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ON INFINITE NIELSEN TRANSFORMATIONS

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Introduction.

Jacob Nielsen's procedure for reducing a finite subset of a free group [8] led to the concept of Nielsen transformation ([6, p. 130]), which has proved useful especially in the case of finitely-generated groups. We shall define here an infinite Nielsen transformation and show that any infinite chain of words can be carried to a Nielsen-reduced chain by an infinite Nielsen transformation. This is equivalent to saying that for any endomorphism α on a free group F of countably infinite rank there exists a free base $\langle a_i \rangle$ such that the set of non-unit images $\langle \alpha a_i \rangle$ is Nielsen-reduced. From this follows immediately the Nielsen–Schreier Subgroup Theorem for F. As another example of the application of our result let us consider the following Proposition ([5, 2.12]): Let f be a homomorphism from F_n onto a free group G; then $F_n = S * Z$, such that f maps S isomorphically onto G and maps Z into the identity. In [4], the proof of the analogous result, with F in place of F_n, consists of a transfinite convergence process making use of Nielsen's procedure for a finite subset. Using our theorem the proof may be simplified, since it may be extended from F_n to F without any essential change. The desirability of transforming infinite subsets has led to other generalisations of Nielsen transformations ([2], [9]); when these are applied to a free system of generators in F, they are examples of our infinite Nielsen transformations. The extended Nielsen operations given in [1] relate only to finite subsets of generators for normal subgroups.

Our definition of the infinite Nielsen transformation is based on an analogy of the result ([6, p. 130]) that the group $\operatorname{Aut} F_n$ is antiisomorphic to the group of Nielsen transformations of rank n. Hence it is natural to consider transformations associated in the same way with automorphisms of F. Our main result shows that for any chain of words $\boldsymbol{u}$ there exists an infinite Nielsen transformation N such that the subset of non-unit words in $N\boldsymbol{u}$ is Nielsen-reduced. By analogy with a similar result for Nielsen transformations of finite rank, it can also be shown, using [7], that an infinite Nielsen transformation is a (possibly infinite) product of elementary Nielsen transformations.

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1. Basic definitions.

Let F be a free group of countable rank on free generators $\{x_i, i \in I\}$, where I is the set of natural numbers. Any infinite sequence of freely reduced words in F is called a *chain*, which we shall denote, for example, by $\boldsymbol{w} = (w_1, w_2, \ldots)$. We call the chain of generators *basic* and denote it by $\boldsymbol{x} = (x_1, x_2, \ldots)$. For each chain $\boldsymbol{n} = (n_1, n_2, \ldots)$ we define a corresponding *transformation* N, on the set of all chains, carrying a chain $\boldsymbol{w} = (w_1, w_2, \ldots)$ into $N\boldsymbol{w} = ((N\boldsymbol{w})_1, (N\boldsymbol{w})_2, \ldots)$, where $(N\boldsymbol{w})_m$ is a reduced word obtained from n_m by substituting w_i for x_i, $i \in I$. For a *sequence of chains* $\langle \boldsymbol{w}^\alpha \rangle$, $\boldsymbol{w}^\alpha = (w_1^\alpha, w_2^\alpha, \ldots)$, the corresponding sequence of transformations will be denoted by $\langle W_\alpha \rangle$, where $W_\alpha \boldsymbol{x} = \boldsymbol{w}^\alpha$, namely $(W_\alpha \boldsymbol{x})_i = w_i^\alpha$ $(i \in I)$. If, for a transformation N with $N\boldsymbol{x} = ((N\boldsymbol{x})_1, (N\boldsymbol{x})_2, \ldots)$, the set $\{(N\boldsymbol{x})_i, i \in I\}$ *generates* F *freely*, then N will be called an *Infinite Nielsen Transformation.* For transformations N_1, N_2, define their product by $(N_2 N_1)\boldsymbol{w} = N_2(N_1 \boldsymbol{w})$. It is easy to see that the product thus defined is associative. Transformation N is called *invertible* if there exists a transformation M such that $MN = NM = E$, where E is the identity transformation. A transformation N is invertible if and only if it is an infinite Nielsen transformation; this follows from the one-to-one correspondence of the group of free substitutions and the group of automorphisms of F, in a similar way to [6, p. 130]. For convenience, similar definitions and results for chains of words and for transformations are given separately.

2. Convergence.

We shall introduce here a concept of convergence for a sequence of chains (or transformations) analogous to coordinate convergence ([3, 10.2]), that is, in a convergent sequence of chains $\boldsymbol{w}^1, \boldsymbol{w}^2, \boldsymbol{w}^3, \ldots$ the mth entries coincide for $\boldsymbol{w}^k$, $k \geqq k_m$. If we have a convergent sequence of invertible transformations, the limit transformation is not necessarily invertible. Indeed, consider transformations $N_1, N_2, N_3, \ldots$ where

$$N_1 \boldsymbol{x} = (x_1 x_2, x_2, x_3, x_4, \ldots)$$

$$N_2 \boldsymbol{x} = (x_1 x_2, x_2 x_3, x_3, x_4, \ldots)$$

$$N_3 \boldsymbol{x} = (x_1 x_2, x_2 x_3, x_3 x_4, x_4, \ldots) .$$

$$\ldots\ldots$$

The limit transformation of the given sequence is N where $N\boldsymbol{x} = (x_1 x_2, x_2 x_3, \ldots, x_k x_{k+1}, \ldots)$ whose entries generate a subgroup in F with elements of even x-length, hence this subgroup is proper in F and N is not a free transformation and hence is not invertible. In Lemma 2 we give a criterion for the limit transformation of a sequence of invertible transformations to be also invertible. For a sequence of transformations we shall also define a sequence of partial products (and a product as its limit), the condition for whose existence is given in Lemma 3. The main theorem of this section provides sufficient conditions for a sequence of Nielsen transformations to have a product which is also a Nielsen transformation.

DEFINITION 1. (i) Let $\langle \boldsymbol{w}^{\alpha}, \alpha<\alpha' \rangle$ be a (transfinite) sequence of chains for a fixed countable limit ordinal α'. We say that $\lim_{\alpha\to\alpha'} \boldsymbol{w}^{\alpha}$ exists if $\forall m \in I$, $\exists \alpha_m<\alpha'$ such that for all α satisfying $\alpha_m \leqq \alpha < \alpha'$, we have $w_m^{\alpha_m} = w_m^{\alpha}$. Then we write $\lim_{\alpha\to\alpha'} \boldsymbol{w}^{\alpha} = \boldsymbol{w}$ for the chain $\boldsymbol{w}$ with $w_m = w_m^{\alpha_m}$ ($\forall m \in I$).

(ii) Let $\langle P_{\alpha}, \alpha<\alpha' \rangle$ be a sequence of transformations. We say that $\lim_{\alpha\to\alpha'} P_{\alpha} = P$ if $\lim_{\alpha\to\alpha'} P_{\alpha}\boldsymbol{x} = P\boldsymbol{x}$; that is, $\forall m \in I$, $\exists \alpha_m<\alpha'$ such that for all α satisfying $\alpha_m \leqq \alpha<\alpha'$, we have $(P_{\alpha_m}\boldsymbol{x})_m = (P_{\alpha}\boldsymbol{x})_m = (P\boldsymbol{x})_m$.

(iii) In (i) and (ii), clearly we can chooze the α_m increasing, and then the limit criteria can be written, respectively, as

$$w_i^{\alpha_m} = w_i^{\alpha}, \quad (P_{\alpha_m}\boldsymbol{x})_i = (P_{\alpha}\boldsymbol{x})_i = (P\boldsymbol{x})_i, \quad \text{for } i \leqq m,\ \alpha_m \leqq \alpha < \alpha' .$$

Let $\beta_1<\beta_2<\beta_3<\ldots$ be a sequence of ordinals such that $\lim_i \beta_i = \alpha'$; then for a sequence of chains $\langle \boldsymbol{w}^{\alpha}, \alpha<\alpha' \rangle$, we shall speak of its β-subsequence $\langle \boldsymbol{w}^{\beta_i}, i = 1,2,\ldots \rangle$.

LEMMA 1. *Let* $\langle \boldsymbol{w}^{\alpha}, \alpha<\alpha' \rangle$ *be a sequence of chains.* (i) *If* $\lim_{\alpha\to\alpha'} \boldsymbol{w}^{\alpha} = \boldsymbol{w}$ *then every* β*-subsequence* $\boldsymbol{w}^{\beta_i}$, $i = 1,2,\ldots$ *has limit* $\boldsymbol{w}$. (ii) *If every* β*-subsequence has a limit, then their limits are the same* ($\boldsymbol{w}$, say), *and* $\lim_{\alpha\to\alpha'} \boldsymbol{w}^{\alpha} = \boldsymbol{w}$.

PROOF. If $\lim_{\alpha\to\alpha'} \boldsymbol{w}^{\alpha} = \boldsymbol{w}$ then, by Definition 1, $\lim_i \boldsymbol{w}^{\beta_i} = \boldsymbol{w}$ for every β-subsequence.

Conversely, let every β-subsequence have a limit and for some fixed β-subsequence let $\lim_i \boldsymbol{w}^{\beta_i} = \boldsymbol{w}^0$; hence $\forall m \in I$, $\exists \beta(m)$ such that $w_m^{\beta_i} = w_m^0$ whenever $\beta_i > \beta(m)$. Suppose now that $\lim_{\alpha\to\alpha'} \boldsymbol{w}^{\alpha} \neq \boldsymbol{w}^0$; then for some $\bar{m}$ and for each β_i there exists $\alpha_i > \beta_i$ such that $w_{\bar{m}}^{\alpha_i} \neq w_{\bar{m}}^0$. Now alternating β_i and α_i gives us a sequence γ_i, $i = 1,2,\ldots$, for which $\lim_i \gamma_i = \alpha'$ but $\langle \boldsymbol{w}^{\gamma_i}, i = 1,2,\ldots \rangle$ has no limit, which is a contradiction.

The same lemma can be formulated for transformations as:

LEMMA 1′. *Let* $\langle P_{\alpha}, \alpha<\alpha' \rangle$ *be a sequence of transformations.* (i) *If* $\lim_{\alpha\to\alpha'} P_{\alpha} = P$ *then every* β*-subsequence* P_{β_i}, $i = 1,2,\ldots$ *has limit* P.

(ii) *If every* β*-subsequence has a limit, then their limits are the same* (P, *say*), *and* $\lim_{\alpha\to\alpha'} P_{\alpha} = P$.

LEMMA 2. *Let* $\langle P_\alpha, \alpha<\alpha'\rangle$ *be a sequence of invertible transformations, and* $\lim_{\alpha\to\alpha'} P_\alpha = P$. *Then there exists* $\lim_{\alpha\to\alpha'} P_\alpha^{-1} = R$ *if and only if* $PR = RP = E$.

PROOF. By Lemma 1′ we can assume that $\alpha'=\omega$. Now let $\lim_k P_k = P$, $\lim_l P_l^{-1} = R$; then $\forall m \in I$, $\exists\, k_m, l_m$ such that, for $k \geqq k_m$, $l \geqq l_m$, we have

$$(P_{k_m}x)_m = (P_k x)_m = (Px)_m \quad \text{and} \quad (P_{l_m}{}^{-1}x)_m = (P_l^{-1}x)_m = (Rx)_m .$$

Choose subsequence T_m, $m \in I$, where $T_m = P_{\max(k_m, l_m)}$; then

$$(T_m x)_i = (Px)_i, \quad (T_m^{-1}x)_i = (Rx)_i \quad (i \leqq m) .$$

Suppose that m first words in Px (and hence in $T_m x$) are expressed through x_i, $i \leqq s$, for some $s \in I$. We can always take $s \geqq m$. Then

$$T_s^{-1}x = \big((Rx)_1, \ldots, (Rx)_m, \ldots, (Rx)_s, y_{s+1}, \ldots\big) ,$$

$$T_m(T_s^{-1}x) = \big((P(Rx))_1, \ldots, (P(Rx))_m, z_{m+1}, \ldots\big) .$$

Since $s \geqq m$, the first m elements in $T_m x$ and $T_s x$ coincide, hence the same is true for $T_m w$ and $T_s w$ for any w. Take $w = T_s^{-1}x$; then

$$T_m w = T_m(T_s^{-1}x) \quad \text{and} \quad T_s w = T_s(T_s^{-1}x) = x;$$

hence $(P(Rx))_i = x_i$ for $i \leqq m$. Since this is true for arbitrary $m \in I$ we get $PR = E$, and in a similar way we get $RP = E$, which gives the required result in one direction.

