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THE SMALLEST SINGULARITY OF A HILBERT SERIES

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Summary.

In this note we prove the following. Let H be any non-commutative finitely generated connected graded algebra over a field $\underline{k}$. Let

$$H(z) = \sum_{n=0}^{\infty} \operatorname{rank}_{\underline{k}}(H_n) z^n$$

be the Hilbert series of H and let $r<\infty$ be the radius of convergence of this series. Then $H(z)$ goes to infinity, as z approaches r, at least as fast as a first order pole. I.e.,

$$\liminf_{z \to r^-} (r-z)H(z) > 0 .$$

The same result holds if $H(z)$ is instead the growth function of any finitely generated group. An application shows that the members of a certain class of finitely presented Hopf algebras always have transcendental Hilbert series.

Let $\underline{k}$ be any field and let

$$M = \bigoplus_{n=0}^{\infty} M_n$$

be a locally finite graded $\underline{k}$-module. Let $M(z)$ denote the Hilbert series of M,

$$M(z) = \sum_{n=0}^{\infty} \operatorname{rank}_{\underline{k}}(M_n) z^n .$$

Hilbert series are intriguing objects because they may be studied from the point of view of algebra, combinatorics, or complex analysis. Consideration of properties of Hilbert series as functions of a complex variable has been limited. The most celebrated result in this direction is that the Krull dimension of a graded commutative algebra R is also the order of the pole of $R(z)$ at $z=1$ [2].

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The question of under what conditions $H(z)$ is a rational function, when H is a finitely presented non-commutative graded algebra [9], [7] or when H is the homology ring of the loop space on a finite CW-complex [8], [1], has also attracted a great deal of interest. Avramov [3] has looked into questions about the radius of convergence r of certain $H(z)$. In this paper we examine the local behavior of $H(z)$ in a neighborhood of $z=r$ and prove that $H(z)$ goes to infinity at r at least as fast as a first order pole if H is finitely generated as an algebra.

$$\text{Let } H = \bigoplus_{n=0}^{\infty} H_n$$

be a finitely generated graded algebra over $\underline{k}$. All graded algebras are assumed to be connected, i.e., $H_0=\underline{k}$. H is a quotient of a free associative polynomial ring $H=\underline{k}\langle x_1,\dots x_t\rangle/I$, where I is a two-sided ideal generated by homogeneous positive degree elements. Let the generator x_i have degree $e_i>0$.

Let $a_n=\operatorname{rank}_{\underline{k}}(H_n)$, so each a_n is a non-negative integer. Let r be the radius of convergence of $H(z)$. If only finitely many of the a_n's are non-zero, $H(z)$ is a polynomial, $r=\infty$, and H is a finite dimensional vector space over $\underline{k}$. We will not be concerned with this case. If infinitely many of the a_n's are non-zero, then the series fails to converge for $z=1$, hence $r\leqq 1$.

Since H is a quotient of $S=\underline{k}\langle x_1,\dots,x_t\rangle$, $H(z)$ is bounded above for $z>0$ by

$$S(z) = \left(1-\sum_{i=1}^{t} z^{e_i}\right)^{-1}$$

If r_0 is the smallest positive root of $1-\sum_{i=1}^{t} z^{e_i}$, then $S(z)$ converges for $|z|<r_0$, hence $0<r_0\leqq r$.

LEMMA 1. *Let H, $\{e_i\}$, $\{a_n\}$ be as above. Suppose each $e_i=1$. Then for all $m\geqq 0$ and $n\geqq 0$, $a_m a_n\geqq a_{m+n}$.*

PROOF. There are pairings μ_{mn}: $H_m\otimes H_n \to H_{m+n}$ for each m,n. We have a commuting diagram

$$\begin{array}{ccc} S_m\otimes S_n & \xrightarrow{\approx} & S_{m+n} \\ \downarrow & & \downarrow \\ H_m\otimes H_n & \xrightarrow{\mu_{mn}} & H_{m+n} \end{array}$$

in which the vertical maps are onto. It follows that μ_{mn} is surjective. Hence

$$\operatorname{rank}(H_m\otimes H_n) \geqq \operatorname{rank}(H_{m+n}),$$

i.e., $a_m a_n\geqq a_{m+n}$.

Our next result shows that this simple criterion is sufficient to imply the desired conclusion.

