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ON THE DOUBLE POINCARÉ SERIES OF THE ENVELOPING ALGEBRAS OF CERTAIN GRADED LIE ALGEBRAS

CALLE JACOBSSON

0. Introduction.

Let N be a graded connected algebra over a field k. It is well known that a basis for the graded vector space $\operatorname{Tor}_1^N(k,k)$ is in a one-to-one correspondence with a minimal system of generators for N. A basis for $\operatorname{Tor}_2^N(k,k)$ corresponds to a minimal system of relations for N, $\operatorname{Tor}_3^N(k,k)$ corresponds to a minimal system of relations between the relations of N, and so on. If $V=\bigoplus_0^\infty V_i$ is a graded vector space, then its *Hilbert series* is $V(z)=\sum_0^\infty |V_i|z^i$, where $|\cdot| = \dim_k(\cdot)$.

The Hilbert series of all the $\operatorname{Tor}_n^N(k,k)$, $n=0,1,2,\ldots$ define the *double Poincaré series* of N:

$$P_N(x,z) \;=\; \sum_{n=0}^{\infty} x^n \operatorname{Tor}_n^N(k,k)(z) \;=\; \sum_{n,i\geqq 0} |\operatorname{Tor}_{n,i}^N(k,k)| x^n z^i \;.$$

The Hilbert series of N is also related to the double Poincaré series by the formula

$$P_N(-1,z) \;=\; (N(z))^{-1} \;.$$

The double Poincaré series thus gives us much information about the graded algebra N. (cf. Roos [12])

Löfwall and Roos [8], [9] have given a method, using extensions of Lie algebras, how to, from a given algebra, construct a new finitely presented Hopf algebra with "much worse" properties. The new algebra e.g. have a transcendental Hilbert series (cf. Anick [3]). In this paper we will study the double Poincaré series of the algebras in the Löfwall–Roos construction, and also give an example of a finitely presented Hopf algebra Λ where $\operatorname{Tor}_n^\Lambda(k,k)(z)$ is a transcendental function for each $n\geqq 3$, thereby answering a question of Lemaire [5] in the negative.

One of the main reasons for studying the series $\operatorname{Tor}_n^\Lambda(k,k)(z)$ is that they

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occur e.g. in the study of the homology algebra of loop spaces of finite, simply-connected CW-complexes of dimension $\leqq 4$ ([5], [6], [11]). These series also occur e.g. in the study of the Yoneda Ext-algebra $\mathrm{Ext}_R^*(k,k)$, where (R,m,k) is a local ring with $m^3=0$ [11].

Details about these applications are given in Section 1.3 below. The proof of the main theorem in Section 1 is given in Section 2, using two technical lemmas, which are proved in Section 3. In Section 4 we use an analytic property of the series $\mathrm{Tor}_n^A(k,k)(z)$ to answer a question of Lemaire. Finally, in Section 5 we mention some open problems.

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1. The main theorem.

1.1 THE LÖFWALL–ROOS CONSTRUCTION. From a given algebra N, we construct a new finitely presented Hopf algebra Ug with transcendental Hilbert series (cf. Anick [3]).

We analyse the double Poincaré series of this Ug, and give a complete result in the case, where $\operatorname{gl}\dim N \leqq 2$. As an example we will take Löfwall–Roos' prime example of an algebra with transcendental Hilbert series. Its double series will be computed in Section 1.2.

It should be noted that Anick [1], [2] was the first to construct finitely presented Hopf algebras with transcendental Hilbert series, using completely different methods but with very similar results. The Löfwall–Roos construction is, however, more suitable for the study of the double Poincaré series. The construction is as follows (for more details, see [8] and [9]):

Take a graded associative algebra $N=\bigoplus_{i\geqq 0} N_i$ with relations of degree $\leqq n$ for some n, where $N^+=\bigoplus_{i\geqq 1} N_i$ is generated by N_1.

Put $f=F\times F'=F(V\oplus W)\times F(V'\oplus ka)$, where $F(\cdot)$ is the "free graded Lie algebra", $V=V'=N_1$

$$W=\{x_w \mid x\in N_2\} \qquad \deg x_w = 1\ ,$$

and a is a symbol of degree 1. We consider N^+ as an abelian Lie algebra, and as an f-module by:

$$W\circ N^+ = a\circ N^+ = 0, \qquad v\circ n = vn, \qquad \text{and}$$

$$v'\circ n = -(-1)^{\deg n} nv', \qquad n\in N^+,\ v\in V,\ v'\in V'\ .$$

The class $[\tilde g]\in H^2(f,N^+)_0$ is defined by:

$$\tilde g(x_w,a) = x \text{ for } x\in N_2 \text{ and } \tilde g = 0 \text{ for other elements of degree } 1\ .$$

This determines an extension of graded Lie algebras

$$0 \to N^+ \to g \to f \to 0$$

such that g is finitely presented, with generators of degree 1 and relations of degree $\leqq n$. The Hilbert series of Ug is the product of the Hilbert series of Uf and that of UN^+, so we have

$$P_{Ug}(-1,z) = P_{Uf}(-1,z) \cdot P_{UN^+}(-1,z) .$$

The entire double series of Ug would also be the product

$$P_{Ug}(x,z) = P_{Uf}(x,z) \cdot P_{UN^+}(x,z) ,$$

if the extension were the trivial one. This would not, however, give a finitely presented Ug, so with the Löfwall–Roos extension we get a more complicated result.