Conversely, suppose that $PR = RP = E$; we want to show that $R = \lim_k P_k^{-1}$. Let

$$(Rx)_m = w(x_1, x_2, \ldots, x_s)$$

be a word in x_i, $i \leqq s$. By convergence of the sequence P_k, $k = 1, 2, 3, \ldots$, to P, there exists k_s such that, for $k \geqq k_s$, we have $(P_k x)_i = (Px)_i$ for $i \leqq s$. Then each transformation $T_k = RP_k$ $(k \geqq k_s)$ carries x_m into

$$(T_k x)_m = \big((RP_k)x\big)_m = w\big((P_k x)_1, \ldots, (P_k x)_s\big) = w\big((Px)_1, \ldots, (Px)_s\big)$$
$$= \big(R(Px)\big)_m = x_m .$$

Now

$$(Rx)_m = \big(T_k(P_k^{-1}x)\big)_m = (P_k^{-1}x)_m \quad \text{for } k \geqq k_s ,$$

so that $R = \lim_k P_k^{-1}$, and the proof is complete.

COROLLARY. *Let* $\langle N_\alpha, \alpha<\alpha'\rangle$ *be a sequence of invertible transformations. Then* $\lim_{\alpha\to\alpha'} N_\alpha = E$ *if and only if* $\lim_{\alpha\to\alpha'} N_\alpha^{-1} = E$.

Now let $\bar{\alpha}$ be any fixed countable ordinal, $\bar{\alpha} \geqq 1$, and denote by Φ the set of ordinals $\langle \alpha, 1 \leqq \alpha \leqq \bar{\alpha} \rangle$. We also write $\Phi = \Phi_0 \cup \Phi_1$, where Φ_0 is the subset of nonlimit ordinals and Φ_1 is the subset of limit ordinals in Φ.

DEFINITION 2. The sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \rangle$ (or the sequence of transformations $\langle P_\alpha, \alpha \in \Phi \rangle$) is called *complete* if $\forall \alpha' \in \Phi_1$ we have $\lim_{\alpha \to \alpha'} \boldsymbol{w}^\alpha = \boldsymbol{w}^{\alpha'}$ ($\lim_{\alpha \to \alpha'} P_\alpha = P_{\alpha'}$), namely $\forall m$, $\exists \alpha_m$, $\alpha_m < \alpha'$, such that $w_m^\alpha = w_m^{\alpha'}$ ($(P_\alpha \boldsymbol{x})_m = (P_{\alpha'} \boldsymbol{x})_m$) whenever $\alpha_m \leqq \alpha < \alpha'$.

DEFINITION 3. A sequence of transformations $\langle P_\alpha, \alpha \in \Phi \rangle$ is called the *sequence of partial products* for the sequence of transformations $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ if $P_1 = N_1$, $P_\alpha = N_\alpha P_{\alpha-1}$ for $\alpha \in \Phi_0 \setminus 1$, $P_{\alpha'} = \lim_{\alpha \to \alpha'} P_\alpha$ for $\alpha' \in \Phi_1$. Obviously every sequence of partial products is complete.

LEMMA 3. *If a sequence $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ has the property $\lim_{\alpha \to \alpha'} N_\alpha = E$ for every $\alpha' \in \Phi_1$, then the sequence of partial products for $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ exists.*

PROOF. Let $P_1 = N_1$ and define $P_\alpha = N_\alpha P_{\alpha-1}$ for $\alpha \in \Phi_0 \setminus 1$. To define $P_{\alpha'}$ for $\alpha' \in \Phi_1$ notice that the condition $\lim_{\alpha \to \alpha'} N_\alpha = E$ implies, for any $m \in I$, the existence of α_m ($\alpha_m < \alpha'$) such that for all α ($\alpha_m \leqq \alpha < \alpha'$), $(N_\alpha \boldsymbol{x})_m = x_m$. Thus define $P_{\alpha'}$ by $(P_{\alpha'} \boldsymbol{x})_m = (P_{\alpha_m} \boldsymbol{x})_m$, $m \in I$. To show that $P_{\alpha'} = \lim_{\alpha \to \alpha'} P_\alpha$ it is enough to verify that for all α ($\alpha_m \leqq \alpha < \alpha'$), $(P_\alpha \boldsymbol{x})_m = (P_{\alpha'} \boldsymbol{x})_m$. Use induction on α. For $\alpha = \alpha_m$ our statement is trivial. For some $\alpha_0 < \alpha'$, let $(P_\alpha \boldsymbol{x})_m = (P_{\alpha'} \boldsymbol{x})_m$ for all α satisfying $\alpha_m \leqq \alpha < \alpha_0$; then we have to show that $(P_{\alpha_0} \boldsymbol{x})_m = (P_{\alpha'} \boldsymbol{x})_m$: if $\alpha_0 \in \Phi_0$ we have

$$(P_{\alpha_0} \boldsymbol{x})_m = (N_{\alpha_0} P_{\alpha_0 - 1} \boldsymbol{x})_m = (P_{\alpha_0 - 1} \boldsymbol{x})_m = (P_{\alpha'} \boldsymbol{x})_m ;$$

while for $\alpha_0 \in \Phi_1$, P_{α_0} was defined by the inductive hypothesis in such a way that $(P_{\alpha_0} \boldsymbol{x})_m = (P_{\alpha_m} \boldsymbol{x})_m$ and hence $(P_{\alpha_0} \boldsymbol{x})_m = (P_{\alpha'} \boldsymbol{x})_m$. It follows that $P_{\alpha'} = \lim_{\alpha \to \alpha'} P_\alpha$ and hence there exists a complete sequence of partial products $\langle P_\alpha, \alpha \in \Phi \rangle$ for $\langle N_\alpha, \alpha \in \Phi_0 \rangle$.

DEFINITION 4. Given a fixed sequence $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ of invertible transformations, let $\hat{\alpha}(m)$ denote the set of indexes of the generators x_i ($i \in I$) appearing in the reduced form of the word $(N_\alpha^{-1} \boldsymbol{x})_m$; thus

$$(N_\alpha^{-1} \boldsymbol{x})_m \in \operatorname{gp} \{x_i, i \in \hat{\alpha}(m)\} .$$

For any $S \subseteq I$ denote $\hat{\alpha}(S) = \bigcup_i \hat{\alpha}(i)$, $i \in S$. Assign to $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ *characteristic sets* $T(m)$, $m \in I$, defined inductively:

$$T_1(m) = \hat{1}(m), \quad T_\alpha(m) = \hat{\alpha}(T_{\alpha-1}(m)) \quad \text{for } \alpha \in \Phi_0 ,$$

$$T_{\alpha'}(m) = \bigcup_{\alpha < \alpha'} T_\alpha(m) \quad \text{for } \alpha' \in \Phi_1 ; \quad T(m) = T_{\bar{\alpha}}(m) .$$

DEFINITION 5. The sequence of invertible transformations $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ is called *regular* if its characteristic sets $T(m)$, $m \in I$, are finite.

THEOREM 1. *Let* $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ *be a regular sequence and* $\lim_{\alpha \to \alpha'} N_\alpha = E$ *for every* $\alpha' \in \Phi_1$. *Then the sequence of partial products for* $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ *consists of infinite Nielsen transformations.*

PROOF. The existence of the sequence $\langle P_\alpha, \alpha \in \Phi \rangle$ of partial products follows from Lemma 3. Thus we have to show that every P_α, $\alpha \in \Phi$, is invertible, which we accomplish by induction.

For $\alpha = 1$, $P_1 = N_1$ is invertible.

For some $\alpha_0 \in \Phi$,

(1) $$\text{suppose all } P_\alpha \ (\alpha < \alpha_0) \text{ invertible;}$$

we then have to show P_{α_0} to be invertible. When $\alpha_0 \in \Phi_0$ this is obvious because $P_{\alpha_0}^{-1} = P_{\alpha_0 - 1}^{-1} N_{\alpha_0}^{-1}$. Now consider the case $\alpha_0 \in \Phi_1$: we first show that for $\alpha < \alpha_0$,

(2) $$(P_\alpha^{-1} \boldsymbol{x})_m \in \mathrm{gp}\{x_i, i \in T_\alpha(m)\},$$

Use induction on α. For $\alpha = 1$,

$$(P_1^{-1} \boldsymbol{x})_m = (N_1^{-1} \boldsymbol{x})_m \in \mathrm{gp}\{x_i, i \in T_1(m)\}.$$

For some $\beta < \alpha_0$, suppose (2) holds for all α, $\alpha < \beta$; then we need to show that

$$(P_\beta^{-1} \boldsymbol{x})_m \in \mathrm{gp}\{x_i, i \in T_\beta(m)\}.$$

For $\beta \in \Phi_0$ denote $N_\beta^{-1} \boldsymbol{x} = \boldsymbol{w}$, so that

$$(N_\beta^{-1} \boldsymbol{x})_j = w_j \in \mathrm{gp}\{x_i, i \in \hat{\beta}(j)\}$$

(see Definition 4). Then

$$\begin{aligned}(P_\beta^{-1} \boldsymbol{x})_m &= ((P_{\beta-1}^{-1} N_\beta^{-1}) \boldsymbol{x})_m = (P_{\beta-1}^{-1} \boldsymbol{w})_m \in \mathrm{gp}\{w_j, j \in T_{\beta-1}(m)\} \\ &\subseteqq \mathrm{gp}\{x_i, i \in \hat{\beta}(j), j \in T_{\beta-1}(m)\} = \mathrm{gp}\{x_i, i \in T_\beta(m)\}.\end{aligned}$$

For $\beta \in \Phi_1$, it follows by (1) that $P_\beta^{-1} = \lim_{\alpha \to \beta} P_\alpha^{-1}$, namely, $\forall m$, $\exists \alpha_m$ $(\alpha_m < \beta)$ such that

$$(P_\beta^{-1} \boldsymbol{x})_m = (P_{\alpha_m}^{-1} \boldsymbol{x})_m \in \mathrm{gp}\{x_i, i \in T_{\alpha_m}(m)\} \subseteqq \mathrm{gp}\{x_i, i \in T_\beta(m)\},$$

where $T_\beta(m) = \bigcup_{\alpha < \beta} T_\alpha(m)$. This proves statement (2).

Notice also that, by the Corollary to Lemma 2, and because of the finiteness of $T(m)$: $\forall m$, $\exists \gamma(m)$ such that for all α $(\gamma(m) \leqq \alpha < \alpha_0)$ and for all $i \in T(m)$, we have $(N_\alpha^{-1} \boldsymbol{x})_i = x_i$. Thus if we denote $N_\beta^{-1} \boldsymbol{x} = \boldsymbol{w}^\beta$, then

(3) $$w_i^\beta = (N_\beta^{-1}x)_i = x_i$$

for all β satisfying $\gamma(m) \leqq \beta < \alpha_0$, and for all $i \in T(m)$.

Our final requirement, that

(4) $$P_{\alpha_0}^{-1} \text{ exists for } \alpha_0 \in \Phi_1$$

is, by Lemma 2, equivalent to the existence of $\lim_{\alpha\to\alpha_0} P_\alpha^{-1}$. Accordingly, we define the transformation R by $(Rx)_m = (P_{\gamma(m)}^{-1}x)_m$ (where $\gamma(m)$ is given above, $m \in I$). Then in order to prove that $R = \lim_{\alpha\to\alpha_0} P_\alpha^{-1}$, it is enough to show that

(5) $$(P_\alpha^{-1}x)_m = (Rx)_m$$

for all α satisfying $\gamma(m) \leqq \alpha < \alpha_0$, which we prove by induction on α. For $\alpha = \gamma(m)$, (5) is trivial. Now suppose that, for some β in $\gamma(m) < \beta < \alpha_0$, (5) holds for $\gamma(m) < \alpha < \beta$; thus we need to show that

(6) $$(P_\beta^{-1}x)_m = (Rx)_m .$$

For $\beta \in \Phi_1$ we have $\lim_{\alpha\to\beta} P_\alpha^{-1} = P_\beta^{-1}$, by (1) and Lemma 2; and hence $(P_\beta^{-1}x)_m$ coincides with $(P_{\beta_m}^{-1}x)_m$ for some nonlimit $\beta_m < \beta$. Thus it remains to prove (6) for $\beta \in \Phi_0$. Since $\beta > 1$,

$$(P_\beta^{-1}x)_m = ((P_{\beta-1}^{-1}N_\beta^{-1})x)_m = (P_{\beta-1}^{-1}(N_\beta^{-1}x))_m = (P_{\beta-1}^{-1}w^\beta)_m ,$$

it follows that the word $(P_\beta^{-1}x)_m$ is obtainable by replacing x_i by w_i^β in the reduced form of $(P_{\beta-1}^{-1}x)_m$. But, by (2),

$$(P_{\beta-1}^{-1}x)_m \in \text{gp}\{x_i,\ i \in T_{\beta-1}(m)\};$$

and, by (3), $w_i^\beta = x_i$ for $i \in T_{\beta-1}(m) \subseteqq T(m)$. Thus it follows that

$$(P_\beta^{-1}x)_m = (P_{\beta-1}^{-1}x)_m = (Rx)_m ,$$

which proves (6).