THEOREM 2. *Suppose*

$$f(z) = \sum_{n=0}^{\infty} c_n z^n$$

satisfies $c_n \in \mathbf{R}$ *and* $c_n \geqq \alpha^n$ *for all* n *for some* $\alpha > 0$ *and* $c_m c_n \geqq c_{m+n}$ *for all* $m, n \geqq 0$. *Let* r *be the radius of convergence of* f. *Then*

$$\liminf_{z \to r^-} (r-z) f(z) > 0 .$$

PROOF. To begin with, $\alpha^n \leqq c_n \leqq c_1^n$ for all n. The series for $f(z)$ converges for $|z| < c_1^{-1}$ and diverges for $|z| \geqq \alpha^{-1}$, and $\alpha^{-1} \geqq r \geqq c_1^{-1}$.

Let $b_n = (c_n)^{1/n} \geqq \alpha$. We prove first that $\lim_{n \to \infty} b_n$ exists. This is essentially proved in [5] in the context of norms in Banach algebras. Quoting from [5, p. 350], let $s > 0$ be arbitrary and let $0 \leqq t < s$. For $n > 0$ we have

$$b_{ns+t}^{ns+t} = c_{ns+t} \leqq c_s^n c_t = b_s^{ns} b_t^t = b_s^{ns+t} \left(\frac{b_t}{b_s} \right)^t ,$$

which leads to

$$b_{ns+t} \leqq b_s \left(\frac{b_t}{b_s} \right)^{\left(\frac{t}{ns+t}\right)} .$$

Letting t vary and $n \to \infty$ yields

$$\limsup_{m \to \infty} b_m \leqq b_s .$$

Letting $s \to \infty$ gives

$$\limsup_{m \to \infty} b_m \leqq \liminf_{s \to \infty} b_s ,$$

hence $\lim_{n \to \infty} b_n$ exists. Let $\beta = \lim_{n \to \infty} b_n$; recall that $r = \beta^{-1}$. We have just shown that $\beta \leqq b_n$ for all $n > 0$. So $c_n \geqq \beta^n$ and consequently

$$\liminf_{z \to r^-} (r-z) f(z) \geq \lim_{z \to r^-} (r-z) \sum_{n=0}^{\infty} \beta^n z^n = \lim_{z \to r^-} \frac{r-z}{1-\beta z} = r > 0 ,$$

as desired.

Combining lemma 1 and theorem 2 and setting $\alpha = 1$, we have at once

COROLLARY 3. *Let H, $\{e_i\}$, $\{a_n\}$, and r be as before. Suppose each $e_i = 1$. Then*

$$\liminf_{z \to r^-} (r-z)H(z) > 0 .$$

We now extend this to the general case.

THEOREM 4. *Let H, $\{e_i\}$, $\{a_n\}$, and r be as before, with no restriction on the $\{e_i\}$. Then*

$$\liminf_{z \to r^-} (r-z)H(z) > 0 .$$

PROOF. We will reduce this problem to the case of Corollary 3. We will embed H in an algebra G which is only slightly bigger than H but has degree one generators.

We may write $H = S/I$, where $S = \underline{k}\langle x_1, \ldots, x_t \rangle$ and $\deg(x_i) = e_i$, and I is a two-sided ideal of S. The first step is to embed S in an algebra T with degree one generators. Let

$$T = \underline{k}\langle y_{ij} \mid 1 \leqq i \leqq t;\ 1 \leqq j \leqq e_i \rangle$$

where each y_{ij} has degree one. Let $\alpha\colon S \to T$ be the injection of graded algebras defined by $\alpha(x_i) = y_{i1} \ldots y_{ie_i}$. Now, however, T is much bigger than S, so we reduce T by dividing out all products which do not appear in $\operatorname{im}(\alpha)$. Let $J \subseteqq T$ be the two-sided ideal generated by all $y_{ij}y_{i'j'}$ for $(i, j, i', j') \notin W$, where

$$W = \{(i, j, i, j+1) \mid j < e_i\} \cup \{(i, e_i, i', 1)\} .$$

Let $\pi\colon T \to T/J$ be the projection. Then T/J has a basis consisting of the π-images of those monomials $y_{i_1 j_1} \ldots y_{i_p j_p}$ with the following property for $k < p$: if $j_k = e_k$, then i_{k+1} is arbitrary and $j_{k+1} = 1$; if $j_k < e_k$, then $i_{k+1} = i_k$ and $j_{k+1} = j_k + 1$. In particular, $\ker(\pi \circ \alpha) = 0$. Let

$$G_1 = \operatorname{Span}\{1;\ y_{ij} \ldots y_{ie_i} \mid \text{any } i;\ j > 1\}$$

and

$$G_2 = \operatorname{Span}\{1;\ y_{i1} \ldots y_{ij} \mid \text{any } i;\ j < e_i\}$$

and

$$G_3 = \operatorname{Span}\{y_{ij} \ldots y_{ik} \mid \text{any } i;\ 1 < j \leqq k < e_i\} .$$

By considering the above described basis for T/J, we have an isomorphism

$$\sigma\colon T/J \xrightarrow{\approx} (G_1 \otimes \operatorname{im}(\alpha) \otimes G_2) \oplus G_3$$

as graded modules. Let $[\alpha(I)]$ be the two-sided ideal of T generated by $\alpha(I)$.