THEOREM: *Assume in the above construction that $N = \bigoplus_0^\infty N_i$ is a Hopf algebra over a field k of characteristic* 0. *Then we have*

$$P_{Ug}(x,z) = x^2 \cdot P_{Uf}(-1,z) \cdot P_{UN^+}(x,z) + (1+x)(1-xz)^{-|N_1|} +$$
$$+ ((|N_1|+|N_2|)z-1)[(x+x^2)(N(z))^{-1} P_{UN^+}(x,z) + xP_N(x,z) - x(N(z))^{-1}]$$
$$+ (x+x^2) \sum_{n \geqq 0} x^n (X_n \otimes_N k)(z) ,$$

where

$$P_{UN^+}(x,z) = \prod_{j \geqq 1} \frac{(1+xz^{2j})^{|N_{2j}|}}{(1-xz^{2j-1})^{|N_{2j-1}|}}$$

$$P_{Uf}(-1,z) = (1-(|N_1|+|N_2|)z)(1-(|N_1|+1)z)$$

$$X_n = \ker [N_1 \otimes_k E_n N^+ \to E_n N^+] ,$$

$E_n N^+$ is the n-th graded exterior product of N^+, and

$$\sum_{n \geqq 0} x^n (X_n \otimes_N k)(z) = \mathrm{Tor}_2^N (k,k)(z) \cdot (N(z))^{-1} (P_{UN^+}(x,z) + x - 1) + |N_1| z$$

if $\operatorname{gl\,dim} N \leqq 2$.

The theorem thus gives a complete series only when we have $\operatorname{gl\,dim} N \leqq 2$.

1.2. APPLICATIONS. If we put $N = k\langle T \rangle$ in the above construction, we get the Löfwall–Roos example of a finitely presented Hopf algebra with transcendental Hilbert series. Since $\operatorname{gl\,dim} N = 1$ in this case, we can apply the theorem ($N(z) = (1-z)^{-1}$ and $|N_i| = 1$). We have:

$$P_{Ug}(x,z) = x^2(1-2z)^2 \prod_{j\geq 1} \frac{1+xz^{2j}}{1-xz^{2j-1}} + (1+x)(1-xz)^{-1} +$$
$$+ (x+x^2)(3z-1)(1-z) \prod_{j\geq 1} \frac{1+xz^{2j}}{1-xz^{2j-1}} + (x+x^2)3z^2$$

The Hilbert series is

$$Ug(z) = (P_{Ug}(-1,z))^{-1} = (1-2z)^{-2} \prod_{j\geq 1} \frac{1+z^{2j-1}}{1-z^{2j}} .$$

The main theorem gives, in particular, the series for $\mathrm{Tor}_3^{Ug}(k,k)$.

COROLLARY 1: *If in the Löfwall–Roos construction, N is a Hopf algebra over a field k of characteristic 0, we have:*

$$\mathrm{Tor}_3^{Ug}(k,k)(z) = P_{Uf}(-1,z)(N(z)-1) + \tfrac{1}{2}(|N_1|^2+|N_1|)(\tfrac{1}{3}(|N_1|+2)z+1)z^2 +$$
$$+ ((|N_1|+|N_2|+1)z-1)\tfrac{1}{2}[N(z)-N(-z^2)(N(z))^{-1} + 2\,\mathrm{Tor}_2^N(k,k)(z)] +$$
$$+ \mathrm{Tor}_2^N(k,k)(z) + (X_2 \otimes_N k)(z) ,$$

where

$$(X_2 \otimes_N k)(z)$$
$$= \mathrm{Tor}_2^N(k,k)(z)\tfrac{1}{2}(N(z)-N(-z^2)(N(z))^{-1} + 2\,\mathrm{Tor}_2^N(k,k)(z) - 2|N_1|z)$$

if $\mathrm{gl\,dim}\, N \leqq 2$.

This gives many examples of infinite-dimensional $\mathrm{Tor}_3^{Ug}(k,k)$, but since $\mathrm{gl\,dim}\, N \leqq 2$ implies that $N(z)$ is rational, all completely computed series are rational. If, however, we could compute $(X_2 \otimes_N k)(z)$ for N = the Ug of Löfwall–Roos above, this would very likely give a transcendent $\mathrm{Tor}_3^{Ug}(k,k)(z)$. In Section 4 we will take an algebra N which is not a Hopf algebra, with relations in degree 2, and show that the corresponding Hopf algebra Ug has transcendent $\mathrm{Tor}_n^{Ug}(k,k)(z)$ for $n \geqq 3$, without actually computing the series.