We have therefore established that (1) implies P_{α_0} invertible, which completes the transfinite induction, and the theorem is proved.

3. Existence of a minimal chain.

Our ultimate aim is to prove the existence of an infinite Nielsen transformation which changes a given chain $\boldsymbol{u}$ into a Nielsen-reduced chain. In this section we shall consider the set $\mathfrak{N}$ of all chains $\boldsymbol{w} = N\boldsymbol{u}$ for all infinite Nielsen transformations N, and introduce a partial order in $\mathfrak{N}$. Using Kuratowski-Zorn's Lemma, we shall then prove the existence of a "minimal chain" with a particular property (Lemma 9). Our partial order $\boldsymbol{w}^{(1)} \succ \boldsymbol{w}^{(2)}$ is based on the existence of a complete sequence $\langle \boldsymbol{w}^\alpha \rangle$ with first element $\boldsymbol{w}^{(1)}$ and last element $\boldsymbol{w}^{(2)}$ such that every $\boldsymbol{w}^\alpha$ can be carried into $\boldsymbol{w}^{\alpha+1}$ by a so-called contracting transformation. These contracting transformations form a sequence which is connecting. We show first that a connecting sequence (Definition 6) has a product and moreover, if this sequence is contracting, its product is an infinite Nielsen transformation which carries the first chain into the last one (Theorem 3). To define a contracting transformation we introduce an ordering (Definition 7) in the set of entries of a chain. We order entries by their x-length, and by their position for words of the same length. In the next section, and subsequently, this ordering will be the only one used, and we shall then change the natural indexes of entries for new transfinite) ones. Lemma 6 of this section will be used in section 4.

DEFINITION 6. We say that a sequence of transformations $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ is a *connecting sequence* for a sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0 \rangle$ if $N_\alpha \boldsymbol{w}^{\alpha-1} = \boldsymbol{w}^\alpha$, $\alpha \in \Phi_0$, and $w_m^\alpha = w_m^{\alpha-1}$ implies $(N_\alpha \boldsymbol{x})_m = x_m$.

LEMMA 4. *If $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ is a connecting sequence for a complete sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0 \rangle$, then $\lim_{\alpha \to \alpha'} N_\alpha = E$ for every $\alpha' \in \Phi_1$.*

PROOF. For every $\alpha' \in \Phi_1$, $\lim_{\alpha \to \alpha'} \boldsymbol{w}^\alpha = \boldsymbol{w}^{\alpha'}$, which implies, for each m, the existence of $\alpha_{m'}$, $\alpha_m < \alpha'$, such that $w_m^{\alpha-1} = w_m^\alpha = w_m^{\alpha'}$ whenever $\alpha_m < \alpha < \alpha'$. This means, by Definition 6, that $(N_\alpha \boldsymbol{x})_m = x_m$ whenever $\alpha_m < \alpha < \alpha'$ and hence $\lim_{\alpha \to \alpha'} N_\alpha = E$ for every $\alpha' \in \Phi_1$, as required.

LEMMA 5. *If $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ is a connecting sequence for a complete sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0 \rangle$, then the sequence of partial products $\langle P_\alpha, \alpha \in \Phi \rangle$ exists, and $P_\alpha \boldsymbol{w}^0 = \boldsymbol{w}^\alpha$ for $\alpha \in \Phi$.*

PROOF. By Lemmas 4 and 3, the sequence $\langle P_\alpha, \alpha \in \Phi \rangle$ of partial products for $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ exists. Now, for $\alpha = 1$, $P_1 \boldsymbol{w}^0 = N_1 \boldsymbol{w}^0 = \boldsymbol{w}^1$. Let $P_\alpha \boldsymbol{w}^0 = \boldsymbol{w}^\alpha$ for all α, $\alpha < \alpha'$. If $\alpha' \in \Phi_0$, then

$$P_{\alpha'} \boldsymbol{w}^0 = (N_{\alpha'} P_{\alpha'-1}) \boldsymbol{w}^0 = N_{\alpha'} \boldsymbol{w}^{\alpha'-1} = \boldsymbol{w}^{\alpha'} .$$

For $\alpha' \in \Phi_1$ notice that $P_{\alpha'} \boldsymbol{w}^0 = \lim_{\alpha \to \alpha'} (P_\alpha \boldsymbol{w}^0)$; then the inductive hypothesis and the completeness of $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0 \rangle$ give

$$P_{\alpha'} \boldsymbol{w}^0 = \lim_{\alpha \to \alpha'} (P_\alpha \boldsymbol{w}^0) = \lim_{\alpha \to \alpha'} \boldsymbol{w}^\alpha = \boldsymbol{w}^{\alpha'} .$$

This completes the proof, which shows in particular that

$$P_{\bar{\alpha}} \boldsymbol{w}^0 = \boldsymbol{w}^{\bar{\alpha}} . \tag{7}$$

LEMMA 6. *If the sequence $\langle N_\alpha, \alpha \in \Phi_0 \rangle$ has the property $\lim_{\alpha \to \alpha'} N_\alpha = E$ for every $\alpha' \in \Phi_1$, then for any $m \in I$ the set $A(m) = \bigcup_{\alpha \in \Phi_0} \hat{\alpha}(m)$ is finite.*

PROOF. We see first that, for all but a finite number of $\alpha \in \Phi_0$, $(N_\alpha \boldsymbol{x})_m = x_m$: for if not, we can find a sequence $\alpha_1 < \alpha_2 < \ldots$ convergent to some $\alpha' \in \Phi_1$ such that $\lim_i (N_{\alpha_i}\boldsymbol{x})_m \neq x_m$ which (by Lemma 1) contradicts the hypothesis $\lim_{\alpha \to \alpha'} N_\alpha = E$. Now $(N_\alpha \boldsymbol{x})_m = x_m$ implies $\hat{\alpha}(m) = \langle m \rangle$, and because of finiteness of every $\hat{\alpha}(m)$, $\alpha \in \Phi_0$, our statement is proved.

DEFINITION 7. For elements w_i, $i \in I$, of a chain $\boldsymbol{w}$ define a new ordering by $w_i \succ w_j$ if and only if either $\ell(w_i) > \ell(w_j)$, or $\ell(w_i) = \ell(w_j)$ and $i > j$, where $\ell(w)$ is the length of a word w in generators x_i.

DEFINITION 8. We say that transformation N *contracts* a chain $\boldsymbol{w}$ if N changes only one element of $\boldsymbol{w}$ (say with index m), decreasing its length by multiplication with elements preceding it in the new ordering. More precisely, we require:

(i) $(N\boldsymbol{x})_i = x_i$, $i \neq m$, $i \in I$;
(ii) $\ell(N\boldsymbol{w})_m < \ell(w_m)$;
(iii) $(N\boldsymbol{x})_m = a(\boldsymbol{x}) x_m^\varepsilon b(\boldsymbol{x})$, where $a(\boldsymbol{x})$, $b(\boldsymbol{x}) \in \operatorname{gp}\{x_i\colon w_i \prec w_m\}$, $\varepsilon = \pm 1$.
If N contracts $\boldsymbol{w}$ let $N(m)$ denote the set of indexes of generators x_i in the words $a(\boldsymbol{x})$, $b(\boldsymbol{x})$.

The first two properties of Definition 8 imply

$$\ell(N\boldsymbol{w})_i = \ell(w_i) \Rightarrow (N\boldsymbol{x})_i = x_i \quad (i \in I)\,. \tag{8}$$

DEFINITION 9. A sequence of transformations $\langle N_\alpha,\ \alpha \in \Phi_0 \rangle$ is called *contracting* if, for some complete sequence of chains $\langle \boldsymbol{w}^\alpha,\ \alpha \in \Phi \cup 0 \rangle$, $N_\alpha \boldsymbol{w}^{\alpha-1} = \boldsymbol{w}^\alpha$ and N_α contracts $\boldsymbol{w}^{\alpha-1}$, $\alpha \in \Phi_0$.

Notice now some properties of a contracting sequence $\langle N_\alpha,\ \alpha \in \Phi_0 \rangle$:

I. Every N_α is invertible, $\alpha \in \Phi_0$. Indeed, $(N_\alpha^{-1}\boldsymbol{x})_i = x_i$ for $i \neq m$, and

$$(N_\alpha^{-1}\boldsymbol{x})_m = [a^{-1}(\boldsymbol{x}) x_m b^{-1}(\boldsymbol{x})]^\varepsilon\,.$$

Here obviously m depends on α.

II. $\lim_{\alpha \to \alpha'} N_\alpha = E$ for every $\alpha \in \Phi_1$. This follows from Lemma 4, since a contracting sequence is obviously connecting.

III. $\hat{\alpha}(i) = \langle i \rangle$ for $i \neq m$, and $\hat{\alpha}(m) = \langle N_\alpha(m), m \rangle$, where $N_\alpha(m)$ was defined in Definition 8 (iii).

A typical element from $\hat{\alpha}(m)$ we will denote by $\alpha(m)$, to indicate that under N_α the element with this index in $\boldsymbol{w}^{\alpha-1}$ is used to obtain w_m^α.

DEFINITION 10. The elements $w_{\alpha(m)}^{\alpha-1}$, $\alpha(m) \in \hat{\alpha}(m)$, we call the *acting elements* under N_α corresponding to indexes $\alpha(m)$.

Notice now some more properties of the contracting sequence $\langle N_\alpha, \alpha \in \Phi_0\rangle$ for the complete sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0\rangle$.

From (8) for $\alpha \in \Phi_0$ it follows that

$$\ell(w_m^{\alpha-1}) = \ell(w_m^\alpha) \Rightarrow (N_\alpha \boldsymbol{x})_m = x_m \quad \text{and} \quad w_m^{\alpha-1} = w_m^\alpha . \tag{9}$$

Now from (i) and (ii) of Definition 8 and the completeness of the sequence $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0\rangle$, if $\alpha, \beta \in \Phi \cup 0$, $\beta < \alpha$, $m \in I$, then

$$\ell(w_m^\beta) \geqq \ell(w_m^\alpha) , \tag{10}$$

$$\ell(w_m^\beta) = \ell(w_m^\alpha) \Rightarrow w_m^\beta = w_m^\alpha . \tag{11}$$

From Definition 8 (iii) it follows that $w_m^\alpha = a(\boldsymbol{w}^{\alpha-1})(w_m^{\alpha-1})^\varepsilon b(\boldsymbol{w}^{\alpha-1})$, where $i \in N_\alpha(m)$ and $w_m^{\alpha-1} \succ w_i^{\alpha-1}$, $i \in N_\alpha(m)$. Thus by III, for any $\alpha(m) \in \hat{\alpha}(m)$ with $\alpha(m) \neq m$, we have $w_m^{\alpha-1} \succ w_{\alpha(m)}^{\alpha-1}$, which means by Definition 7 that

$$\text{either} \quad \ell(w_m^{\alpha-1}) > \ell(w_{\alpha(m)}^{\alpha-1}) , \tag{12}$$
$$\text{or} \quad \ell(w_m^{\alpha-1}) = \ell(w_{\alpha(m)}^{\alpha-1}) \quad \text{and} \quad m > \alpha(m) .$$

LEMMA 7. *Let $\alpha, \beta \in \Phi_0$, $\beta < \alpha$. Fix $\beta(m) \in \hat{\beta}(m)$, $\alpha(\beta(m)) \in \hat{\alpha}(\beta(m))$. Let $\alpha(\beta(m)) \neq \beta(m)$. Then for the corresponding acting elements (see Definition* 10) *it follows that*

$$\ell(w_{\beta(m)}^{\beta-1}) \geqq \ell(w_{\alpha(\beta(m))}^{\alpha-1}) , \tag{13}$$

$$\ell(w_{\beta(m)}^{\beta-1}) = \ell(w_{\alpha(\beta(m))}^{\alpha-1}) \Rightarrow \beta(m) > \alpha(\beta(m)) . \tag{14}$$

PROOF. By (12),

$$\ell(w_{\beta(m)}^{\alpha-1}) \geqq \ell(w_{\alpha(\beta(m))}^{\alpha-1})$$

and, by (10),

$$\ell(w_{\beta(m)}^{\beta-1}) \geqq \ell(w_{\beta(m)}^{\alpha-1}) ,$$

which gives (13). If

$$\ell(w_{\beta(m)}^{\beta-1}) = \ell(w_{\alpha(\beta(m))}^{\alpha-1}) ,$$

then both the previous relations are equalities and

$$\ell(w_{\beta(m)}^{\alpha-1}) = \ell(w_{\alpha(\beta(m))}^{\alpha-1})$$

implies $\beta(m) > \alpha(\beta(m))$, by (12), which proves (14).