Since $\alpha(I)$ is already a two-sided ideal of $\mathrm{im}(\alpha)$,

$$\sigma(\pi([\alpha(I)])) = G_1 \otimes \alpha(I) \otimes G_2 .$$

Let $G = T/(J + [\alpha(I)])$. The algebra G has degree one generators and

$$G \approx (G_1 \otimes \mathrm{im}(\alpha) \otimes G_2 \oplus G_3)/(G_1 \otimes \alpha(I) \otimes G_2) \approx (G_1 \otimes H \otimes G_2) \oplus G_3$$

as graded modules.

From this we deduce that $G(z) = G_1(z)H(z)G_2(z) + G_3(z)$. Also

$$G_1(z) = 1 + \sum_{i=1}^{t} \sum_{j=2}^{e_i} z^{e_i - j + 1}$$

and

$$G_2(z) = 1 + \sum_{i=1}^{t} \sum_{j=1}^{e_i - 1} z^j$$

and

$$G_3(z) = \sum_{i=1}^{t} \sum_{1 < j \leqq k < e_i} z^{k-j+1}$$

are polynomials in z which are positive for $z \geqq 0$. $G(z)$ and $H(z)$ have the same radius of convergence r.

$$\begin{aligned} \liminf_{z \to r^-} (r-z)H(z) &= \liminf_{z \to r^-} (r-z)G(z)G_1(z)^{-1}G_2(z)^{-1} \\ &= G_1(r)^{-1}G_2(r)^{-1} \liminf_{z \to r^-} (r-z)G(z) > 0 \end{aligned}$$

by Corollary 3. This is the desired result.

EXAMPLES. That $\liminf_{z \to r^-} (r-z)H(z)$ can be finite is exemplified by $H = \underline{k}\langle x_1, \ldots, x_t \rangle$ with $\{e_i\}$ arbitrary. Let

$$f(z) = H(z)^{-1} = 1 - \sum_{i=1}^{t} z^{e_i}$$

and let r_0 be the smallest positive root of $f(z)$. r_0 is the radius of convergence of $H(z)$, and r_0 is a simple pole, because it is not a double root of f ($f'(z) < 0$ for $z > 0$). At the other extreme, $H(z)$ can have an essential singularity at r, even if H is finitely presented. Shearer [9, see "note added in proof"] gives an example of a finitely presented algebra whose Hilbert series is

$$H(z) = \left[(1-z)(1-z^2) \prod_{n=1}^{\infty} (1-z^n) \right]^{-1},$$

for which $r=1$. The function $H(z)$ has an essential singularity at $z=1$. In fact, the unit circle is the natural boundary of this analytic function.

Theorem 2 has another application to the growth functions of finitely generated groups. Let G be any finitely generated group and let C be a finite generating set. For $g \in G$, let

$$L(g) = L_C(g) = \inf\{n \mid x_{i_1}^{\varepsilon_1} x_{i_2}^{\varepsilon_2} \ldots x_{i_n}^{\varepsilon_n} = g \text{ for some } x_{i_j} \in C \text{ and } \varepsilon_j = \pm 1\} .$$

$L(g)$ can be thought of as the length of the shortest path from the identity to g in the graph of G. The growth function of G (relative to C) is defined by

$$f(z) = f_C(z) = \sum_{g \in G} z^{L(g)} .$$

If $f(z)=\sum_{n=0}^{\infty} a_n z^n$, then $a_0=1$, $a_1 \leqq 2\#(C)$, and $a_n=\#\{g \in G \mid L(g)=n\}$.

LEMMA 5. *Let $f(z)=f_C(z)$ be the growth function of the group G generated by the finite set C. Let*

$$f(z) = \sum_{n=0}^{\infty} a_n z^n .$$

Then $a_m a_n \geqq a_{m+n}$ for all $m, n \geqq 0$.

