring

$$\mathsf{R}[\{X_{ij};\ 1 \leqq i<j \leqq 2d+3\}]/(\{\mathrm{Pf}_i^{2d+3}\ ;\ 1 \leqq i \leqq 2d+3\}) \;=\; P_{2d+3}$$

is a "generic" graded Gorenstein ring over R of embedding dimension $2d^2+5d+3$ and dimension $2d^2+5d$ [3]. We will shows that $R_{\partial C(2d+3,2d)}$ is a specialization of P_{2d+3} by means of factoring out a P_{2d+3}-sequence of *variables* of length $2d^2+3d$.

Let B_{d+2} be the $(d+2)\times(d+2)$-matrix (b_{ij}) with $b_{ij}=X_{ij}$, where X_{ij} are variables. Let $m_{i,j}^{d+2}$ be the submaximal minors of B_{d+2} (row i and column j is deleted). The ring

$$S_{d+2} \;=\; \mathsf{R}[\{X_{i,j}\ ;\ 1 \leqq i,j \leqq d+2\}]/(\{m_{i,j}^{d+1}\ ;\ 1 \leqq i,j \leqq d+2\})$$

is a graded Gorenstein ring of embedding dimension d^2+4d+4 and dimension d^2+4d of a rather general type [7]. We will show that $R_{\partial(2d+4,2d)}$ is a specialization of S_{d+2} by means of factoring out an S_{d+2}-sequence of *variables* of length d^2+2d. Moreover these variables are chosen in the "same" way as in the former case.

Suppose that $\{X_1,\ldots,X_n\}$ is a totally ordered set of points on $C(n,d)$, i.e. $X_i = X(t_i)$ and $t_1 < t_2 < \ldots < t_n$. For simplicity we denote the set $\{X_{i_1},\ldots,X_{i_k}\}$ by its index set $\{i_1,\ldots,i_k\}$ and will always mean that $\{i_1,\ldots,i_k\}$ is totally ordered. In [12] there is an algorithm described how to decide if $\{i_1,\ldots,i_k\}$ is a face in $C(n,d)$. We will describe that algorithm briefly. A subset $\{i,i+1,i+2,\ldots,j\}$ of $\{1,2,\ldots,n\}$ is called *contiguous* if $1<i$ and $j<n$. A set $\{1,2,3,\ldots,\mathrm{k}\}$ or $\{k,k+1,k+2,\ldots,n\}$ is called an *end-set*. It is clear that any subset $I=\{i_1,\ldots,i_k\}$ can be written uniquely in the form $I=Y_1\cup X_1\cup\ldots\cup X_t\cup Y_2$ with X_i contiguous, Y_i end-sets or empty and t minimal. I is said to be of type s if in the minimal representation there are exactly s of the X_i's of odd cardinality. The result is: If Card $(I)\geqq d+1$, then I is a face of $C(n,d)$ if and only if I is of type s for some $0\leqq s\leqq d-\mathrm{card}\,(I)$. Since all $R_{\partial C(n,2d)}$ are extremal Gorenstein rings, all (minimal) relations are of the same degree [15], in fact of degree $d+1$. So to get the relations in $R_{\partial C(n,2d)}$ we should determine all subsets of $\{1,2,\ldots,2d+3\}$ (respectively $\{1,2,\ldots,d+4\}$) which are of cardinality $d+1$ and which are non-

faces. So we should determine all subsets of $\{1,2,\ldots,2d+3\}$ (respectively $\{1,2,\ldots,2d+4\}$) of cardinality $d+1$ and of type $s>d-1$. It is easily seen that these are

(*) all subsets of $\{1,2,\ldots,2d+3\}$ (respectively $\{1,2,\ldots,2d+4\}$) which modulo $2d+3$ (respectively modulo $2d+4$) contains no pair of adjacent numbers (so $(2d+3,1)$ respectively $(2d+4,1)$ are also considered to be adjacent).