1.3. APPLICATIONS TO THE HOMOLOGY OF LOOP SPACES AND OF LOCAL RINGS. Let Y be a finite, simply-connected CW-complex with $\dim Y \leqq 4$, ΩY the loop space on Y and $H_*(\Omega Y, \mathbf{Q})$ the homology algebra of ΩY. It is known that Y (at least over $\mathbf{Q}$) can essentially be obtained as the mapping cone of a map

$$\bigvee S^3 \to \bigvee S^2$$

between finite wedges of spheres. It is also known that the algebra image of the natural map

$$H_*(\Omega S^2, \mathbf{Q}) \to H_*(\Omega Y, \mathbf{Q})$$

is a finitely presented Hopf algebra Λ with generators in degree 1 and relations in degree 2, and that all such $\Lambda - s$ (over $\mathbf{Q}$) occur in this way ([11]). Furthermore, under weak conditions ([5], [6]) we have

$$(*) \qquad \operatorname{Tor}_{i,*}^{H_*(\Omega Y,\mathbf{Q})}(\mathbf{Q},\mathbf{Q}) = \operatorname{Tor}_{i,*}^{\Lambda}(\mathbf{Q},\mathbf{Q}) \oplus \operatorname{Tor}_{i+2,*-1}^{\Lambda}(\mathbf{Q},\mathbf{Q}).$$

It follows, in particular ($i=1$), that a minimal set of generators for the algebra $H_*(\Omega Y, \mathbf{Q})$ is formed by the generators (of degree 1) for Λ, *and* some "strange" generators corresponding to a basis for $\operatorname{Tor}_3^{\Lambda}(\mathbf{Q},\mathbf{Q})$. These "strange" generators can occur in a very irregular manner, since we in Section 4 will show that all the series $\operatorname{Tor}_i^{\Lambda}(k,k)(z)$ for $i \geqq 3$ can be transcendental for some $\Lambda - s$. This also shows that the relations (and higher relations) between the generators can occur in a "transcendental" way.

Let (R,m,k) be a local commutative, noetherian ring with $m^3=0$. Consider the Yoneda Ext-algebra $\operatorname{Ext}_R^*(k,k)$, and let Λ be the subalgebra generated by $\operatorname{Ext}_R^1(k,k)$. Then Λ is a finitely presented Hopf algebra, with generators in degree 1 and relations in degree 2, and all such $\Lambda - s$ occur in this way. The following formula, very similar to $(*)$ above, is proved in [11], assuming R equicharacteristic.

$$(*') \qquad \operatorname{Tor}_i^{\operatorname{Ext}_R^*(k,k)}(k,k)^* = \operatorname{Tor}_i^{\Lambda}(k,k)^* \oplus \operatorname{Tor}_{i+2}^{\Lambda}(k,k)^{*+1}$$

(Λ is considered here to be upper graded).

It follows in the same way as above that $\operatorname{Ext}_R^*(k,k)$ besides generators of degree 1, needs some "strange" generators, and that these can occur in a transcendental way. The Hopf algebra Ug of Section 4 corresponds (cf. Roos [11]) to a local ring (R,m,k) with embedding dimension $|m/m^2|=27$, having 168 relations of degree 2. There ought to exist smaller examples.

2. Proof of the main theorem.

2.1. THE HOCHSCHILD–SERRE SPECTRAL SEQUENCE. We analyse $\operatorname{Tor}_n^{Ug}(k,k)$ by means of the Hochschild–Serre spectral sequence ([4], [8], [9]) in the graded case, We have

$$E_{p,q}^2 = \operatorname{Tor}_p^{Uf}(k, \operatorname{Tor}_q^{UN^+}(k,k)) \xrightarrow{p} \operatorname{Tor}_{p+q}^{Ug}(k,k).$$

As N^+ is considered as an abelian Lie algebra, it is easy to see that $\operatorname{Tor}_q^{UN^+}(k,k) = E_q N^+$, the q-th graded exterior product of N^+, where $E_0 N^+ = k$.

Since $\operatorname{gl\,dim} Uf = 2$, we have $E_{p,q}^2 = 0$ for $p \neq 0,1,2$, $d_{p,q}^2 = 0$ for $p \neq 2$ and also $E_{1,q}^{\infty} = E_{1,q}^2$.

The mapping

$$d^2_{2,q} : \operatorname{Tor}_2^{Uf}(k, E_q N^+) \to \operatorname{Tor}_0^{Uf}(k, E_{q+1} N^+)$$

is given by

$$(\alpha_1, \alpha_2) \otimes_{Uf} \langle x_1, \ldots, x_q \rangle \mapsto \langle \tilde{g}(\alpha_1, \alpha_2), x_1, \ldots, x_q \rangle ,$$

where $[\tilde{g}] \in H^2(f, N^+)_0$ as above determines the extension. It is easily shown ([9]), that

$$\operatorname{Im} d^2_{2,q} = E^2_{0,q+1} - \{\text{all elements of degree } q+1\}$$

and so

$$\begin{aligned} E^\infty_{0,q+1} &= \{\text{all elements of degree } q+1\} \\ &= \{\langle x_1, \ldots, x_{q+1} \rangle \mid x_j \in N_1 \text{ for all } j\} . \end{aligned}$$

Thus we can see that

$$\sum_{n \geqq 0} x^n \cdot E^\infty_{0,n}(z) = (1 - xz)^{-|N_1|} .$$

This shows, in particular, that we always have $\operatorname{gl\,dim} Ug = \infty$ if $N^+ \neq 0$.