PROOF. Let $G_n=\{g \in G \mid L(g)=n\}$. For fixed m and n, define a map

$$\nu : G_m \times G_n \to \bigcup_{i=0}^{m+n} G_i$$

by $\nu(g_1, g_2)=g_1 g_2$. We now show that $G_{m+n} \subseteqq \operatorname{im} \nu$. Any $h \in G_{m+n}$ may be written as

$$h = x_{i_1}^{\varepsilon_1} \ldots x_{i_{m+n}}^{\varepsilon_{m+n}} .$$

Let

$$g_1 = x_{i_1}^{\varepsilon_1} \ldots x_{i_m}^{\varepsilon_m} \quad \text{and} \quad g_2 = x_{i_{m+1}}^{\varepsilon_{m+1}} \ldots x_{i_{m+n}}^{\varepsilon_{m+n}} .$$

If $L(g_1)<m$, then $g_1 g_2=h$ would have a representation as a product of $L(g_1)+n$ or fewer generators, contradicting the fact that $L(h)=m+n$. Thus $L(g_1)=m$ and likewise $L(g_2)=n$. $h=\nu(g_1, g_2)$ and hence $G_{m+n} \subseteqq \operatorname{im} \nu$. It follows that

$$a_{m+n} = \#(G_{m+n}) \leqq \#(G_m)\#(G_n) = a_m a_n .$$

THEOREM 6. *Let $f(z)=f_c(z)$ be the growth function of an infinite group G generated by the finite set C. Let r be the radius of convergence of $f(z)$. Then $r \in (0,1]$ and $\liminf_{z \to r^-} (r-z)f(z)>0$.*

PROOF. Since G is infinite and C is finite, none of the coefficients of $f(z)$ are zero and $r \leqq 1$. The result is a direct application of Lemma 5 and Theorem 2 with $\alpha = 1$.

In the remainder of this paper we use Theorem 4 to show that all members of a certain class of finitely presented Hopf algebras have transcendental Hilbert series. The class consists of those algebras H which arise from the construction in the following theorem, which is proved in [1].

THEOREM 7. *Let N be any finitely presented connected graded algebra over $\underline{k}$, with Hilbert series*

$$N(z) = 1 + \sum_{n=1}^{\infty} a_n z^n .$$

Let $P(z^i)$ denote $1 + z^i$, if i is odd and char $(\underline{k}) \neq 2$ *and $P(z^i) = (1 - z^i)^{-1}$, if i is even or* char $(\underline{k}) = 2$. *Then there is a finitely presented Hopf algebra H whose Hilbert series is the product of a rational function with $\prod_{n=1}^{\infty} P(z^n)^{a_n}$.*

By choosing $N = \underline{k}[x]$, one obtains a transcendental infinite product, implying that $H(z)$ is also transcendental. This example settled three long-standing interrelated problems in topology and algebra. These were the possible irrationality of the three series $H(z)$ for H a finitely presented Hopf algebra; $\sum_{n=0}^{\infty} \operatorname{rank}(H_n(\Omega X ; Q))z^n$ for X a finite CW-complex; and $\sum_{n=0}^{\infty} \operatorname{rank}(\operatorname{Tor}_n^A(\underline{k}, \underline{k}))z^n$ for A a local Artin ring with residue field $\underline{k}$. We are now in a position to show that the series $H(z)$ which are obtained through Theorem 7 are transcendental for any infinite dimensional algebra N. It suffices to show that

$$g(z) = \prod_{n=1}^{\infty} P(z^n)^{a_n}$$

is transcendental. This will answer a question of Roos and Jacobsson [oral communication] as to whether or not $g(z)$ might possibly be rational or algebraic for some choices of the algebra N.

We first prove (cf. also [4])

PROPOSITION 8. *Let $\{a_n\}_{n \geqq 1}$ be any sequence of integers and suppose $\sum_{n=1}^{\infty} a_n z^n$ has radius of convergence $r \in (0, 1]$. Then $\prod_{n=1}^{\infty} P(z^n)^{a_n}$ converges in the open disk $|z| < r$ and converges uniformly on any closed disk $|z| \leqq r_1$ for $r_1 < r$.*

PROOF. Let

$$g(z) = \prod_{n=1}^{\infty} P(z^n)^{a_n} ,$$

so

$$\log g(z) = \sum_{n=1}^{\infty} a_n \log P(z^n) .$$

If $P(z^n) = 1 + z^n$, then

$$|\log P(z^n)| = |\log(1+z^n)| \leqq \sum_{i=1}^{\infty} \frac{|z^n|^i}{i} \leqq \frac{|z|^n}{1-|z|^n} \quad \text{for } |z|<1 .$$