Now consider A_{2d+3}. Specialize by setting all elements in the first d diagonals to the right of the main diagonal equal to zero. (The first diagonal to the right of the main diagonal consists of the pairs $(1,2),(2,3),\ldots,(2d+2,2d+3)$ and $(2d+3,1)$ and so on with the other diagonals.) Then number the remaining variables according to the following rule: Let X_1 be $X_{1,d+2}$ and let X_{i+1} be the remaining element in the same row (respectively column) as X_i if i is odd (respectively even). We give an example ($d=2$):

$$\begin{bmatrix} 0 & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} \\ -X_{12} & 0 & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} \\ -X_{13} & -X_{23} & 0 & X_{34} & X_{35} & X_{36} & X_{37} \\ -X_{14} & -X_{24} & -X_{34} & 0 & X_{45} & X_{46} & X_{47} \\ -X_{15} & -X_{25} & -X_{35} & -X_{45} & 0 & X_{56} & X_{57} \\ -X_{16} & -X_{26} & -X_{36} & -X_{46} & -X_{56} & 0 & X_{67} \\ -X_{17} & -X_{27} & -X_{37} & -X_{47} & -X_{57} & -X_{67} & 0 \end{bmatrix}$$

specializes to

$$\begin{bmatrix} 0 & 0 & 0 & X_1 & X_2 & 0 & 0 \\ 0 & 0 & 0 & 0 & X_3 & X_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & X_5 & X_6 \\ -X_1 & 0 & 0 & 0 & 0 & 0 & X_7 \\ -X_2 & -X_3 & 0 & 0 & 0 & 0 & 0 \\ 0 & -X_4 & -X_5 & 0 & 0 & 0 & 0 \\ 0 & 0 & -X_6 & -X_7 & 0 & 0 & 0 \end{bmatrix}$$

Now consider B_{d+2}. Specialize by setting all elements in d diagonals to the right of the main diagonal equal to zero and number the remaining variables according to the same rule as above. An example ($d=5$): (in a similar way as above a diagonal consists of $d+2$ pairs.)

$$\begin{bmatrix} X_{11} & X_{12} & X_{13} & X_{14} & X_{15} & X_{16} & X_{17} \\ X_{21} & X_{22} & X_{23} & X_{24} & X_{25} & X_{26} & X_{27} \\ X_{31} & X_{32} & X_{33} & X_{34} & X_{35} & X_{36} & X_{37} \\ X_{41} & X_{42} & X_{43} & X_{44} & X_{45} & X_{46} & X_{47} \\ X_{51} & X_{52} & X_{53} & X_{54} & X_{55} & X_{56} & X_{57} \\ X_{61} & X_{62} & X_{63} & X_{64} & X_{65} & X_{66} & X_{67} \\ X_{71} & X_{72} & X_{73} & X_{74} & X_{75} & X_{76} & X_{77} \end{bmatrix}$$

specializes to

$$\begin{bmatrix} X_2 & 0 & 0 & 0 & 0 & 0 & X_1 \\ X_3 & X_4 & 0 & 0 & 0 & 0 & 0 \\ 0 & X_5 & X_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & X_7 & X_8 & 0 & 0 & 0 \\ 0 & 0 & 0 & X_9 & X_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & X_{11} & X_{12} & 0 \\ 0 & 0 & 0 & 0 & 0 & X_{13} & X_{14} \end{bmatrix}$$

Now it is a completely elementary exercise, using the definition of determinant, to check that the minors of the specialized A_{2d+3} when deleting a row and the same column (respectively the minors of the specialized B_{d+2} when deleting a row and a column) are exactly $\pm$ the squares of the elements $X_{i_1}\cdot\ldots\cdot X_{i_{d+1}}$, where $\{i_1,\ldots,i_{d+1}\}$ fulfills (*) (respectively $\pm$ the elements $X_{i_1}\cdot\ldots\cdot X_{i_{d+1}}$, where $\{i_1,\ldots,i_{d+1}\}$ fulfills (*)).

NOTE. $\mathbf{R}$ could be replaced by any field (respectively any field of characteristic $\neq 2$).

Monomial rings.

Call a ring $R=k[X_1,\ldots,X_n]/(M_1,\ldots,M_r)$, where the M_i's are monomials in $\{X_i\}$ and k a field, a monomial ring. If the M_i's are squarefree we say that R is a squarefree monomial ring. We denote $k[X_1,\ldots,X_n]$ by A. Let $p=\sum\alpha_{(i)}X^{(i)}$ belong to A. We say that the monomial $X^{(i)}$ appears in p if $\alpha_{(i)}\neq 0$. Let M_{ij} denote the least common multiple of M_i and M_j and let $|M_i|$ denote the degree of M_i.