We know that

$$\operatorname{Tor}_n^{Ug}(k,k) = E^\infty_{0,n} \oplus E^\infty_{1,n-1} \oplus E^\infty_{2,n-2} \qquad \text{for } n \geqq 2 ,$$

and that

$$0 \to E^\infty_{2,n-2} \to E^2_{2,n-2} \xrightarrow{d^2} E^2_{0,n-1} \to E^\infty_{0,n-1} \to 0$$

is an exact sequence for $n \geqq 2$. Since $E^\infty_{1,n-1} = E^2_{1,n-1}$ this gives us;

(2*) $$\begin{aligned} \operatorname{Tor}_n^{Ug}(k,k)(z) = {} & E^\infty_{0,n}(z) + E^2_{1,n-1}(z) + E^2_{2,n-2}(z) - \\ & - E^2_{0,n-1}(z) + E^\infty_{0,n-1}(z) \end{aligned}$$

for $n \geqq 2$. It thus remains to compute the series $E^2_{p,q}(z)$.

2.2. THE TERMS $E^2_{p,q}$ OF THE SPECTRAL SEQUENCE. We can analyse $E^2_{p,q} = \operatorname{Tor}_p^{Uf}(k, E_q N^+)$ by means of the "small" — only four terms — spectral sequence associated to the trivial extension of graded Lie algebras;

$$0 \to F \to f \to F' \to 0 \qquad (f = F \times F')$$

We have

$$\begin{aligned} E^2_{0,n-1} &= \operatorname{Tor}_0^{UF}(k, \operatorname{Tor}_0^{UF'}(k, E_{n-1} N^+)) \\ E^2_{1,n-1} &= \operatorname{Tor}_0^{UF}(k, \operatorname{Tor}_1^{UF'}(k, E_{n-1} N^+)) \oplus \operatorname{Tor}_1^{UF}(k, \operatorname{Tor}_0^{UF'}(k, E_{n-1} N^+)) \\ E^2_{2,n-2} &= \operatorname{Tor}_1^{UF}(k, \operatorname{Tor}_1^{UF'}(k, E_{n-2} N^+)) . \end{aligned}$$

We want to compute $E^2_{2,n-2}(z)+E^2_{1,n-1}(z)-E^2_{0,n-1}(z)$. It is useful to compute $E^2_{0,n-1}(z)$ together with the second term of $E^2_{1,n-1}(z)$. Since V, or V', acts on E_nN^+ as N_1 from the left, or right, respectively, it is convenient to have the following notation;

$$X_n = \ker(N_1 \otimes E_nN^+ \to E_nN^+) .$$

(All tensor products are over k, except when especially stated.) As N is a Hopf algebra, the series will not be affected by putting N_1 to the right, instead of to the left. We will allow this ambiguity, but by $X_n \otimes_N k$ it will be understood that this X_n has N_1 to the left, and vice versa. Since ka operates trivally on N^+, we have:

$$\begin{aligned}\operatorname{Tor}_0^{UF'}(k, E_nN^+) &= \operatorname{coker}((V' \oplus ka) \otimes E_nN^+ \to E_nN^+) \\ &= \operatorname{coker}(E_nN^+ \otimes N_1 \to E_nN^+) \\ &= E_nN^+ \otimes_N k .\end{aligned}$$

And similarly

$$\operatorname{Tor}_1^{UF'}(k, E_nN^+) = \ker((V' \oplus ka) \otimes E_nN^+ \to E_nN^+) = E_nN^+ \otimes ka \oplus X_{n-1} .$$

So $E^2_{0,n-1} = \operatorname{Tor}_0^{UF}(k, E_{n-1}N^+ \otimes_N k)$, the second term of $E^2_{1,n-1}$ is equal to $\operatorname{Tor}_1^{UF}(k, E_{n-1}N^+ \otimes_N k)$, and since we want to compute the difference between the series, we can instead compute the difference;

$$\begin{aligned}&(V \oplus W) \otimes (E_{n-1}N^+ \otimes_N k)(z) - (E_{n-1}N^+ \otimes_N k)(z) \\ &\qquad = ((|N_1| + |N_2|)z - 1)(E_{n-1}N^+ \otimes_N k)(z) .\end{aligned}$$

As W operates trivially on N^+, the fist term of $E^2_{1,n-1}$ is;

$$\begin{aligned}\operatorname{Tor}_0^{UF}(k, (E_{n-1}N^+ \otimes ka \oplus X_{n-1})) &= k \otimes_N (E_{n-1}N^+ \otimes ka \oplus X_{n-1}) \\ &= (k \otimes_N E_{n-1}N^+) \otimes ka \oplus k \otimes_N X_{n-1}\end{aligned}$$

Finally we compute the term $E^2_{2,n-2}$;

$$\begin{aligned}E^2_{2,n-2} &= \operatorname{Tor}_1^{UF}(k, (E_{n-2}N^+ \otimes ka \oplus X_{n-2})) \\ &= \ker((V \oplus W) \otimes E_{n-2}N^+ \to E_{n-2}N^+) \otimes ka \oplus \\ &\quad \oplus \ker((V \oplus W) \otimes X_{n-2} \to X_{n-2}) \\ &= (W \otimes E_{n-2}N^+ \oplus X_{n-2}) \otimes ka \oplus W \otimes X_{n-2} \oplus \\ &\quad \oplus \ker(N_1 \otimes X_{n-2} \to X_{n-2}) .\end{aligned}$$