THEOREM 2. *Any contracting sequence $\langle N_\alpha, \alpha \in \Phi_0\rangle$ is regular.*

PROOF. According to Definition 5 we have to prove that the characteristic sets $T(m)$ $(m \in I)$ of $\langle N_\alpha, \alpha \in \Phi_0\rangle$ are finite.

Consider at first the set $T_\omega(m)$ for some fixed $m \in I$. Now $T_\omega(m)=\bigcup_k T_k(m)$ consists of indexes t,

$$t \in T_k(m) = \hat{k}\big((k-1)\hat{}(\ldots \hat{2}(\hat{1}(m))\ldots)\big), \qquad k=1,2,3,\ldots,$$

so that, for some fixed indexes $1(m) \in \hat{1}(m)$, $2(1(m)) \in \hat{2}(1(m)),\ldots,t$ can be written in the form $k\big(k-1(\ldots 2(1(m))\ldots)\big)$. If it happens inside this iteration that $i(r)=r$ then instead of $i+1(i(r))$ we shall write $i+1(r)$. Then every $t \in T_\omega(m)$ can be written in shortened form $t=k_s\big(k_{s-1}(\ldots k_1(m)\ldots)\big)$ such that in the sequence of internal indexes $k_1(m), k_2(k_1(m)),\ldots,t$, every two neighbouring indexes are different natural numbers and $k_1(m)\neq m$.

The same can be done for elements of $T(m)=T_{\bar{\alpha}}(m)$: every $t \in T(m)$ can be written in a form $t=\alpha_k\big(\alpha_{k-1}(\ldots \alpha_2(\alpha_1(m))\ldots)\big)$, where $\alpha_i \in \Phi_0$ and $\alpha_k>\alpha_{k-1}>\ldots>\alpha_2>\alpha_1$, and such that the sequence of internal indexes

$$\alpha_1(m),\ \alpha_2(\alpha_1(m)),\ \alpha_3\big(\alpha_2(\alpha_1(m))\big),\ldots,t \tag{15}$$

has different neighbouring elements and $\alpha_1(m)\neq m$. We shall then say that our index t is of *weight* k and has the *core* $\alpha_1(m)\neq m$. To prove that $|T(m)|<\infty$ we shall show that the number of different sequences (15) is finite.

For every $t \in T(m)$ the core of t is in $A(m)$ which, by II and Lemma 6, is finite. Hence it is enough to consider a subset $C(m)\subseteqq T(m)$, consisting of indexes with the same fixed core $\alpha_1(m)$, and to show its finiteness.

For the sequence (15) of internal indexes for t consider the sequence of the lengths of corresponding acting elements (Definition 10), which by Lemma 7 is non-increasing:

$$l(w^{\alpha_1-1}_{\alpha_1(m)}) \geqq l(w^{\alpha_2-1}_{\alpha_2(\alpha_1(m))}) \geqq \ldots \geqq l(w^{\alpha_k-1}_t)\,. \tag{16}$$

The number d of strict inequalities in (16) will be called the *defect* of the index t. Obviously $0\leqq d\leqq l(w^{\alpha_1-1}_{\alpha_1(m)})$ for every $t \in C(m)$. Also, by Lemma 7, for each equality in (16) there is a strict inequality (decrease) between corresponding indexes in (15).

For any fixed m and $\alpha_1(m)$, denote by $C^d(m)$ the subset of $C(m)$ consisting of indexes of defect d. It is enough to show that $C^d(m)$ is finite. Let us show first that $C^0(m)$ is finite. Denote by $C^0_k(m)$ the subset of $C^0(m)$ consisting of indexes of weight k; then $C^0(m)=\bigcup_k C^0_k(m)$. Use induction on k. For $k=1$, $C^0_1(m)$ consists of only one index $\alpha_1(m)$ and hence is finite. Suppose $C^0_{k-1}(m)$ is finite and $t \in C^0_k(m)$, that is,

$$t = \alpha_k\big(\alpha_{k-1}(\ldots \alpha_1(m)\ldots)\big) \in \hat{\alpha}_k\big(\alpha_{k-1}(\ldots \alpha_1(m)\ldots)\big)\,.$$

Then $C^0_k(m)\subseteqq \bigcup A(s)$, $s \in C^0_{k-1}(m)$. By II and Lemma 6, every $A(s)$ is finite, and by the inductive hypothesis, $C^0_{k-1}(m)$ is finite; hence $C^0_k(m)$ is finite. We show now that for $k>\alpha_1(m)$ all $C^0_k(m)$ are empty: for a given $t \in C^0(m)$, $d=0$ means that in the corresponding sequence (16) all lengths are equal and hence, by Lemma 7, sequence (15) of inner indexes for t is strictly decreasing, and hence has not more than $\alpha_1(m)$ terms. Consequently $C^0_k(m)$ is empty for $k>\alpha_1(m)$. It now follows that $\bigcup_k C^0_k(m)=C^0(m)$ is finite for any m and $\alpha_1(m)$.

Let us again fix some m and $\alpha_1(m)$ and, using induction on d, suppose that $C^{d-1}(m)$ is finite. Let $t \in C^d(m)$ and $t=\alpha_k(\alpha_{k-1}(\ldots \alpha_1(m)\ldots))$. Suppose that in the corresponding sequence (16) the last strict inequality occurs for

$$l(w_r^{\alpha_{s-1}-1}) > l(w_{\alpha_s(r)}^{\alpha_s-1}), \quad \text{where} \quad r = \alpha_{s-1}(\ldots\alpha_1(m)\ldots) \in C^{d-1}(m);$$

then $t=\alpha_k(\ldots \alpha_s(r)\ldots) \in C^0(r)$ and $C^d(m)\subseteq \bigcup_r C^0(r)$, $r \in C^{d-1}(m)$. As shown above, $C^0(r)$ is finite for every $r \in I$ and every $\alpha_s(r)$, while $C^{d-1}(m)$ is finite by the inductive hypothesis; hence $C^d(m)$ is also finite. The proof is therefore complete.

THEOREM 3. *If a complete sequence of chains $\langle w^\alpha, \alpha \in \Phi \cup 0\rangle$ has a contracting sequence $\langle N_\alpha, \alpha \in \Phi_0\rangle$ then there exists an infinite Nielsen transformation P such that $w^{\bar\alpha}=Pw^0$.*

PROOF. By (7) of Lemma 5 (since a contracting sequence is connecting), $P=P_{\bar\alpha}$ is the last element of the sequence of partial products for $\langle N_\alpha, \alpha \in \Phi_0\rangle$. By property II and Theorem 2 our contracting sequence satisfies the conditions of Theorem 1 and hence $P_{\bar\alpha}$ is an infinite Nielsen transformation.

Now take some fixed chain u, and denote by $\mathfrak{N}$ the set of chains $w=Nu$ for all infinite Nielsen transformations N.

DEFINITION 11. (Partial order in $\mathfrak{N}$). Let $w^{(1)}, w^{(2)} \in \mathfrak{N}$. If, for some Φ with last element $\bar\alpha$, there is a complete sequence of chains $\langle w^\alpha, \alpha \in \Phi \cup 0\rangle$, possessing a contracting sequence of transformations, and such that $w^0=w^{(1)}$, $w^{\bar\alpha}=w^{(2)}$, then we write $w^{(1)}\succ w^{(2)}$.

By properties (10) and (11) of a contracting sequence, $w^{(1)}\succ w^{(2)}$ implies

$$(17) \qquad \ell(w^{(1)}_m) \geqq \ell(w^{(2)}_m), \qquad m \in I$$

$$(18) \qquad \ell(w^{(1)}_m) = \ell(w^{(2)}_m) \Rightarrow w^{(1)}_m = w^{(2)}_m, \qquad m \in I\,.$$

LEMMA 8. *Every linearly ordered subset $\mathfrak{L}$ of chains in $\mathfrak{N}$ has a lower bound in $\mathfrak{N}$.*

PROOF. Denote by $\langle w_m, w \in \mathfrak{L}\rangle$ the set of different mth entries for the chains in $\mathfrak{L}$. Because of linear ordering in $\mathfrak{L}$ and properties (17), (18): for each $m \in I$, the set $\langle w_m, \, \boldsymbol{w} \in \mathfrak{L} \rangle$ has exactly one element of minimal length, which will be denoted by $\bar{w}_m$. Let us show now that the chain $\bar{\boldsymbol{w}} = (\bar{w}_1, \bar{w}_2, \ldots)$ has the property $\bar{\boldsymbol{w}} \prec \boldsymbol{w}'$ for any $\boldsymbol{w}' \in \mathfrak{L}$. If $w_1' \neq \bar{w}_1$ then $\ell(w_1') > \ell(\bar{w}_1)$. Denote by $\boldsymbol{w}^1$ any chain in $\mathfrak{L}$ for which $w_1^1 = \bar{w}_1$; then because of (17) and linear ordering in $\mathfrak{L}$ we have $\boldsymbol{w}' \succ \boldsymbol{w}^1$. If $w_1' = \bar{w}_1$ we take $\boldsymbol{w}^1 = \boldsymbol{w}'$. Proceeding inductively, let $\boldsymbol{w}^n \in \mathfrak{L}$ be defined such that $w_i^n = \bar{w}_i$, $i \leqq n$, $w_{n+1}^n \neq \bar{w}_{n+1}$, which implies $\ell(w_{n+1}^n) > \ell(\bar{w}_{n+1})$. Then there exists $\boldsymbol{w}^{n+1} \in \mathfrak{L}$ for which $w_{n+1}^{n+1} = \bar{w}_{n+1}$; hence $\ell(w_{n+1}^n) > \ell(w_{n+1}^{n+1})$ which implies $\boldsymbol{w}^n \succ \boldsymbol{w}^{n+1}$ and so, by (17) and the inductive hypothesis,

$$\ell(w_i^{n+1}) \leqq \ell(w_i^n) = \ell(\bar{w}_i) \quad \text{for } i \leqq n \, .$$

It follows now that $w_i^{n+1} = \bar{w}_i$ for $i \leqq n+1$. In case $w_{n+1}^n = \bar{w}_{n+1}$ define $\boldsymbol{w}^{n+1} = \boldsymbol{w}^n$.

In this way we obtain a sequence

$$\text{(19)} \qquad \boldsymbol{w}' \succ \boldsymbol{w}^1 \succ \boldsymbol{w}^2 \succ \ldots$$

which is convergent to $\bar{\boldsymbol{w}}$ and which, together with $\bar{\boldsymbol{w}}$, therefore gives us a complete sequence. Relation $\boldsymbol{w}^n \succ \boldsymbol{w}^{n+1}$ means, by Definition 11, that there exists some complete sequence of chains beginning with $\boldsymbol{w}^n$ and ending with $\boldsymbol{w}^{n+1}$, having a contracting sequence of transformations. Inserting all these intermediate complete sequences, we extend (19) to a complete sequence satisfying Definition 11; this gives $\boldsymbol{w}' \succ \bar{\boldsymbol{w}}$.

By Theorem 3 there exists an infinite Nielsen transformation P such that $\bar{\boldsymbol{w}} = P\boldsymbol{w}'$, and since $\boldsymbol{w}' \in \mathfrak{N}$, we get $\bar{\boldsymbol{w}} \in \mathfrak{N}$. This completes the proof.

Now by Kuratowski–Zorn's Lemma there exists a minimal chain $\boldsymbol{w}^0$ in $\mathfrak{N}$ such that, for every $\boldsymbol{w}' \in \mathfrak{N}$, either $\boldsymbol{w}^0 \prec \boldsymbol{w}'$, or $\boldsymbol{w}^0 = \boldsymbol{w}'$, or $\boldsymbol{w}^0$ and $\boldsymbol{w}'$ are not comparable.