PROPOSITION. *If $R=A/I$ is a monomial ring, there is an $N\geqq n$ and a squarefree monomial ring $R'=k[X_1,\ldots,X_N]/(N_1,\ldots,N_r)$, such that $R=R'/(f_1,\ldots,f_{N-n})$, where $f_1,\ldots,f_{N-n}$ is a regular sequence of forms of degree one in R'. Furthermore,*

$$|M_i| = |N_i| \quad \text{and} \quad \left|\frac{M_iM_j}{M_{ij}}\right| = \left|\frac{N_iN_j}{N_{ij}}\right| \quad \text{for any } i, j.$$

PROOF. Let

$$I = (M_1, \ldots, M_r) = (X_1^{i_1}M'_1, \ldots, X_1^{i_k}M'_k, M_{k+1}, \ldots, M_r)\,,$$

where X_1 neither divides M'_i, $1 \leqq i \leqq k$, nor M_j, $k+1 \leqq j \leqq r$, and where all $i_j > 0$. Let $R_1 = A/I'$, where

$$I' = (X_0X_1^{i_1-1}M'_1, \ldots, X_0X_1^{i_k-1}M'_k, M_{k+1}, \ldots, M_r)\,.$$

Then $R = R_1/(X_0 - X_1)$. It suffices to show that $X_0 - X_1$ is a non-zerodivisor in R_1, since the construction could then be repeated until a squarefree monomial ring is achieved. Suppose that $p(X_0 - X_1) \in I'$, where $p \in A$. We shall show that $p \in I'$.

Let M be the monomial of lowest lexicographical order appearing in p such that $X_0M \notin I'$, if there is any such monomial, and let $M = X_1^jM'$, where X_1 does not divide M'. But since $X_0M \notin I'$ and $(X_0 - X_1)p \in I'$, $X_0X_1^{j-1}M'$ must appear in p. But this contradicts the minimality of M since $X_0^2X_1^{j-1}M' \notin I'$ if $X_0X_1^{j-1}M' \notin I'$. Hence X_0 multiply all monomials appearing in p into I, so they are all contained in $(X_1^{i_1-1}M'_1, \ldots, X_1^{i_k-1}M'_k)$. But then also X_1 must multiply all monomials appearing in p into I', whence they are all contained in (X_0). It follows that all monomials appearing in p are contained in

$$(X_1^{i_1-1}M'_1, \ldots, X_1^{i_k-1}M'_k) \cap (X_0) \subseteqq I'\,.$$

COROLLARY. *With the notations above*

a) $P_{R'} = (1+Z)^{N-n}P_R$.

b) *R is a complete intersection iff R' is.*

c) *R is Gorenstein iff R' is.*

d) *R is Cohen–Macaulay iff R' is.*

e) *R is Golod iff R' is.*

PROOF. If x is a non-zerodivisor in $m \setminus m^2$ in (S, m), it is well known that $(1+Z)P_{S/(x)}(Z) = P_S(Z)$ so a) holds. Complete intersections are characterized by their Poincaré series, so b) follows from a).

It is well known that if $x \in m$ is a non-zerodivisor in (S, m), then $S/(x)$ is Gorenstein (Cohen–Macaulay) iff S is Gorenstein (Cohen–Macaulay) whence c) and d) hold.

Given their Koszul homology, Golod rings are characterized by their Poincaré series; since R and R' have the same Koszul homology, e) follows.

REMARK. We have been informed that a similar proposition has independently been proved by Weyman.

Let $R=k[X_1,\ldots,X_n]/I$ be a monomial ring and let x_i denote the image of X_i in R. An element of the form

$$M = x_1^{i_1}\ldots x_n^{i_n}T_1^{j_1}\ldots T_n^{j_n}$$

in the Koszul complex $K^R=R\langle T_1,\ldots,T_n;\ dT_i=x_i\rangle$ is called a Koszul monomial. Let $\mathrm{Deg}(M)=(i_1+j_1,\ldots,i_n+j_n)$, then the differential in K^R is homogenous in Deg of Deg $=(0,\ldots,0)$, so the homology of $K^R, H(K^R)$, can be represented by multihomogenous cycles, i.e. elements which are sums of Koszul monomials of the same Deg. We say that a Koszul monomial M contains a multisquare if $i_k+j_k\geqq 2$ for some k. Otherwise we call M multisquarefree.