The series of X_{n-2} can be computed from the exact sequence:

$$0 \to X_{n-2} \to N_1 \otimes E_{n-2}N^+ \to E_{n-2}N^+ \to k \otimes_N E_{n-2}N^+ \to 0$$

and we get

$$X_{n-2}(z) = (|N_1|z-1)(E_{n-2}N^+)(z) + (k \otimes_N E_{n-2}N^+)(z) .$$

Similarly, we use the exact sequence;

$$0 \to \ker (N_1 \otimes X_{n-2} \to X_{n-2}) \to N_1 \otimes X_{n-2} \to X_{n-2} \to k \otimes_N X_{n-2} \to 0$$

to get

$$\ker (N_1 \otimes X_{n-2} \to X_{n-2})(z) = (|N_1|z-1)X_{n-2}(z) + (k \otimes_N X_{n-2})(z) .$$

Summing up, the series for the term $E^2_{2,n-2}$ is;

$$E^2_{2,n-2}(z) = (1-(|N_1|+|N_2|)z)(1-(|N_1|+1)z)(E_{n-2}N^+)(z) +$$
$$+ ((|N_1|+|N_2|+1)z-1)(k \otimes_N E_{n-2}N^+)(z) + (k \otimes_N X_{n-2})(z) .$$

Since UN^+ is the enveloping algebra of the abelian Lie algebra N^+, we have

$$P_{UN^+}(x,z) = \sum_{n \geqq 0} x^n (E^n N^+)(z) = \prod_{j \geqq 1} \frac{(1+xz^{2j})^{|N_{2j}|}}{(1-xz^{2j-1})^{|N_{2j-1}|}} .$$

The proof of the main theorem is completed by the two lemmas;

LEMMA 1: *The double series of* $k \otimes_N E_* N^+$ *is given by;*

$$\sum_{n \geqq 0} x^n (k \otimes_N E_n N^+)(z) = (N(z))^{-1} P_{UN^+}(x,z) + (1+x)^{-1} \big(P_N(x,z) - (N(z))^{-1}\big) ,$$

where $P_N(x,z)$ *is the double Poincaré series of* N.

LEMMA 2: *For* $n=0,1$, *we have*

$$(k \otimes_N X_n)(z) = \mathrm{Tor}^N_{n+1}(k,k)(z) ,$$

and if $\mathrm{gl\,dim}\, N \leqq 2$, *the double series of* $k \otimes_N X_*$ *is given by:*

$$\sum_{n \geqq 0} x^n (k \otimes_N X_n)(z) = (N(z))^{-1} \mathrm{Tor}^N_2(k,k)(z) \cdot (P_{UN^+}(x,z) + x - 1) + |N_1|z .$$

3. Proofs of the two lemmas.

3.1. COMPUTATION OF THE SERIES OF $E_n N^+ \otimes_N k$. We use a lemma shown to us by Roos;

LEMMA 3: *When* N *is a Hopf algebra, the n-th graded exterior product* $E_n N$ *is free as a right or left* N*-module, for its natural* N*-module structure.*

Here we observe that the lemma treats the exterior product of $N=k\oplus N^+$. Assuming N to have generators of degree 1, the (left) N-module structure is given by the action of N_1:

$$T_i\circ\langle x_1,\ldots,x_n\rangle = \langle T_ix_1,x_2,\ldots,x_n\rangle+(-1)^{\deg x_1}\langle x_1,T_ix_2,x_3,\ldots,x_n\rangle+ \\ +\ldots+(-1)^{\Sigma_1^{n-1}\deg x_j}\langle x_1,\ldots,x_{n-1},T_ix_n\rangle .$$

PROOF OF LEMMA 3. Since N is a Hopf algebra $N=Uh$ for some graded Lie algebra h, and $\bigotimes_1^n N=U(\bigoplus_1^n h)$. This is a free N-module, since the diagonal embedding $h\hookrightarrow\bigoplus_1^n h$ gives an injection of Hopf algebras $N=Uh\hookrightarrow U(\bigoplus_1^n h)$, and as a Hopf algebra is a free module over any sub-Hopf algebra.

Now $\bigotimes_1^h N$ can be considered as the direct sum of E_nN and the ideal C of $\bigotimes_1^n N$ generated by all elements of the form $u_1\otimes u_2\otimes\ldots\otimes u_n-(-1)^{|\sigma|}u_{\sigma(1)}\otimes u_{\sigma(2)}\otimes\ldots\otimes u_{\sigma(n)}$ in the tensor algebra. ($|\sigma|$ is defined with respect to the degrees of u_i.) We can do this since we have the exact sequence

$$0\to C\to\bigotimes_1^n N \underset{s}{\overset{p}{\rightleftarrows}} E_nN\to 0$$

where the "splitting" map $s\colon E_nN\to\bigotimes_1^h N$ is defined by

$$\langle u_1,\ldots,u_n\rangle \overset{s}{\mapsto} \frac{1}{n!}\sum_\sigma(-1)^{|\sigma|}u_{\sigma(1)}\otimes\ldots\otimes u_{\sigma(n)} ,$$

where the sum runs over all permutations of $(1,2,\ldots,n)$. This map is a left and right N-module homomorphism (shown by direct computation). Since projectives—and even flats—are free for Hopf algebras, this completes the proof of Lemma 3.