LEMMA 9. *For some fixed $m \in I$, let $w_m^0 \in \boldsymbol{w}^0$, where $\boldsymbol{w}^0$ is a minimal chain in $\mathfrak{N}$. Let $g \in F$ have the form $g = a(\boldsymbol{w}^0)(w_m^0)^\varepsilon b(\boldsymbol{w}^0)$, where $\varepsilon = \pm 1$ and*

$$a(\boldsymbol{w}^0), b(\boldsymbol{w}^0) \in \operatorname{gp}\{w_i^0, \; w_i^0 \prec w_m^0 \text{ (as in Definition 7)}\} \, .$$

Then

$$\ell(g) \geqq \ell(w_m^0) \, .$$

PROOF. We define an infinite Nielsen transformation N by $(N\boldsymbol{x})_i = x_i$ $(i \neq m)$, $(N\boldsymbol{x})_m = a(\boldsymbol{x})x_m^\varepsilon b(\boldsymbol{x})$. Suppose we had in addition that $\ell(g) \equiv \ell(N\boldsymbol{w}^0)_m < \ell(w_m^0)$; then by Definition 8, N would be contracting for $\boldsymbol{w}^0$. This implies, by Definition 11, that $\boldsymbol{w}^0 \succ N\boldsymbol{w}^0$, which contradicts the minimality of $\boldsymbol{w}^0$.

4. An algorithm.

According to [6, Lemma 3.1], a set w is Nielsen-reduced if it satisfies properties 1* and 2* given below. In this section we show that for the minimal chain w^0 with the property given in Lemma 9, there exists a Nielsen transformation P such that in Pw^0 the set of non-unit elements satisfies 1*. The transformation P will appear as a product of a connecting sequence of transformations $\langle N_\alpha \rangle$, where N_α changes w^α to $w^{\alpha+1}$ and the sequence leads from w^0 to $w^{\bar{\alpha}}$. We achieve the connecting sequence $\langle N_\alpha \rangle$ by performing the algorithm similar to the one given in [6, Theorem 3.1]. In every step from w^α to $w^{\alpha+1}$ the length of entries will be unchanged and hence the ordering given in Definition 7 will be the same in every w^α as in w^0. According ro the algorithm, in the step from w^α to $w^{\alpha+1}$ only elements with new indexes $\lambda \leqq \alpha+1$ can be changed and then only if their length is equal to $\ell(w^0_{\alpha+1})$. For each step the corresponding chain $w^{\alpha+1}$ is such that the subset of its elements with indexes μ, $\mu \leqq \alpha$, satisfies 1*.

DEFINITION 12. We shall say that subset $w = \langle w_i \rangle$ of words in F satisfies *property* 1* if for each non-unit word of even length its right half is isolated in w, that is, does not occur as a terminal segment of any other w_j^ε, $\varepsilon = \pm 1$.

DEFINITION 13. Consider the elements w_i^0, $i \in I$, of the chain w^0, and rearrange them according to their lengths by the ordering $w_i^0 \prec w_j^0$ given in Definition 7. This rearrangement therefore assigns an element w_i^0 the new position $\varphi(i)$, where $\alpha = \varphi(i)$ is a nonlimit ordinal, $1 \leqq \alpha \leqq \bar{\alpha} \leqq \omega^2$, and $\varphi\colon I \to \Phi_0 \subseteqq \Phi = \langle \xi, 1 \leqq \xi \leqq \bar{\alpha} \rangle$ is a bijection. For example, in w^0 let $w^0_{2k-1} = x_{2k-1}$, $w^0_{2k} = x_{2k}x_{2k+1}$ $(k = 1, 2, \ldots)$; then $\bar{\alpha} = 2\omega$ and $\varphi(2k-1) = k$, $\varphi(2k) = \omega + k$ $(k = 1, 2, \ldots)$. In what follows we shall assume, unless specified to the contrary, that the set w^0 has been re-indexed by $w_i^0 \to w^0_{\varphi(i)}$, where the $\varphi(i)$ will be denoted by Greek letters.

REMARK 1. Let $\alpha, \beta \in \Phi_0$. Then

$$(20) \qquad \alpha < \beta \quad \text{implies} \quad \ell(w^0_\alpha) \leqq \ell(w^0_\beta)\,.$$

Moreover, if α' is a limit ordinal and $\alpha < \alpha' < \beta$, then by Definition 13, $\ell(w^0_\alpha) < \ell(w^0_\beta)$.

REMARK 2. If for some chain $w = (w_1, w_2, \ldots)$, we have $\ell(w_i) = \ell(w_i^0)$ for all $i \in I$, then w and w^0 induce the same rearrangement φ.

THEOREM 4. *Let w^0 be a minimal chain in $\mathfrak{N}$ then there exists a complete sequence of chains $\langle w^\alpha, \alpha \in \Phi \cup 0 \rangle$ which satisfies* (a)–(e) *below for all $\alpha \in \Phi \cup 0$.*

(a) $\ell(w_i^\alpha)=\ell(w_i^0)$ *for all* $i \in I$, *which, by Remark* 2, *is the same as* $\ell(w_\lambda^\alpha)=\ell(w_\lambda^0)$ *for all* $\lambda \in \Phi_0$.

(b) $\lambda>\alpha \Rightarrow w_\lambda^\alpha=w_\lambda^0$, $\lambda \in \Phi_0$, i.e., *elements of* $\boldsymbol{w}^\alpha$ *with indexes* $\lambda>\alpha$ *coincide with those in* $\boldsymbol{w}^0$.

(c) $\lambda\leqq\alpha$, $w_\lambda^\alpha \neq w_\lambda^{\alpha-1} \Rightarrow w_\lambda^\alpha=a(\boldsymbol{w}^0)(w_\alpha^0)^\varepsilon b(\boldsymbol{w}^0)$, *where* $\varepsilon=\pm 1$, $a(\boldsymbol{w}^0), b(\boldsymbol{w}^0) \in \operatorname{gp}\{w_\xi^0,\ \xi<\alpha\}$, *and hence*

$$w_\lambda^\alpha \in \operatorname{gp}\{w_\xi^0,\ \xi\leqq\alpha\} \quad \text{for } \lambda\leqq\alpha\,. \tag{21}$$

(d) *Subset* $\langle w_\lambda^\alpha,\ \lambda\leqq\alpha\rangle$ *satisfies property* 1* (*Definition* 12).

(e) *If* $\lambda<\alpha$, $w_\lambda^{\alpha-1} \neq w_\lambda^\alpha$ *then* (i) $\ell(w_\lambda^\alpha)=\ell(w_\alpha^\alpha)$, (ii) $w_\lambda^\alpha=w_\alpha^\alpha w_\lambda^{\alpha-1}$, (iii) *the left half of* w_λ^α *lexicographically precedes that of* $w_\lambda^{\alpha-1}$ *and coincides with that of* w_α^α, (iv) *the left half of* w_λ^β *coincides with that of* w_α^β *for* $\beta\geqq\alpha$.

Proof. We proceed by induction on α. For $\alpha=0$, $\boldsymbol{w}^0$ obviously satisfies (a)–(e). For some $\alpha \in \Phi \cup 0$, suppose the chains $\boldsymbol{w}^\beta$, $\beta<\alpha$, are defined to satisfy (a)–(e). We then have to construct $\boldsymbol{w}^\alpha$ for the two possibilities where α is a nonlimit or a limit ordinal.

Consider $\alpha \in \Phi_0$. Then $\boldsymbol{w}^{\alpha-1}$ is defined and we shall obtain $\boldsymbol{w}^\alpha$ as in [6, Theorem 3.1], with suitable modifications. Denote the subset of elements $w_\mu^{\alpha-1}$, $\mu\leqq\alpha-1$, by $\boldsymbol{w}'$, and its elements by w'_μ. By the inductive hypothesis we get from (d) and (b) that $\boldsymbol{w}'$ has property 1*, and $w_\alpha^{\alpha-1}=w_\alpha^0$. To obtain $\boldsymbol{w}^\alpha$ we isolate the right halves of words in $\boldsymbol{w}'$ from w_α^0 and its inverse.

Suppose that $w'_\lambda=u_\lambda v_\lambda$, $w_\alpha^0=s_1 v_\lambda$, where u_λ, v_λ are the left and right halves of w'_λ. Then replace w_α^0 by $w'_\alpha=w_\alpha^0(w'_\lambda)^{-1}=s_1 u_\lambda^{-1}$. Now $s_1 u_\lambda^{-1}$ is freely reduced, and hence $\ell(w'_\alpha)=\ell(w_\alpha^0)$; for otherwise, $\ell(w'_\alpha)<\ell(w_\alpha^0)$ and $w'_\alpha=w_\alpha^0(w'_\lambda)^{-1}$, where (by assumption (21) for $\boldsymbol{w}^{\alpha-1}$) $w'_\lambda \in \operatorname{gp}\{w_\xi^0,\ \xi\leqq\alpha-1\}$; and the choice $g=w'_\alpha$ contradicts Lemma 9.

If w'_α contains as a terminal segment the right half v_μ of w'_μ in $\boldsymbol{w}'$, then v_μ properly contains u_λ^{-1}; for v_μ cannot be a terminal segment of w_λ^{-1}, and not of u_λ^{-1}. Therefore, $w'_\alpha=s_2 v_\mu$, where $\ell(s_2)<\ell(s_1)$. Now replace w'_α by $w''_\alpha=w'_\alpha(w'_\mu)^{-1}$. Continuing in this way we arrive in a finite number of steps (because of the relation $\frac{1}{2}\ell(w_\alpha^0)\leqq\ell(s_{k+1})<\ell(s_k)$) at a $\bar{w}_\alpha$ which does not terminate with a right half of a word in $\boldsymbol{w}'$. However, $\bar{w}_\alpha^{-1}$ may end with such a right half. If it does, repeat the above procedure with $\bar{w}_\alpha^{-1}$ in place of w_α^0. In a finite number of steps we finally arrive at a word $\tilde{w}_\alpha^{-1}$ which does not end with a right half from $\boldsymbol{w}'$. For in going from $\bar{w}_\alpha^{-1}$ to $\tilde{w}_\alpha^{-1}$ only right halves of words in $\boldsymbol{w}'$ are deleted; therefore, since $\ell(\bar{w}_\alpha)\geqq\ell(w'_\lambda)$, every initial segment of $\bar{w}_\alpha^{-1}$ of length $\leqq\frac{1}{2}\ell(\bar{w}_\alpha)$ is unaltered. Hence, the terminal segments of $\tilde{w}_\alpha$ with length $\leqq\frac{1}{2}\ell(\tilde{w}_\alpha)$ are the same as those of $\bar{w}_\alpha$. Consequently neither $\tilde{w}_\alpha$ nor $\tilde{w}_\alpha^{-1}$ ends with a right half from $\boldsymbol{w}'$. Moreover,

$$\tilde{w}_\alpha = a(\boldsymbol{w}')w_\alpha^0 b(\boldsymbol{w}'), \quad \text{where } a(\boldsymbol{w}'), b(\boldsymbol{w}') \in \operatorname{gp}\{w'_\xi,\ \xi<\alpha\}\,. \tag{22}$$

By the inductive hypothesis for $w^{\alpha-1}$ it follows from (21) that

$$a(w'), b(w') \in \mathrm{gp}\{w^0_\xi,\ \xi \leqq \alpha-1), \qquad \lambda<\alpha .$$

By Lemma 9 (with $g=\tilde{w}_\alpha$) it follows that

$$\ell(\tilde{w}_\alpha) = \ell(w^0_\alpha), \tag{23}$$

since $\ell(\tilde{w}_\alpha)>\ell(w^0_\alpha)$ is not admissible by our construction. If $\tilde{w}_\alpha$ has odd length, then w' together with $\tilde{w}_\alpha$ satisfies 1* and w^α can be defined by $w^\alpha_\mu=w^{\alpha-1}_\mu$ $(\mu\neq\alpha)$, $w^\alpha_\alpha=\tilde{w}_\alpha$, to satisfy properties (a)–(e).