LEMMA. *Suppose R is a squarefree monomial ring. Then a basis over k of $H(K^R)$ can be represented by multisquarefree multihomogenous cycles in K^R.*

PROOF. Suppose z is a multihomogenous cycle with $\mathrm{Deg}(z)=(d_1,\ldots,d_n)$, and say $d_1\geqq 2$. Then we can write $z=x_1^2Y_1+x_1T_1Y_2$, where neither x_1 nor T_1 divide Y_2. Now

$$z' = z-d(x_1T_1Y_1) = x_1^2Y_1+x_1T_1Y_2-x_1^2Y_1+x_1T_1dY_1 = x_1T_1(Y_2+dY_1)$$

and

$$0 = dz' = x_1^2(Y_2+dY_1)-x_1T_1dY_2\ .$$

But then $x_1^2(Y_2+dY_1)=0$, since x_1 cannot divide dY_2. Thus $x_1(Y_2+dY_1)=0$, since the m_i's are squarefree, so $z'=0$, i.e. z is a boundary.

THEOREM 3. *Let $R=k[X_1,\ldots,X_n]/I$, I generated by squarefree monomials of degree $\geqq d$ $(d\geqq 2)$. If*

(a) *all squarefree monomials of degree $2d-2$ belong to I or if*
(b) $2d>n$,

then R is a Golod ring. In particular

$$P_R(Z) = (1+z)^n/(1-c_1z^2-c_2z^3-\ldots-c_nz^{n+1})\ ,$$

where $c_i=\dim_k(H_i(K^R))$.

PROOF. Choose multisquarefree elements z_i in $Z(K)$ representing a basis for $H(K)$. If $z_i\cdot z_k$ contains a multisquare, it is a boundary according to the lemma (this is always the case in (b)). If $z_i\cdot z_k$ does not contain a multisquare it is zero since ring deg $(z_i z_k) \geqq 2d-2$. The same argument applies to higher Massey products since

$$\deg(\gamma(z_1,\ldots,z_j)) = \deg(z_1\cdot\ldots\cdot z_j)$$

and ring

$$\deg(j(z_1,\ldots,z_j)) \geqq j(d-2)+2$$

which is $\geqq 2d-2$ if $j \geqq 2$ and $d \geqq 2$.

Reisner gave an example in [13] of a squarefree monomial ring which is Cohen–Macaulay if char $(k) \neq 2$, but not Cohen–Macaulay if char $(k)=2$. We will show that also the Poincaré series of this ring depends on char (k).

Reisner's example was

$$k[X_1,\ldots,X_6]/(X_1X_2X_3, X_1X_2X_4, X_1X_3X_5, X_1X_4X_6, X_1X_5X_6, X_2X_3X_6, X_2X_4X_5, X_2X_5X_6, X_3X_4X_5, X_3X_4X_6)\,.$$

The Krull dimension of this ring is 3, since its Hilbert polynomial is $1+5n^2$. We have $\dim_k H_1(K)=10$, $\dim_k \mathrm{H}_2(\mathrm{K})=15$, $\dim_k H_3(K)=6$, and $\dim H_i(K)=0$ if $i>3$ for char $(k) \neq 2$, but we have $\dim_k H_1(K)=10$, $\dim_k H_2(K)=15$, $\dim_k H_3(K)=7$, $\dim_k H_4(K)=1$ and $\dim_k H_i(K)=0$ if $i>4$ for char $(k)=2$. Thus the depth is 3 if char $(k) \neq 2$ and 2 if char $(k)=2$.

COROLLARY. *The Reisner ring has different Poincaré series for* char $(k) \neq 2$ *and for* char $(k)=2$. *In fact*

$$P_R(Z) = (1+Z)^6/(1-10Z^2-15Z^3-6Z^4) \qquad \textit{if } \operatorname{char}(k) \neq 2$$

and

$$P_R(Z) = (1+Z)^6/(1-10Z^2-15Z^3-7Z^4-Z^5) \qquad \textit{if } \operatorname{char}(k)=2\,.$$

PROOF. All squarefree monomials of degree 4 are zero, so the theorem applies.

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