We are now able to compute the series of $E_nN^+\otimes_N k$. No element u_i of even degree can occur twice in $\langle u_1,\ldots,u_n\rangle\in E_nN$, so we have

$$E_nN = E_nN^+\oplus\langle E_{n-1}N^+,1\rangle$$

as vector spaces, and that

$$0\to E_nN^+\to E_nN\to E_{n-1}N^+\to 0 \quad (n\geqq 1,\ E_0N^+=k)$$

is an exact sequence of N-modules. If we apply the functor $\cdot\otimes_N k$, and make use of Lemma 3, we get the exact sequence;

$$0\to \mathrm{Tor}_1^N(E_{n-1}N^+,k)\to E_nN^+\otimes_N k\to E_nN\otimes_N k\to E_{n-1}N^+\otimes_N k\to 0$$

and the isomorphism

$$\mathrm{Tor}_{i+1}^N(E_{n-1}N^+,k)\xrightarrow{\sim}\mathrm{Tor}_i^N(E_nN^+,k) \quad (i\geqq 1,\ n\geqq 1)$$

which immediately gives us

$$\mathrm{Tor}_1^N(E_{n-1}N^+, k) = \mathrm{Tor}_n^N(k,k)\,, \tag{3*}$$

and that

$$0 \to \mathrm{Tor}_n^N(k,k) \to E_nN^+ \otimes_N k \to E_nN \otimes_N k \to E_{n-1}N^+ \otimes_N k \to 0$$

is an exact sequence of N-modules for $n \geqq 1$. Since E_nN is free, we have

$$(E_nN \otimes_N k)(z) = (N(z))^{-1}(E_nN)(z) = (N(z))^{-1}((E_nN^+)(z) + (E_{n-1}N^+)(z))\,.$$

These formulas put together implies;

$$(E_nN^+ \otimes_N k)(z) = (N(z))^{-1}(E_nN^+)(z) + \\ + (-1)^n \sum_{j=0}^{n} (-1)^j \mathrm{Tor}_j^N(k,k) - (-1)^n (N(z))^{-1}\,.$$

We have proved;

LEMMA 1. *If N is a Hopf algebra, the double series of $E_*N^+ \otimes_N k$ is given by the formula*:

$$\sum_{n \geqq 0} x^n (E_nN^+ \otimes_N k)(z) = (N(z))^{-1} P_{UN^+}(x,z) + (1+x)^{-1}\left(P_N(x,z) - (N(z))^{-1}\right),$$

where $P_N(x,z)$ is the double Poincaré series of N.

The lemma is thus valid for arbitrary N, but the formula is particularly simple when N has finite global dimension, since (3*) shows us that; If $n \geqq \mathrm{gl\,dim}\, N$, then E_nN^+ is free as an N-module, and

$$(E_nN^+ \otimes_N k)(z) = (N(z))^{-1}(E_nN^+)(z)\,.$$

3.2. COMPUTATION OF THE SERIES OF $X_n \otimes_N k$. We recall the definition of X_n; $X_n = \ker (N_1 \otimes E_nN^+ \to E_nN^+)$. For $n=0$, we have — as $E_0N^+ = k$ — the exact sequence;

$$0 \to X_0 \to N_1 \otimes k \to k \to k \to 0$$

so we see that

$$X_0 = X_0 \otimes_N k = N_1 = \mathrm{Tor}_1^N(k,k)\,.$$

For $n=1$ we study $0 \to X_1 \to N_1 \otimes N^+ \to N^+ \to N_1 \to 0$. Since $N_1 \otimes k \to N_1$ is an isomorphism, we can substitute N for N^+, and get $0 \to X_1 \to N_1 \otimes N \to N \to k \to 0$, which is the exact sequence to the right in:

$$\ldots \to \operatorname{Tor}_3^N(k,k)\otimes N \to \operatorname{Tor}_2^N(k,k)\otimes N \longrightarrow N_1 \otimes N \to N \to k \to 0$$

$$\begin{array}{ccc} \operatorname{Tor}_2^N(k,k)\otimes N & & N_1\otimes N \\ & \searrow \; X_1 \; \nearrow & \\ & \nearrow \quad \searrow & \\ 0 & & 0 \end{array}$$

(cf. [7]). Applying $\cdot \otimes_N k$ on the exact sequence to the left, we get the exact sequence

$$\operatorname{Tor}_3^N(k,k) \to \operatorname{Tor}_2^N(k,k) \to X_1 \otimes_N k \to 0\,.$$

Since the sequence above is a minimal free resolution of the N-module k, we have $X_1 \otimes_N k = \operatorname{Tor}_2^N(k,k)$.

When $n \geqq 2$ we are forced, reluctantly, to restrict ourselves to the case where $\operatorname{gl\,dim} N \leqq 2$.