Suppose $\tilde{w}_\alpha$ has even length and let u_α, v_α be its left and right halves. If v_α (or u_α) is isolated then again w' together with $\tilde{w}_\alpha$ (or with $\tilde{w}_\alpha^{-1}$) satisfies 1*. And we can define $w^\alpha_\mu=w^{\alpha-1}_\mu$ $(\mu\neq\alpha)$, $w^\alpha_\alpha=\tilde{w}_\alpha$ (or $\tilde{w}_\alpha^{-1}$). Assume that neither u_α nor v_α is isolated and that u_α precedes v_α^{-1} lexicographically (if necessary substitute $\tilde{w}_\alpha^{-1}$ for $\tilde{w}_\alpha$). Then $\tilde{w}_\alpha=u_\alpha v_\alpha$, where u_α and v_α must be initial and terminal segments of elements, or inverses of elements of w'. Now v_α cannot be a terminal segment of a w'_λ since $\ell(w'_\lambda)\leqq\ell(\tilde{w}_\alpha)$, and a right half of a w'_λ cannot be a terminal segment of $\tilde{w}_\alpha$. Similarly, u_α^{-1} cannot be a terminal segment of w'_λ, so that u_α cannot be the initial segment of $(w'_\lambda)^{-1}$. Therefore, for some ϱ and $\sigma<\alpha$, $u_\alpha=u_\varrho$, $v_\alpha=u_\sigma^{-1}$, where u_ϱ, u_σ are initial segments of w'_ϱ, w'_σ. Now modify w' so as to isolate the right half u_σ^{-1} of $\tilde{w}_\alpha$ from those $(w'_\lambda)^{-1}$ which end with u_σ^{-1}. If $w'_\lambda=u_\sigma v_\lambda$ replace w'_λ by $\tilde{w}_\lambda=\tilde{w}_\alpha w'_\lambda=u_\varrho v_\lambda$, and otherwise let $\tilde{w}_\lambda=w'_\lambda$, $\lambda<\alpha$. In any case, this ensures that

$$\ell(\tilde{w}_\lambda) \leqq \ell(w'_\lambda). \tag{24}$$

Define w^α by $w^\alpha_\lambda=\tilde{w}_\lambda$ $(\lambda\leqq\alpha)$, $w^\alpha_\lambda=w^{\alpha-1}_\lambda$ $(\lambda>\alpha)$. Before checking the properties (a)–(e) for w^α notice that, by (24), the inductive hypotheses (a) for $w^{\alpha-1}$, Remark 1, and (23), we have for $\lambda<\alpha$,

$$\ell(w^\alpha_\lambda) = \ell(\tilde{w}_\lambda) \leqq \ell(w'_\lambda) = \ell(w^0_\lambda) \leqq \ell(w^0_\alpha) = \ell(\tilde{w}_\alpha) = \ell(w^\alpha_\alpha). \tag{25}$$

Let $\tilde{w}_\lambda\neq w'_\lambda$; then, by (22) and the inductive hypothesis (21) for $w^{\alpha-1}_\lambda$, $\lambda\leqq\alpha-1$, we have

$$\tilde{w}_\lambda = \tilde{w}_\alpha w'_\lambda = a(w')(w^0_\alpha)^\varepsilon b(w')\cdot w'_\lambda,$$

where $a(w')$, $b(w')$, $w'_\lambda \in \mathrm{gp}\{w^0_\xi,\ \xi\leqq\alpha-1<\alpha\}$. Now if we suppose that $\ell(\tilde{w}_\lambda)<\ell(w^0_\alpha)$ then we contradict Lemma 9 by taking $g=\tilde{w}_\lambda$ and so (25) becomes

$$\ell(w^\alpha_\lambda) = \ell(\tilde{w}_\lambda) = \ell(w'_\lambda) = \ell(w^0_\lambda) = \ell(w^0_\alpha) = \ell(\tilde{w}_\alpha) = \ell(w^\alpha_\alpha). \tag{26}$$

From (26) we have immediately property (a) for w^α, namely $\ell(w^\alpha_\lambda)=\ell(w^0_\lambda)$, $\lambda\leqq\alpha$; while for $\lambda>\alpha$, $w^\alpha_\lambda=w^{\alpha-1}_\lambda=w^0_\lambda$ by the inductive hypothesis, which gives (b).

By (22), $w_\alpha^\alpha = a(w^{\alpha-1})(w_\alpha^0)^\varepsilon b(w^{\alpha-1})$. By inductive hypothesis (21) for $w^{\alpha-1}$:

$$a(w^{\alpha-1}),\ b(w^{\alpha-1}) \in \mathrm{gp}\{w_\xi^0, \xi \leqq \alpha-1\}\ ,$$

which gives property (c) for $\lambda=\alpha$. In case $\lambda<\alpha$: either

$$w_\lambda^\alpha = w_\alpha^\alpha w_\lambda^{\alpha-1} = a(w^{\alpha-1})(w_\alpha^0)^\varepsilon b(w^{\alpha-1})\cdot w_\lambda^{\alpha-1}$$

and (c) is true by the same reason as above, or else $w_\lambda^\alpha = w_\lambda^{\alpha-1}$ and (c) is true by the inductive hypothesis for $w^{\alpha-1}$.

We show now that the set $\langle \tilde{w}_\mu, \mu \leqq \alpha\rangle$ satisfies property 1*. Let v_λ $(\lambda<\alpha)$ be the right half of $\tilde{w}_\lambda$ and therefore of w'_λ. Then v_λ cannot be the terminal segment of a word of odd length (nor its inverse) in $\tilde{w}$, since words of odd length were not altered in going from w' to $\tilde{w}$. Also v_λ is not a terminal segment of $\tilde{w}_\alpha$ or $\tilde{w}_\alpha^{-1}$ (by construction of $\tilde{w}_\alpha$). Suppose now that v_λ is a terminal segment of $\tilde{w}_\mu^\varepsilon$ $(\varepsilon=\pm 1,\ \mu<\alpha,\ \mu\neq\lambda)$: in the case $\tilde{w}_\mu = w'_\mu$, we then have a contradiction to property 1* for w'; in the case $\tilde{w}_\mu = \tilde{w}_\alpha w'_\mu$, we have, by (26), Remark 1 for $\lambda \leqq \alpha$, and property (a), that

$$\ell(\tilde{w}_\mu) = \ell(w_\alpha^0) \geqq \ell(w_\lambda^0) = \ell(\tilde{w}_\lambda)\ ,$$

and hence v_λ is a terminal segment of the right half of $\tilde{w}_\mu^\varepsilon$. Therefore v_λ must be a terminal segment of v_μ or u_ϱ^{-1} (since $\tilde{w}_\mu = \tilde{w}_\alpha w'_\mu = u_\varrho v_\mu$). But then v_λ is a terminal segment of w'_μ or $(w'_\varrho)^{-1}$, contrary to w' satisfying 1*. Thus the right half of $\tilde{w}_\lambda$ $(\lambda<\alpha)$ is isolated in $\tilde{w}$. Similarly the right half u_σ^{-1} of $\tilde{w}_\alpha$ cannot be a terminal segment of a word of odd length (nor of its inverse) in $\tilde{w}$. If u_σ^{-1} is a terminal segment of $\tilde{w}_\lambda^\varepsilon$ $(\varepsilon=\pm 1,\ \lambda<\alpha)$, then u_σ^{-1} must be the right half of $\tilde{w}_\lambda^\varepsilon$. Now $\varepsilon \neq 1$ since we have just shown that the right half of $\tilde{w}_\lambda$ $(\lambda<\alpha)$ cannot be a terminal segment of $\tilde{w}_\alpha$. Moreover, $\varepsilon \neq -1$, since otherwise u_σ is the left half of $\tilde{w}_\lambda$, contrary to the construction of $\tilde{w}_\lambda$ $(\lambda<\alpha)$. Consequently $\langle \tilde{w}_\lambda, \lambda \leqq \alpha\rangle$ satisfies 1*, which gives us property (d) for w.

We now show that (e) holds. Notice that $w_\lambda^\alpha \neq w_\lambda^{\alpha-1}$ means $\tilde{w}_\lambda \neq w'_\lambda$, so that by (26), $\ell(w_\lambda^\alpha) = \ell(w_\alpha^\alpha)$, which is (e) (i). Now $\tilde{w}_\lambda = \tilde{w}_\alpha w'_\lambda = u_\varrho v_\lambda$ gives (e) (ii) and (e) (iii), because the left half of $w_\lambda^\alpha = \tilde{w}_\lambda$, namely $u_\varrho = u_\alpha$, precedes lexicographically the left half $u_\sigma = v_\alpha^{-1}$ of $w_\lambda^{\alpha-1} = w'_\lambda$; while the left halves of $\tilde{w}_\lambda$ and $\tilde{w}_\alpha$ coincide. To obtain (e) (iv) we notice that while constructing the chain w^α we changed element $w_\lambda^{\alpha-1}$ $(\lambda<\alpha)$ if and only if its left half coincided with v_α^{-1}. This implies that if in $w^{\alpha-1}$ the elements with indexes λ and μ $(\lambda,\mu<\alpha)$ have common left half, then they have a (possibly new) common left half in w^α.

Thus for $\alpha \in \Phi_0$, the chain w^α is defined to satisfy properties (a)–(e), which completes the induction for α a nonlimit ordinal.

Consider $\alpha \in \Phi_1$. Suppose the chains w^β, $\beta<\alpha$, are defined to satisfy properties (a)–(e). We shall show that w^α can be defined as $\lim_{\beta\to\alpha} w^\beta$. Indeed, for a fixed $\mu>\alpha$ and any $\beta<\alpha$, the inductive hypothesis (b) for w^β gives $w_\mu^\beta = w_\mu^0$.

For a fixed $\mu<\alpha$ and any β with $\mu<\beta<\alpha$, $w_\mu^\beta \neq w_\mu^{\beta+1}$ implies, by inductive hypothesis (e) (iii) for $\boldsymbol{w}^\beta$, that the left half of $w_\mu^{\beta+1}$ precedes lexicographically that of w_μ^β. This means that for all but a finite number of β $(\mu<\beta<\alpha)$, $w_\mu^\beta = w_\mu^{\beta+1}$ holds and hence the sequence $\langle \boldsymbol{w}^\beta,\ \beta<\alpha\rangle$ is convergent. We may therefore denote $\boldsymbol{w}^\alpha = \lim_{\beta\to\alpha} \boldsymbol{w}^\beta$ and we now check properties (a)–(e) for $\boldsymbol{w}^\alpha$. Properties (a)–(c) are obviously satisfied, while (e) has no sense, because $\alpha \notin \Phi_0$. To check that $\langle w_\xi^\alpha, \xi<\alpha\rangle$ satisfies 1* we take any two elements $w_\mu^\alpha, w_\nu^\alpha$, with $\mu, \nu<\alpha$. Because of the definition $\boldsymbol{w}^\alpha = \lim_{\beta\to\alpha} \boldsymbol{w}^\beta$, there exists γ $(\mu,\nu<\gamma<\alpha)$ such that in $\boldsymbol{w}^\gamma$ we have $w_\mu^\gamma = w_\mu^\alpha$ and $w_\nu^\gamma = w_\nu^\alpha$. By the inductive hypothesis for $\boldsymbol{w}^\gamma$ $(\gamma<\alpha)$ the right half of w_μ^γ is isolated from $(w_\nu^\gamma)^{\pm 1}$, and hence the same holds in $\boldsymbol{w}^\alpha$. This completes the induction for α a limit ordinal, and the theorem is therefore proved.

REMARK 3. Notice now that the procedure of the algorithm given in the proof of Theorem 4 allows us to define, for each $\alpha \in \Phi_0$, an infinite Nielsen transformation N_α, such that $N_\alpha \boldsymbol{w}^{\alpha-1} = \boldsymbol{w}^\alpha$. That is, N_α is a product of two transformations, $N_\alpha = N_{\alpha 2} N_{\alpha 1}$, where

$$
\begin{aligned}
(N_{\alpha 1}\boldsymbol{x})_\lambda &= x_\lambda, && \lambda \neq \alpha\,,\\
(N_{\alpha 1}\boldsymbol{x})_\alpha &= a(\boldsymbol{x}) x_\alpha^\varepsilon b(\boldsymbol{x}), && \text{where } a(\boldsymbol{x}), b(\boldsymbol{x}) \in \operatorname{gp}\{x_\xi,\ \xi<\alpha\}\,,\\
(N_{\alpha 2}\boldsymbol{x})_\lambda &= x_\lambda, && \lambda \geqq \alpha\,,\\
(N_{\alpha 2}\boldsymbol{x})_\lambda &= x_\alpha x_\lambda && \text{for some } \lambda<\alpha \text{ (according to the algorithm)}\,,\\
(N_{\alpha 2}\boldsymbol{x})_\lambda &= x_\lambda && \text{for the rest of the } \lambda, \lambda<\alpha\,.
\end{aligned}
$$

In this way we can define a connecting sequence of transformations $\langle N_\alpha, \alpha \in \Phi_0\rangle$ (Definition 6) for the complete sequence of chains constructed in Theorem 4. Obviously $N_{\alpha 1}, N_{\alpha 2}$, and hence N_α, $\alpha \in \Phi_0$, are invertible.