If $P(z^n) = (1-z^n)^{-1}$, then

$$|\log P(z^n)| = |\log(1-z^n)| \leqq \frac{|z|^n}{1-|z|^n} \quad \text{for } |z|<1 .$$

Hence

$$\begin{aligned} \left|\sum_{n=1}^{\infty} a_n \log P(z^n)\right| &\leqq \sum_{n=1}^{\infty} |a_n|\,|\log P(z^n)| \\ &\leqq \sum_{n=1}^{\infty} |a_n|\left(\frac{|z|^n}{1-|z|^n}\right) \\ &\leqq \left(\frac{1}{1-|z|}\right) \sum_{n=1}^{\infty} |a_n z^n| , \end{aligned}$$

which converges for $|z|<r$ and converges uniformly on $|z| \leqq r_1$ for $r_1 < r$.

To obtain the transcendentality of the product, we need a lower bound on $g(z)$ as well.

PROPOSITION 9. *Let*

$$f(z) = \sum_{n=1}^{\infty} a_n z^n$$

have radius of convergence $r \in (0,1]$, *where* $\{a_n\}_{n \geqq 1}$ *are non-negative integers. Let*

$$g(z) = \prod_{n=1}^{\infty} P(z^n)^{a_n} .$$

Then

$$g(z) \geqq e^{\frac{1}{2} f(z)} \quad \textit{for all } z \in [0,r) .$$

PROOF.

$$\log g(z) = \sum_{n=1}^{\infty} a_n \log P(z^n) .$$

Fix $z \in [0,r)$. If $P(z^n)=1+z^n$,

$$\log P(z^n) = \log(1+z^n) \geqq z^n - \frac{z^{2n}}{2}$$

and if $P(z^n)=(1-z^n)^{-1}$,

$$\log P(z^n) = -\log(1-z^n) \geqq z^n .$$

In either case,

$$\log P(z^n) \geqq z^n - \frac{z^{2n}}{2} \geqq \frac{z^n}{2} .$$

$$\log g(z) = \sum_{n=1}^{\infty} a_n \log P(z^n) \geqq \sum_{n=1}^{\infty} \tfrac{1}{2} a_n z^n = \tfrac{1}{2} f(z) .$$

Hence $g(z) \geqq e^{\frac{1}{2}f(z)}$.

THEOREM 10. *Let N be any infinite dimensional finitely generated connected graded algebra over $\underline{k}$. Let*

$$N(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$$

have radius of convergence r. Let

$$g(z) = \prod_{n=1}^{\infty} P(z^n)^{a_n} .$$

Then $g(z)$ is a transcendental analytic function defined on $|z|<r$. If N is finitely presented and H is the algebra constructed from N as in Theorem 7, then $H(z)$ is also transcendental.

PROOF. We have already established through Propoisition 8 that $g(z)$ is analytic in the open disk $|z|<r$. By Theorem 4, there is some $\lambda>0$ such that

$$\liminf_{z \to r^-} (r-z)N(z) \geqq \lambda .$$

If $g(z)$ were algebraic, $\lim_{z \to r^-} (r-z)^m g(z)$ would be zero for some sufficiently large m. But for any $m \geqq 0$,

$$\lim_{z \to r^-} (r-z)^m g(z) \geqq \liminf_{z \to r^-} (r-z)^m e^{(\frac{1}{2})(N(z)-1)}$$
$$\geqq e^{-1/2} \lim_{z \to r^-} (r-z)^m e^{(\frac{1}{2})\lambda(r-z)^{-1}} = \infty .$$

One further observation on the singularities of their Hilbert series may be made for this class of algebras. We have shown that $g(z)$ is transcendental, but the Hopf algebra H constructed from N always has a Hilbert series which is a rational function times $g(z)$. By its construction, this rational function always has a finite order pole at some point strictly smaller than r, the radius of convergence of $N(z)$. There seems to be no way of avoiding these additional singularities. This observation provokes the following open question.

QUESTION. Let r be the radius of convergence of a finitely presented graded Hopf algebra H. Is r always a finite order pole of $H(z)$?

We have already seen that the Hopf algebra requirement cannot be omitted, and it is easy to construct finitely generated (but not finitely presented!) Hopf algebras with only essential singularities.

NOTE ADDED IN PROOF. We have since discovered finitely presented Hopf algebras with smallest singularity essential. The question remains open, however, for Hopf algebras with degree one generators and degree two relations.

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