We know that $E_n N^+$ is a free N-module for $n \geqq \operatorname{gl\,dim} N$, so in the same way as above we have the exact sequences of right N-modules;

$$\ldots \to \operatorname{Tor}_3^N(k,k)\otimes E_n N^+ \to \operatorname{Tor}_2^N(k,k)\otimes$$

$$\otimes E_n N^+ \longrightarrow N_1 \otimes E_n N^+ \to E_n N^+ \to k \otimes_N E_n N^+ \to 0$$

$$\begin{array}{ccc} \otimes E_n N^+ & & N_1\otimes E_n N^+ \\ & \searrow \; X_n \; \nearrow & \\ & \nearrow \quad \searrow & \\ 0 & & 0 \end{array}$$

Applying $\cdot \otimes_N k$ on the sequence to the left, we get the sequence:

$$\operatorname{Tor}_3^N(k,k)\otimes (E_n N^+ \otimes_N k) \to \operatorname{Tor}_2^N(k,k)\otimes (E_n N^+ \otimes_N k) \to X_n \otimes_N k \to 0\,.$$

If $\operatorname{gl\,dim} N \leqq 2$, this of course gives us;

$$X_n \otimes_N k \;=\; \operatorname{Tor}_2^N(k,k)\otimes (E_n N^+ \otimes_N k) \qquad \text{for } n \geqq 2\,.$$

Observing that $(E_n N^+ \otimes_N k)(z) = (N(z))^{-1}(E_n N^+)(z)$, if $n \geqq \operatorname{gl\,dim} N$, we have proved;

LEMMA 2. *If N is a finitely presented Hopf algebra with generators in degree* 1, *we have*:

For $n=0$ and $n=1$, $X_n \otimes_N k = \operatorname{Tor}_{n+1}^N(k,k)$, and if $\operatorname{gl\,dim} N \leqq 2$, the double series for $X_* \otimes_N k$ is given by:

$$\sum_{n \geqq 0} x^n (X_n \otimes_N k)(z) \;=\; (N(z)^{-1} \operatorname{Tor}_2^N(k,k)(z)\cdot (P_{UN^+}(x,z) + x - 1) + |N_1| z\,.$$

4. A question of Lemaire.

Lemaire ([5], [6]) gave examples of finitely presented Hopf algebras Λ, where $\operatorname{Tor}_3^\Lambda(k,k)(z)$ were rational functions but not polynomials. He asked among other things:

—Is the series $\mathrm{Tor}_3^A(k,k)(z)$ always a rational function?

We will answer this question in the negative by taking an algebra constructed by Löfwall (a modification of an algebra of Shearer [13]) as N in the Löfwall–Roos construction. The resulting Ug is a finitely presented Hopf algebra, with generators in degree 1 and relations in degree 2, where all the series $\mathrm{Tor}_n^{Ug}(k,k)(z)$ $(n \geqq 3)$ are transcendental analytic functions defined for $|z|<1$. We first prove the following lemma:

LEMMA 4. *If the finitely generated algebra N in the Löfwall–Roos construction has a Hilbert series $N(z)$ with radius of convergence r, $0<r\leqq 1$, then all the series $\mathrm{Tor}_n^{Ug}(k,k)(z)$ for $n\geqq 3$ also have radius of convergence r, and moreover, for functions on $0\leqq z<r$, we have the inequality*:

$$\mathrm{Tor}_n^{Ug}(k,k)(z) \geqq (N(z)-3)\tfrac{1}{2}z^{n-2} \qquad (n\geqq 3)\,.$$

PROOF. When we analyse $\mathrm{Tor}_n^{Ug}(k,k)$ by means of the Hochschild–Serre spectral sequence as in Sections 2.1 and 2.2, we easily see that each E^2-term is majorated on $0\leqq z<r$ by

$$p(z)(E_nN^+)(z) \leqq p(z)(N(z))^n$$

for some polynomial $p(z)$. This shows that $\mathrm{Tor}_n^{Ug}(k,k)(z)$ converges in the open disc $|z|<r$.

Since all the coefficients of the different terms are non-negative, we have the following inequalities for functions defined on $0\leqq z<r$:

$$\begin{aligned}\mathrm{Tor}_n^{Ug}(k,k)(z) &= E^{\infty}_{0,n}(z)+E^2_{1,n-1}(z)+E^{\infty}_{2,n-2}(z) \geqq E^2_{1,n-1}(z)\\ &= \mathrm{Tor}_0^{UF}(k,\mathrm{Tor}_1^{UF'}(k,E_{n-1}N^+))(z)+\mathrm{Tor}_1^{UF}(k,\mathrm{Tor}_0^{UF'}(k,E_{n-1}N^+))(z)\\ &\geqq \mathrm{Tor}_0^{UF}(k,\mathrm{Tor}_1^{UF'}(k,E_{n-1}N^+))(z) = z(k\otimes_N E_{n-1}N^+)(z)\\ &\quad +(k\otimes_N X_{n-1})(z)\\ &\geqq z(k\otimes_N E_{n-1}N^+)(z)\,.\end{aligned}$$

Clearly

$$N(z)(k\otimes_N E_{n-1}N^+)(z) \geqq (E_{n-1}N^+)(z) \geqq (E_2N^+)(z)z^{n-3}$$

and since

$$(E_2N^+)(z) = \tfrac{1}{2}((N(z)-1)^2-N(-z^2)+1) \geqq \tfrac{1}{2}((N(z))^2-3N(z))$$

$(N(z)\geqq N(-z^2)$ for $0\leqq z<r)$, we get the desired inequality by dividing with $N(z)$ which is a strictly positive function on the interval. The proof is completed.

We can now use a graded algebra constructed by Löfwall, which is a modification of an algebra of Shearer ("Note added in proof" in [13]). The algebra is described in greater detail in the following Appendix by Löfwall.