LEMMA 10. *Let $\langle N_\alpha, \alpha \in \Phi_0\rangle$ be the connecting sequence from Remark 3. If $\lambda<\alpha<\beta$, where $\beta \in \hat{\beta}(\alpha)$, $\alpha \in \hat{\alpha}(\lambda)$ (Definition 4), then $\beta \in \hat{\beta}(\lambda)$.*

PROOF. Notice first that if $\lambda<\alpha$, then N_α acts on x_λ in the same way as $N_{\alpha 2}$. This means that either $\hat{\alpha}(\lambda) = \langle \lambda, \alpha\rangle$, or $\hat{\alpha}(\lambda) = \langle \lambda\rangle$. Moreover, if $\alpha \in \hat{\alpha}(\lambda)$ then $w_\lambda^\alpha \neq w_\lambda^{\alpha-1}$, and so by (e) (iii) the left halves of w_λ^α and w_α^α coincide. By (e) (iv) the left halves of $w_\lambda^{\beta-1}$ and $w_\alpha^{\beta-1}$ coincide for $\beta>\alpha$. Now $\beta \in \hat{\beta}(\alpha)$ means that the element of $\boldsymbol{w}^{\beta-1}$ with index α changes its left half under N_β $(\beta>\alpha)$ and hence so does the element with index λ; this implies $\beta \in \hat{\beta}(\lambda)$, as required.

LEMMA 11. *The connecting sequence of invertible transformations $\langle N_\alpha, \alpha \in \Phi_0\rangle$, given in Remark 3 for the sequence of chains constructed in Theorem 4, is regular (Definition 5).*

PROOF. By Definition 5 we have to show that all characteristic sets $T(\mu)$, $\mu \in \Phi_0$, for our sequence, are finite. Notice first that for some fixed μ and all $\alpha \leqq \mu-1$, we have $(N_\alpha \boldsymbol{x})_\mu = (N_{\alpha 2}\boldsymbol{x})_\mu = x_\mu$. This implies that $\hat{\alpha}(\mu) = \langle \mu \rangle$ for $\alpha \leqq \mu - 1$. Now by induction on α $(1 \leqq \alpha \leqq \mu - 1)$ we get $T_{\mu-1}(\mu) = \langle \mu \rangle$. Consider now $T_\mu(\mu) = \hat{\mu}(T_{\mu-1}(\mu)) = \hat{\mu}(\mu)$. Notice that $T_\mu(\mu)$ consists of indexes of generators in

$$(N_\mu^{-1}\boldsymbol{x})_\mu \;=\; (N_{\mu 1}^{-1}N_{\mu 2}^{-1}\boldsymbol{x})_\mu \;=\; (a^{-1}(N_{\mu 2}^{-1}\boldsymbol{x})x_\mu b^{-1}(N_{\mu 2}^{-1}\boldsymbol{x}))^\varepsilon \in \mathrm{gp}\{x_\xi,\ \xi \leqq \mu\};$$

thus if we denote $T_\mu(\mu) = \Lambda$, then $\lambda \in \Lambda$ implies $\lambda \leqq \mu$, and Λ is obviously finite. Now let $\alpha > \mu$; then $\alpha > \lambda$ $(\lambda \in \Lambda)$ and, as we noticed in the proof of Lemma 10, either $\hat{\alpha}(\lambda) = \langle \lambda, \alpha \rangle$ or $\hat{\alpha}(\lambda) = \langle \lambda \rangle$. Then $T_{\mu+1}(\mu) = (\mu+1)\hat{}(\Lambda) \subseteqq \langle \Lambda, \mu+1 \rangle$, and for any natural k,

$$T_{\mu+k}(\mu) \;=\; (\mu+k)\hat{}(T_{\mu+k-1}(\mu)) \;\subseteqq\; \langle \Lambda, \mu+1, \mu+2, \ldots, \mu+k \rangle\,.$$

This implies

$$T_{\mu+\omega}(\mu) \;\subseteqq\; \langle \Lambda, \mu+1, \mu+2, \ldots \rangle \;\subseteqq\; \langle \xi,\ \xi < \mu+\omega \rangle\,.$$

We shall show now that $T_{\mu+\omega}(\mu) = T(\mu)$. Notice that for α satisfying $\xi < \mu + \omega < \alpha$, by Remark 1 (before Theorem 4) we have $\ell(w_\xi^\alpha) < \ell(w_\alpha^\alpha)$ and hence, by Theorem 4(e)(i), $w_\xi^{\alpha-1} = w_\xi^\alpha$, which gives $\hat{\alpha}(\xi) = \langle \xi \rangle$ for $\xi \in T_{\mu+\omega}(\mu)$, $\alpha > \mu+\omega$. Now simple transfinite induction on α, where $\mu+\omega+1 \leqq \alpha \leqq \bar{\alpha}$, gives

$$T(\mu) \;=\; T_{\mu+\omega}(\mu) \;\subseteqq\; \langle \Lambda, \mu+1, \mu+2, \ldots \rangle\,.$$

We shall show now that $T(\mu)$ is finite. By Lemmas 4 and 6, for each $\lambda \in \Lambda$ the set

$$A(\lambda) \;=\; \bigcup_{\alpha \in \Phi_0} \hat{\alpha}(\lambda)$$

is finite, and hence the set $A(\Lambda) = \bigcup A(\lambda)$, $\lambda \in \Lambda$, is finite. To prove finiteness of $T(\mu)$ we shall show that $T(\mu) \subseteqq A(\Lambda)$; we consider the sets $T_{\mu+k}(\mu)$ and use induction on k to show that $T_{\mu+k}(\mu) \subseteqq (\mu+k)\hat{}(\Lambda)$. For

$$k \;=\; 1, \qquad T_{\mu+1}(\mu) \;=\; (\mu+1)\hat{}(T_\mu(\mu)) \;=\; (\mu+1)\hat{}(\Lambda)\,.$$

Now suppose that $T_{\mu+k-1}(\mu) \subseteqq (\mu+k-1)\hat{}(\Lambda)$; then

$$T_{\mu+k}(\mu) \;=\; (\mu+k)\hat{}(T_{\mu+k-1}(\mu)) \;=\; \bigcup_{\xi} (\mu+k)\hat{}(\xi), \qquad \xi \in (\mu+k-1)\hat{}(\Lambda)\,.$$

By Lemma 10, $t \in (\mu+k)\hat{}(\xi)$ and $\xi \in (\mu+\mathrm{k}-1)\hat{}(\Lambda)$ imply $t \in (\mu+k)\hat{}(\Lambda)$, which gives $T_{\mu+k}(\mu) \subseteqq (\mu+k)\hat{}(\Lambda)$. So for any $k \geqq 1$ we have $T_{\mu+k}(\mu) \subseteqq (\mu+k)\hat{}(\Lambda) \subseteqq A(\Lambda)$; hence $T(\mu) = T_{\mu+\omega}(\mu) \subseteqq A(\Lambda)$, as required. This completes the proof of the regularity of $\langle N_\alpha, \alpha \in \Phi_0 \rangle$.

THEOREM 5. *For a minimal chain* $\boldsymbol{w}^0$ *there exists an infinite Nielsen transformation* P *such that the chain* $P\boldsymbol{w}^0 = \boldsymbol{w}^*$ *satisfies property* 1^*.

PROOF. In Theorem 4 we constructed a complete sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0\rangle$, beginning with $\boldsymbol{w}^0$ and ending with $\boldsymbol{w}^{\bar{\alpha}}$, (put $\boldsymbol{w}^{\bar{\alpha}} = \boldsymbol{w}^*$), satisfying property 1*, and it was shown in Lemma 11 that the corresponding connecting sequence of invertible transformations $\langle N_\alpha, \alpha \in \Phi_0\rangle$ is regular. Then by (7) of Lemma 5, $P\boldsymbol{w}^0 = \boldsymbol{w}^*$, where P is the last element of the corresponding sequence of partial products for $\langle N_\alpha, \alpha \in \Phi_0\rangle$. By Lemma 4 and 11, it follows from Theorem 1 that P is an infinite Nielsen transformation, as required.

5. The main theorem.

In section 3 we proved that, for any given chain $\boldsymbol{u}$, there exists a chain $\boldsymbol{w}^0 = N\boldsymbol{u}$ which satisfies the property given in Lemma 9, where N is an infinite Nielsen transformation. In section 4 we proved the existence of an infinite Nielsen transformation P such that the chain $P\boldsymbol{w}^0 = \boldsymbol{w}^*$ satisfies the property 1*. We now show that $\boldsymbol{w}^*$ also satisfies 2* (Definition 14) which leads to Theorem 6.

DEFINITION 14. We say that a set $\boldsymbol{w}$ of words in F satisfies *property* 2* if both the major initial and major terminal segments (see [6, p. 123]) of each non-unit word in $\boldsymbol{w}$ are isolated, i.e., if neither occurs as an initial or terminal segment, respectively, of any other $w_\lambda^{\pm 1}$, $w_\lambda \in \boldsymbol{w}$.

If w_β is not isolated from $w_\gamma^{\pm 1}$ then one of the following holds:

(a) the major initial segment of w_β is an initial segment of $(w_\gamma)^{\pm 1}$, in which case either

$$\ell(w_\beta^{-1} w_\gamma) < \ell(w_\gamma) \tag{27}$$

or

$$\ell(w_\gamma w_\beta) < \ell(w_\gamma);$$

(b) the major terminal segment of w_β is a terminal segment of $(w_\gamma)^{\pm 1}$, in which case either

$$\ell(w_\gamma w_\beta^{-1}) < \ell(w_\gamma)$$

or

$$\ell(w_\beta w_\gamma) < \ell(w_\gamma)\,.$$

Our object now is to show (Lemma 14) that the chain $\boldsymbol{w}^*$ satisfies property 2*, and for this we need two further lemmas.

LEMMA 12. *If one of the words w_β^*, w_γ^* in $\boldsymbol{w}^*$ is not isolated from the other or its inverse, and if $\beta < \gamma$, then w_β^* is not isolated from $(w_\gamma^*)^{\pm 1}$.*

PROOF. By Theorem 4(a), $\ell(w_\lambda^0)=\ell(w_\lambda^*)$ $(\lambda \in \Phi_0)$, and then by Remarks 2 and 1, $\beta<\gamma$ implies $\ell(w_\beta^*)\leqq\ell(w_\gamma^*)$. This implies that if, in addition, w_γ^* is not isolated from $(w_\beta^*)^{\pm 1}$, then also w_β^* is not isolated from $(w_\gamma^*)^{\pm 1}$, which proves the lemma.

LEMMA 13. *For each fixed $\beta \in \Phi_0$ consider the sequence $\langle w_\beta^\alpha, \alpha \in \Phi \cup 0\rangle$ of entries with index β from the sequence of chains $\langle \boldsymbol{w}^\alpha, \alpha \in \Phi \cup 0\rangle$ constructed in Theorem 4. Then there exists exactly one $\beta_0 \in \Phi_0 \cup 0$ (either $\beta_0=0$ or $\beta_0\geqq\beta$) such that:*

(28) $$w_\beta^\alpha \neq w_\beta^* \ (\alpha<\beta_0), \qquad w_\beta^\alpha = w_\beta^* \ (\alpha\geqq\beta_0)\,.$$

PROOF. Recall that, from Theorem 4(b), the element with index β is equal to w_β^0 in the chains $\boldsymbol{w}^\alpha$, $1\leqq\alpha\leqq\beta-1$. If $\alpha=\beta$ we have, by (22), $w_\beta^\beta=a(\boldsymbol{w}^{\beta-1})(w_\beta^{\beta-1})^\varepsilon b(\boldsymbol{w}^{\beta-1})$. For the chains $\boldsymbol{w}^\alpha$ with $\alpha>\beta$, the element with index β can change only a finite number of times by decreasing its left half (lexicographically) to obtain its last limit value w_β^*.

In case $w_\beta^0\neq w_\beta^*$ there obviously therefore exists only one β_0 (with $\beta_0\geqq\beta$) satisfying (28).

In case $w_\beta^0=w_\beta^*$ we shall now show that $w_\beta^\alpha=w_\beta^*$ for any $\alpha \in \Phi\cup 0$. Notice first that if we denote $w_\mu^\alpha=u_\mu^\alpha v_\mu^\alpha$, where u and v are left and right halves of w, then, because an element with index μ can alter its right half only under N_μ, we have $v_\mu^\alpha=v_\mu^*$ for all $\alpha\geqq\mu$. If $w_\beta^0=w_\beta^*$ it is easy to see that $v_\beta^\alpha=v_\beta^*$ for all $\alpha \in \Phi\cup 0$. Suppose however, that the left half of the element with index β can change; then $u_\beta^*=u_\beta^{\beta-1}\neq u_\beta^\beta$ implies that for some $\lambda<\beta$, the right half v_λ^* of some $w_\lambda^{\beta-1}$ is a terminal segment of $(u_\beta^*)^{-1}$. Observe also that if, for some $\beta_1>\beta$, $u_\beta^{\beta_1-1}\neq u_\beta^{\beta_1}=u_\beta^*$, then by Theorem 4(e)(iii), $u_{\beta_1}^{\beta_1}=u_\beta^*$. Moreover, in $\boldsymbol{w}^{\beta_1-1}$ the right halves v_λ^* of all elements $w_\lambda^{\beta_1-1}$ are isolated from $w_{\beta_1}^{\beta_1}$ and hence v_λ^* cannot be a terminal segment of $(u_\beta^*)^{-1}$. This contradiction establishes that the element with index β has constant value w_β^* in each chain $\boldsymbol{w}^\alpha$, $\alpha \in \Phi\cup 0$; so that (28) follows, with $\beta_0=0$. This proves the lemma.

