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APPENDIX TO "ON THE DOUBLE POINCARÉ SERIES OF THE ENVELOPING ALGEBRAS OF CERTAIN GRADED LIE ALGEBRAS"

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Shearer gave in [4] the following example:

$$N = k\langle a,b,c\rangle/(aba-bc, ac-ca, b^2)$$

where k is a field and $k\langle a,b,c\rangle$ means the free non-commutative algebra on the variables a,b,c, and where $\deg(a)=\deg(b)=1$ and $\deg(c)=2$. The Hilbert series of N is transcendental and has radius of convergence 1. By adding one new variable, Shearer modified the example to get an algebra generated by elements of degree one, with relations of degree two and three. Here we give another modification in order to get the relations homogeneous of *degree two*, and still get a transcendental series with radius of convergence 1. Following Anick [1] and Löfwall–Roos [2], [3], we obtain a local ring R with $m^3=0$ and it is proved in this paper in section 4 (see also end of section 1.3) that R has the property that the graded vector space of indecomposable elements of $\mathrm{Ext}_R(k,k)$ has transcendental Hilbert series and moreover a minimal set of generators of the ideal of relations between a minimal set of generators of $\mathrm{Ext}_R(k,k)$ has transcendental Hilbert series. The modified example is the following

$$N = k\langle a,b,c,d,e\rangle/(ba-cd, ac-be, ad-da, ae-ea, \\ b^2, c^2, d^2, e^2, bc, bd, cb, ce, eb, ec).$$

A basis for N_2 (the elements of degree two in N) is

$$aa, ab, ac, ad, ae, ba, ca, db, dc, de, ed.$$

The corresponding local ring is a quotient of a polynomial ring in 27 variables with 168 quadratic relations and all monomials of degree three:

$$R = k[t_a, t_b, \ldots, t_e, t'_a, \ldots, t'_e, v_a, \ldots, v_e, u, s_{aa}, s_{ab}, \ldots, s_{ed}]$$

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(one s_x for each basis element of N_2) *modulo* $t_x t_y, t_x s_y, s_x s_y, t'_x t'_y, t'_x u, uu$ (all possible choices of x and y) and all monomials ofm degree three and the following 11 relations

$$t_a v_a + v_a t'_a + s_{aa} u = 0 \qquad t_a v_b + v_a t'_b + s_{ab} u = 0$$

$$t_c v_a + v_c t'_a + s_{ca} u = 0 \qquad t_d v_b + v_d t'_b + s_{db} u = 0$$

$$t_d v_c + v_d t'_c + s_{dc} u = 0 \qquad t_d v_e + v_d t'_e + s_{de} u = 0$$

$$t_e v_d + v_e t'_d + s_{ed} u = 0$$

$$t_a v_c + v_a t'_c + t_b v'_e + v_b t'_e + s_{ac} u = 0$$

$$t_a v_d + v_a t'_d + t_d v_a + v_d t'_a + s_{ad} u = 0$$

$$t_a v_e + v_a t'_e + t_e v_a + v_e t'_a + s_{ae} u = 0$$

$$t_b v_a + v_b t'_a + t_c v_d + v_c t'_d + s_{ba} u = 0$$

LEMMA 1. *N is generated as a vector space by*

$$|d|(ed)^{n_1}|c|a^{n_2}(ba^{n_3})(ba^{n_4})\ldots(ba^{n_t})|e| \qquad n_i \geqq 0,\ t \geqq 0 .$$

If $t \leqq 2$ *no b occurs,* $|\cdot|$ *means that the element inside the bracket may be counted or not.*

PROOF. It is proved by induction on s that N_s is generated by the given words. It is obviously true for $s=0,1$. Suppose it is true for $s-1$. Then N_s is generated by $\{xa, xb, xc, xd, xe;\ x$ a generator of right type of degree $s-1\}$. It is now easily proved by means of the relations in N that this is of the right type.

LEMMA 2. *A basis for N as a vector space is given by*

(i) $|d|(ed)^{n_1}a^{n_2}(ba^{i_1})(ba^{i_2})\ldots(ba^{i_t})e^{\varepsilon} \quad n_1, n_2 \geqq 0,\ \varepsilon = 0, 1$ *and*
$i_1 > i_2 > \ldots > i_{t-2} > i_{t-1} + \varepsilon > i_t + \varepsilon \quad t \geqq 0,\ i_t \geqq 0$
(*if* $t=0$ *no group* ba^{i_k} *occurs*)

and

(ii) $|d|(ed)^{n_1}ca^{i_1}ba^{i_2}\ldots ba^{i_t}e^{\varepsilon} \quad n_1 \geqq 0,\ t \geqq 2,\ \varepsilon = 0, 1$ *and*
$i_1 > i_2 > \ldots > i_{t-2} > i_{t-1} + \varepsilon > i_t + \varepsilon$

and

(iii) $|d|(ed)^{n_1}ca^{n_2} \quad n_1 \geqq 0,\ n_2 \geqq 0$ (*c does occur*).