Take the algebra

$$N = k\langle a,b,c,d,e\rangle/(ba-cd,\ ac-be,\ ad-da,\ ae-ea,\ b^2,\ c^2,$$
$$d^2,\ e^2,\ bd,\ bc,\ eb,\ ec,\ ce,\ cb)$$

with generators in degree 1 and relations in degree 2. Its Hilbert series

$$N(z) = R_1(z) \prod_{j \geqq 1} (1-z^{2j-1})^{-1} + R_2(z)$$

(where R_1 and R_2 are rational functions) has radius of convergence $r=1$, and, since

$$\lim_{z \to 1^-} (1-z)^m N(z) > 0 \qquad \text{for all } m\,,$$

it has an essential singularity at $z=1$.

The corresponding Löfwall–Roos Ug is a finitely presented Hopf algebra, also with generators in degree 1 and relations in degree 2 ([8], [9]). Lemma 4 immediately gives us the transcendency of all the series $\mathrm{Tor}_n^{Ug}(k,k)(z)$ $n \geqq 3$. We have proved:

COROLLARY 2. *There exist finitely presented Hopf algebras* Λ (i.e. $\mathrm{Tor}_1^{\Lambda}(k,k)(z)$ *and* $\mathrm{Tor}_2^{\Lambda}(k,k)(z)$ *are polynomials*), *where for each* $n \geqq 3$, $\mathrm{Tor}_n^{\Lambda}(k,k)(z)$ *is a transcendental analytic function.*

The algebra Ug constructed above is applied to the case of local rings and to the homology of loop spaces in Section 1.3.

5. Related problems.

The obvious problem is to compute the series of $X_n \otimes_N k$, when $\mathrm{gl\,dim}\, N \geqq 3$. The series of $X_2 \otimes_N k$ is, in view of the corollary in Section 1.2 and the applications in Section 1.3, especially interesting.

Since the Ug-s in the Löfwall–Ross construction always have $\mathrm{gl\,dim}\, Ug = \infty$, they cannot be used to answer

—If Λ is a finitely presented Hopf algebra with finite global dimension, is the series $\mathrm{Tor}_n^{\Lambda}(k,k)(z)$ always a rational function?

This would make the double Poincaré series $P_{\Lambda}(x,z)$ a rational function in two variables, whenever $\mathrm{gl\,dim}\, \Lambda < \infty$, and give a partially positive answer to (the already negatively answered) Problem 2 of Roos [11].

The algebras of Shearer and Löfwall has an essential singularity as the "smallest singularity" (at $z=1$), which is necessary for the use of Lemma 4.

These algebras are not Hopf algebras, however, and indeed Anick in [3] asked the following question:

—Is the smallest singularity of a finitely presented Hopf algebra with generators of degree 1 and relations of degree 2 always a pole of finite order?

REFERENCES

1. D. Anick, *A counterexample to a conjecture of Serre*, Thesis, M.I.T. 1980, and also Annals of Math. 115 (1982), 1–33.
2. D. Anick, *Construction d'espaces de lacets et d'anneaux locaux à séries de Poincaré–Betti non rationelles*, C. R. Acad. Sci. Paris Sér. A 290 (1980), 729–732.
3. D. Anick, *The smallest singularity of a Hilbert series*, Reports 1981–7, Department of Mathematics, University of Stockholm, Sweden, to appear in Math. Scand.
4. G. Hochschild and J.-P. Serre, *Cohomology of Lie algebras*, Ann. of Math. 57 (1953), 591–603.
5. J.-M. Lemaire, *A finite complex whose rational homotopy is not finitely generated* in *H-spaces*, ed. F. Sigrist, (Lecture Notes in Math. 196), pp. 114–120, Springer-Verlag, Berlin - Heidelberg - New York, 1971.
6. J.-M. Lemaire, *Algèbres connexes et homologie des espaces de lacets* (Lecture Notes in Math. 422), Springer-Verlag, Berlin - Heidelberg - New York, 1974.
7. C. Löfwall, *Une algèbre nilpotente dont la série de Poincaré–Betti est non rationelle*, C. R. Acad. Sci. Paris Sér. A 288 (1979), 327–330.
8. C. Löfwall and J.-E. Roos, *Cohomologie des algèbres de Lie graduées et séries de Poincaré–Betti non rationelles*, C. R. Acad. Sci. Paris Sér. A 290 (1980), 733–736.
9. C. Löfwall and J.-E. Roos, Detailed version of [8], to appear.
10. J. Milnor and J. C. Moore, *On the structure of Hopf algebras*, Ann. of Math. 81 (1965), 211–264.
11. J.-E. Roos, *Relations between the Poincaré–Betti series of loop spaces and of local rings* in *Séminaire d'Algèbre Paul Dubreil*, Proceedings, 1977–78, ed. M.-P. Malliavin (Lecture Notes in Math. 740), pp. 285–322, Springer-Verlag, Berlin - Heidelberg - New York, 1979.
12. J.-E. Roos, *Homology of loop spaces and of local rings*, Reports 1980-15, Department of Mathematics, University of Stockholm, Sweden, and also Proceedings of the 18th Scand. Congr. of Math. 1980, Århus, Denmark. Birkhäuser-Verlag, in the series Progress in Mathematics.
13. J. B. Shearer, *A graded algebra with non-rational Hilbert series*, J. Algebra 62 (1980), 228–231.

MATEMATISKA INSTITUTIONEN
THUNBERGSVÄGEN 3
S-752 38 UPPSALA
SWEDEN