REMARK 4. If, in Lemma 13, $\beta_0=0$, then

(29) $$w_\beta^* = w_\beta^0 = w_\beta^\alpha \quad \text{for any } \alpha \in \Phi \cup 0\,.$$

If $\beta_0\neq 0$ then $\beta_0\geqq\beta$ and

(30) $$w_\beta^{\beta_0-1} \neq w_\beta^{\beta_0} = w_\beta^*$$

and hence, by Theorem 4(a) and (e)(i),

(31) $$\ell(w_\beta^0) = \ell(w_\beta^{\beta_0}) = \ell(w_{\beta_0}^{\beta_0}) = \ell(w_{\beta_0}^0);$$

if, moreover, $\beta_0 > \beta$, then by (30) and Theorem 4(e)(ii),

$$(32) \qquad w_\beta^{\beta_0} = w_{\beta_0}^{\beta_0} w_\beta^{\beta_0 - 1} .$$

In addition we have, by (31) and (20),

$$(33) \qquad \ell(w_\beta^0) = \ell(w_\xi^0) = \ell(w_{\beta_0}^0) \quad \text{for any } \xi \text{ in } \beta < \xi < \beta_0 .$$

Also, by Theorem 4(a),

$$(34) \qquad \ell(w_\xi^0) = \ell(w_\xi^*) \quad \text{for any } \xi \in \Phi_0 .$$

LEMMA 14. *The chain* w^* *satisfies property* 2^* (*Definition* 14).

PROOF. We shall suppose that w_β^* is not isolated from w_γ^*, as in case (27) (for the other cases the proof is similar). We denote $g = (w_\beta^*)^{-1} w_\gamma^*$, so that (27) and (34) give

$$(35) \qquad \ell(g) < \ell(w_\gamma^*) = \ell(w_\gamma^0) ,$$

and we now show that (35) contradicts the conclusion of Lemma 9 for w^0. By Lemma 12 we can suppose that $\beta < \gamma$. By Lemma 13, $\beta \leqq \beta_0$ or $\beta_0 = 0$, and also $\gamma \leqq \gamma_0$ or $\gamma_0 = 0$. Then there are the following possibilities, some of which will be considered together.

(a) $\beta_0 = 0, \quad \gamma_0 = 0$;
(b) $\beta \leqq \beta_0, \quad \gamma \leqq \gamma_0, \quad \beta_0 < \gamma_0$;
(c) $\beta \leqq \beta_0, \quad \gamma \leqq \gamma_0, \quad \gamma_0 < \beta_0$;
(d) $\beta \leqq \beta_0, \quad \gamma \leqq \gamma_0, \quad \beta_0 = \gamma_0$, (i) $\gamma = \gamma_0$, (ii) $\gamma < \gamma_0$;
(e) $\beta \leqq \beta_0, \quad \gamma_0 = 0$, (i) $\gamma < \beta_0$, (ii) $\gamma = \beta_0$, (iii) $\gamma > \beta_0$;
(f) $\beta_0 = 0, \quad \gamma \leqq \gamma_0$.

Cases (a) *and* (e)(iii). In the cases: $\gamma_0 = 0$, which implies, by (29), that $w_\gamma^* = w_\gamma^0$; and $\beta_0 < \gamma$, which implies (trivially in case (a), and by (21) in case (e)(iii)) that $w_\beta^{\beta_0} \in \mathrm{gp}\{w_\xi^0, \xi < \beta_0 < \gamma\}$. Then

$$g = (w_\beta^*)^{-1} w_\gamma^* = (w_\beta^{\beta_0})^{-1} w_\gamma^{\gamma_0} = (w_\beta^{\beta_0})^{-1} w_\gamma^0 ,$$

which, together with (35), gives a contradiction to Lemma 9.

Cases (b) *and* (f). Here $\beta_0 < \gamma_0$. By (35) and (31), $\ell(g) < \ell(w_\gamma^0) = \ell(w_{\gamma_0}^0)$. By (30) and Theorem 4(c),

$$g = (w_\beta^*)^{-1} w_\gamma^* = (w_\beta^{\beta_0})^{-1} w_\gamma^{\gamma_0} = (w_\beta^{\beta_0})^{-1} a(w^0)(w_{\gamma_0}^0)^\varepsilon b(w^0) ,$$

where $a(w^0), b(w^0) \in \mathrm{gp}\{w_\xi^0, \xi < \gamma_0\}$; also (trivially for (f), and by (21) for (b)) $w_\beta^{\beta_0} \in \mathrm{gp}\{w_\xi^0, \xi \leqq \beta_0 < \gamma_0\}$. Again a contradiction with Lemma 9 ensues.

Cases (c) *and* (e)(i). Here $\gamma_0<\beta_0$ and $\beta<\gamma<\beta_0$. Thus, by (35) and (33), $\ell(g)<\ell(w_\gamma^0)=\ell(w_{\beta_0}^0)$. By (30) and Theorem 4(c),

$$g = (w_\beta^*)^{-1}w_\gamma^* = (w_\beta^{\beta_0})^{-1}w_\gamma^{\gamma_0} = a(\boldsymbol{w}^0)(w_{\beta_0}^0)^\varepsilon b(\boldsymbol{w}^0)w_\gamma^{\gamma_0},$$

where $a(\boldsymbol{w}^0), b(\boldsymbol{w}^0) \in \operatorname{gp}\{w_\xi^0, \xi<\beta_0\}$; also (trivially for (e)(i), and by (21) for (c)) $w_\gamma^{\gamma_0} \in \operatorname{gp}\{w_\xi^0, \xi\leqq\gamma_0<\beta_0\}$. We obtain a contradiction to Lemma 9.

Cases (d)(i) *and* (e)(ii). Here $\beta<\gamma=\beta_0$. We shall see that in these cases, $w_\gamma^*=w_\gamma^\gamma$. Indeed, in (d)(i), $\gamma=\gamma_0$ and hence $w_\gamma^*=w_\gamma^{\gamma_0}=w_\gamma^\gamma$. In (e)(ii), $\gamma_0=0$ and so, by (29), $w_\gamma^*=w_\gamma^0=w_\gamma^\gamma$. Hence $w_\gamma^*=w_\gamma^\gamma=w_{\beta_0}^{\beta_0}$, and then, by (32),

$$g^{-1} = (w_\gamma^*)^{-1}w_\beta^* = (w_{\beta_0}^{\beta_0})^{-1}w_\beta^{\beta_0} = w_\beta^{\beta_0-1},$$

which implies that $\ell(g)=\ell(w_\beta^{\beta_0-1})=\ell(w_\beta^0)$. On the other hand, by (35) and (33), $\ell(g)<\ell(w_\gamma^0)=\ell(w_\beta^0)$, which is a contradiction.

Case (d)(ii). Here $\beta<\gamma<\gamma_0=\beta_0$. This relation means that the elements with indexes β and γ change their left halves simultaneously under $N_{\beta_0}=N_{\gamma_0}$ to become equal to w_β^* and w_γ^* respectively; hence they have common left halves in $\boldsymbol{w}^{\beta_0-1}$ and (by Theorem 4(e)(iv)) in all succeeding chains—in particular in $\boldsymbol{w}^*$. It follows also, by Theorem 4(e)(i) and (a), that $\ell(w_\beta^0)=\ell(w_{\beta_0}^0)=\ell(w_\gamma^0)$. Let δ be the minimal ordinal (obviously nonlimit) such that, in all chains $\boldsymbol{w}^\alpha$ ($\alpha\geqq\delta$), elements with indexes β and γ have common left halves, while (if $\delta>0$) $w_\beta^{\delta-1}$, $w_\gamma^{\delta-1}$ have different left halves. The element with index β can alter its right half only under N_β, so the right half of w_β^α coincides with that of w_β^* for all $\alpha\geqq\beta$ and hence for $\alpha=\gamma$.

Let $\delta\leqq\gamma$. Then it now follows that elements w_β^γ and w_γ^γ have common left halves, and have the same right halves as w_β^* and w_γ^*, respectively. Thus

$$(w_\beta^*)^{-1}w_\gamma^* = (w_\beta^\gamma)^{-1}w_\gamma^\gamma. \tag{36}$$

In case $w_\beta^\gamma\neq w_\beta^{\gamma-1}$, we have, by Theorem 4(e)(ii), $w_\beta^\gamma=w_\gamma^\gamma w_\beta^{\gamma-1}$; hence

$$g = (w_\beta^*)^{-1}w_\gamma^* = (w_\beta^\gamma)^{-1}w_\gamma^\gamma = (w_\beta^{\gamma-1})^{-1}(w_\gamma^\gamma)^{-1}w_\gamma^\gamma = (w_\beta^{\gamma-1})^{-1}$$

and $\ell(g)=\ell(w_\beta^{\gamma-1})=\ell(w_\beta^0)=\ell(w_\gamma^0)$, which contradicts (35).

In case $w_\beta^\gamma=w_\beta^{\gamma-1}$,

$$g = (w_\beta^*)^{-1}w_\gamma^* = (w_\beta^\gamma)^{-1}w_\gamma^\gamma = (w_\beta^{\gamma-1})^{-1}w_\gamma^\gamma.$$

Now by Theorem 4(c), $w_\gamma^\gamma=a(\boldsymbol{w}^0)(w_\gamma^0)^\varepsilon b(\boldsymbol{w}^0)$, where

$$a(\boldsymbol{w}^0), b(\boldsymbol{w}^0) \in \operatorname{gp}\{w_\xi^0, \xi<\gamma\};$$

and, by (21), $w_\beta^{\gamma-1} \in \operatorname{gp}\{w_\xi^0, \xi\leqq\gamma-1<\gamma\}$. Taken together with (35), this gives a contradiction to Lemma 9.

Now let $\beta<\gamma<\delta$. Then, analogously to (36), $(w_\beta^*)^{-1}w_\gamma^*=(w_\beta^\delta)^{-1}w_\gamma^\delta$. The choice of δ implies that $w_\beta^{\delta-1}$ and $w_\gamma^{\delta-1}$ have different left halves, hence only one of the elements (say with index γ) changes under N_δ. Then $w_\beta^\delta=w_\beta^{\delta-1}$ and $w_\gamma^\delta=w_\delta^\delta w_\gamma^{\delta-1}$. By Theorem 4(c),

$$w_\delta^\delta = a(\mathbf{w}^0)(w_\delta^0)^\varepsilon b(\mathbf{w}^0)\,,$$

where $a(\mathbf{w}^0)$, $b(\mathbf{w}^0)\in \mathrm{gp}\{w_\xi^0,\ \xi<\delta\}$; hence

$$g = (w_\beta^*)^{-1}w_\gamma^* = (w_\beta^\delta)^{-1}w_\delta^\delta w_\gamma^{\delta-1} = (w_\beta^{\delta-1})^{-1}a(\mathbf{w}^0)(w_\delta^0)^\varepsilon b(\mathbf{w}^0)w_\gamma^{\delta-1}\,,$$

where, by (21), $w_\beta^{\delta-1}, w_\gamma^{\delta-1}\in \mathrm{gp}\{w_\xi^0,\ \xi\leqq\delta-1<\delta\}$. Now, by (35) and Theorem 4(e)(i) and (a), $\ell(g)<\ell(w_\gamma^0)=\ell(w_\delta^0)$, which gives a contradiction to Lemma 9. This completes the proof of Lemma 14.

To recapitulate: for any fixed chain $\mathbf{u}$ we proved the existence of some minimal chain $\mathbf{w}^0$ in the set $\mathfrak{N}$ of all chains $N\mathbf{u}$ for all infinite Nielsen transformations N. Hence $\mathbf{w}^0=N_0\mathbf{u}$ for some infinite Nielsen transformation N_0. Later we constructed an infinite Nielsen transformation P such that $P\mathbf{w}^0=\mathbf{w}^*$, where the chain $\mathbf{w}^*$ satisfied properties 1* and 2* (Definitions 12 and 14). By [6, Lemma 3.1], the set of non-unit elements in $\mathbf{w}^*$ is a Nielsen-reduced set. We have therefore proved:

THEOREM 6. *For any chain $\mathbf{u}$ in F there exists an infinite Nielsen transformation N such that the subset of non-unit words of $N\mathbf{u}$ is Nielsen-reduced.*

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