PROOF. Firstly, if the vectors are linearly dependent then some of them is zero or two of them are equal. This follows easily from the fact that the relations are monomials and binomials. We consider the given vectors in $k\langle a,b,c,d,e\rangle$. For each vector we give a list of all vectors which are equivalent to the given one modulo the relations defining N. The proof is finished by observing that the lists are disjoint and that no list contains zero.

(i) $|d|(ed)^{n_1}a^{n_2-k_1}\overset{cd}{(ba)}a^{j_1}(ed)^{k_1}\overset{cd}{(ba)}a^{j_2}(ed)^{k_2}\dots\overset{cd}{(ba)}a^{j_t}(ed)^{k_t}e^{\varepsilon}$

or (when $\varepsilon=1$)

same beginning ... $\overset{cd}{(ba)}a^{j_{t-1}}(ed)^{k_{t-1}}acaa^{i_t}$

$\overset{cd}{(ba)}$ means that ba may be replaced by cd. The exponents satisfy the following relations

$j_s \geqq -1$ (-1 only if there is an a which kills a^{-1})
$j_{t-1} \geqq -2$ in the second case
$k_1 \leqq n_2$, $k_s \geqq 0$, $1 \leqq s \leqq t$
$j_s + k_s > j_{s+1} + k_{s+2}$, $s \neq t-2$
$j_{t-2} + k_{t-2} > j_{t-1} + k_t + \varepsilon$
$i_s = 1 + j_s + k_s + k_{s+1}$, $1 \leqq s \leqq t$
(i_s, j_s, k_s are defined to be zero if $s > t$).

This is not quite a complete list since the relations $ae=ea$ and $ad=da$ may be used. However zero is not in any of the lists if one can prove that $j_s+k_s\geqq 0$ for $s\leqq t-1$. This follows from above since $j_{s+1}+k_{s+2}\geqq -1$ if $\varepsilon=0$ and $j_{t-1}+k_t+1\geqq -1$. Also the lists are disjoint, since the sequence $n_1, n_2, i_1, \dots, i_t, \varepsilon$ may be found from the list. To see that the list is complete (modulo commutation of a by d or e) one must see that an application of the relation $be=ac$ gives a new element of the list. But the effect of changing ac to be is that for some s, j_{s-1} and j_s are lowered by one while k_s is raised by one (and the opposite if be is changed to ac). It is easily seen that all the relations between the exponents are preserved under such a change. The second class of vectors (ii) are treated in the same way. The only difference is that $n_2=0$ and the first b is replaced by c. It is also clear that every word in any list contains c without a to the left and this is not the case in (i). Hence the lists are disjoint. In (iii) the lists consist of just one element. They are disjoint from the lists in (i) as before and from the lists in (ii) since in that case the vectors contain at least one b or two c. Lemma 2 is proved.

PROPOSITION. *Let* $P(z)=\prod_{i=1}^{\infty}(1+z^i)$, *then the Hilbert series of* N *is*

$$N(z) = (2-z)(1+z+z^2)z^{-1}(1+z)^{-1}(1-z)^{-2}P(z)-2z^{-1}(1-z)^{-2}$$

and N *has radius of convergence* 1 *and is a transcendental function.*

REMARK. $P(z)=\sum p'(n)z^n$, where

$$p'(n) = \operatorname{card}\{j_1>j_2>\ldots>j_t\geqq 1;\ \sum j_i=n\},$$

also $P(z)=\prod_{i=1}^{\infty}(1-z^{2i-1})^{-1}$.

PROOF. It follows easily from Lemma 2 that

$$\begin{aligned} N(z) = {} & (1+z)[(1-z^2)^{-1}(1-z)^{-1}P(z)+ \\ & +(1-z^2)^{-1}(1-z)^{-1}z^{-1}(P(z)(1+z)^{-1}-1)]+ \\ & +(1+z)[(1-z^2)^{-1}(P(z)-(1-z)^{-1})+(1-z^2)^{-1}(z^{-1}(P(z)(1+z)^{-1}-1)- \\ & -z(1-z)^{-1})]+(1+z)z(1-z^2)^{-1}(1-z)^{-1}, \end{aligned}$$

and this gives the result after some calculations. The radius of convergence is one, since $P(z)$ has radius of convergence one and the singularity at $z=0$ is removable. The transcendency of N follows from the fact that

$$\lim_{z\to 1-}(1-z)^m N(z) > 0 \quad \text{for all } m$$

(which is a consequence of Remark above) and this cannot hold for an algebraic function.

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