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POWERS OF PARTIALLY ORDERED SETS: THE AUTOMORPHISM GROUP

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1. Introduction.

The purpose of this paper is to prove the following result.

THE PRINCIPAL THEOREM. *Suppose P is a bounded, directly indecomposable poset that satisfies the descending chain condition, and suppose $P \cong A^C$, where A is exponentially indecomposable. Then*

$$\mathrm{Aut}\,(P) \cong \mathrm{Aut}\,(A) \times \mathrm{Aut}\,(C)\,.$$

To say that a poset (partially ordered set) is bounded means that it has a smallest element 0 and a largest element 1. The poset A^C consists of all isotone functions from C to A, ordered by pointwise inclusion,

$$f \le g \quad \text{iff } f(x) \le g(x) \quad \text{for all } x \in C\,.$$

To say that A is exponentially indecomposable means that there are no posets X and Y with $A \cong X^Y$ and $|Y| > 1$. Aut (P) is of course the automorphism group of P.

For any posets A and C, there is a natural map

$$\pi:\ \mathrm{Aut}\,(A) \times \mathrm{Aut}\,(C) \to \mathrm{Aut}\,(A^C)\,,$$

where for $\xi \in \mathrm{Aut}\,(A)$, $\gamma \in \mathrm{Aut}\,(C)$ and $f \in A^C$,

$$\pi(\xi,\gamma)(f) = \xi \circ f \circ \gamma^{-1}\,.$$

This map is always an embedding, and it is our objective to show that in the case under consideration it is an isomorphism. Obviously this would not be true without some restrictions on the posets A and C. The particular conditions assumed here are best motivated by recalling the canonical representation property that holds for a large class of posets P. To simplify the discussion, we assume that P is finite.

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First, P is the disjoint sum of its (connected) components,

$$P = \sum (P_i, i \in I),$$

and this representation is obviously unique. Secondly, each P_i is isomorphic to a direct product of directly indecomposable posets,

$$P_i \cong \prod (P_{i,j}, j \in J_i),$$

and by Hashimoto's Theorem ([1]), this representation is unique up to isomorphism. Finally, each $P_{i,j}$ can be represented as a power,

$$P_{i,j} \cong A_{i,j}^{C_{i,j}}$$

with $A_{i,j}$ exponentially indecomposable. It is not known whether, in general, this representation is unique up to isomorphism, but by either Theorem 5.2 or 8.2 of Jónsson, McKenzie [2] we know that the representation is unique if $P_{i,j}$ is either upper or lower bounded.

Roughly speaking, the automorphism group of P is determined by the automorphism groups of the components P_i, and these in turn are determined by the automorphism groups of the factors $P_{i,j}$. More precisely, if we partition I in such a way that two indices belong to the same block just in case the corresponding components are isomorphic, then the automorphism group of P is isomorphic to the direct product of the automorphism groups of the posets $P_K = \sum (P_i, i \in K)$, taken over all the blocks K, and the automorphism group of P_K is isomorphic to the wreath product of the automorphism group of one of its components P_i and of the full symmetric group on K. In exactly the same manner, the automorphism group of P_i can be represented as a direct product of wreath products of automorphism groups of factors $P_{i,j}$ and of full symmetric groups, although this is a less trivial result and depends on the strict refinement property for direct products of connected posets.

In view of this it is natural to ask what information we can obtain about the automorphism group of A^C if we know the automorphism groups of A and C, under the assumptions that A^C is connected and directly indecomposable, and that A is exponentially indecomposable. In particular, one would like to know whether the natural map is onto. Unfortunately this is not always true, even for finite posets. In Example 11.1 in Jónsson and McKenzie [2], A consists of a 2-element chain and a 3-element chain, with their bottom elements identified, while C is the 2-element chain **2**. Thus both A and C have trivial automorphism groups but, as noted there, the automorphism group of A^C has order 2.

It is clear from the above example that we cannot drop the boundedness assumption from our theorem, and cannot even weaken it by just assuming the existence of one of the two bounds. It is of course possible that, even when the natural embedding is not onto, something can be said about the relationship between the three groups. Indeed, Theorems 11.2 and 11.4 in Jónsson, McKenzie [2] are examples of results of that kind. However, these are quite special, and at the present we have no idea what kind of general results one could hope to obtain.

2. The logarithmic property.

We shall make extensive use of the results, techniques and notation from Jónsson and McKenzie [2], and henceforth that paper will be referred to, briefly, as [JM]. In particular, the version of the logarithmic property introduced there will play a fundamental role. We begin by recalling some of the basic facts, and introducing certain assumptions and notation that will be in effect throughout this section and the next three.

We shall be concerned with an isomorphism

(2.1) $$\varphi:\ A^C \cong B^C,$$

where it is assume that

(2.2) A is bounded, and $\nabla J(A) = A$.

(2.3) C and D are connected and directly indecomposable, and satisfy the ascending chain condition.

Recall that $J(A)$ is the set of all strictly join irreducible elements of A, and that $\nabla J(A)$ is the set of all those elements of A that are least upper bounds of subsets of $J(A)$. Because A is assumed to be bounded, the sets $J(A)$ and $J'(A)$ are equal. (Cf. Section 6 in [JM].) Before introducing further notation, we pause to indicate how the investigation of this isomorphism φ will aid in the proof of the Principal Theorem. It will be shown that if A is directly indecomposable, then one of two things must happen. One possibility is that φ is induced by an isomorphism ξ from A to B and an isomorphism γ from C to D, in the sense that $\varphi(f) = \xi \circ f \circ \gamma^{-1}$ for all $f \in A^C$. The second possibility is that A and B have representations λ: $A \cong F^D$ and μ: $B \cong E^C$, and that there is an isomorphism ϱ from F to E such that the diagram

$$\begin{array}{ccc} A^C & \xrightarrow{\varphi} & B^D \\ \downarrow & & \downarrow \\ (F^D)^C & \rightarrow & (E^C)^D \end{array}$$

commutes, where the unlabelled isomorphisms are obtained in an obvious manner from λ, μ, and ϱ. It is clear that this is closely related to various results in [JM], e.g. Theorem 8.2. The principal difference is that there we were concerned with the existence of certain isomorphisms, but here we need to know their explicit form.

The next step is much easier. We consider a power A^{C^n}, where A is directly indecomposable and is not of the form X^C. Of course we also assume that A is bounded, with $\nabla J(A)=A$, and that C is connected and directly indecomposable, and satisfies the ascending chain condition. We show by induction on n that the natural map from Aut $(A)\times$ Aut (C^n) to Aut (A^{C^n}) is in this case an isomorphism. From this the Principal Theorem follows by an easy induction on the number of non-isomorphic factors in the exponent.

The various notational conventions introduced in [JM] will be freely used. E.g., we write $\bar{f}$ for $\varphi(f)$; if $a \in A$, then $\langle a\rangle$ is the constant function on C whose sole value is a; for $a, a' \in A$ and $S \subseteq C$, $\langle a[S], a'\rangle$ is the function on C that takes on the value a on S and a' on $C-S$; and for $a \in A$ and $s \in C$, $j(a,s)=\langle a[x \geq s], 0\rangle$. $R(\varphi)$ is the set of all $a \in A$ such that $\varphi(\langle a\rangle)$ is constant, and $\mathring{\varphi}$ is the bijection from $R(\varphi)$ to $R(\varphi^{-1})$ such that $\varphi(\langle a\rangle)=\langle\mathring{\varphi}(a)\rangle$. For $a, a' \in R(\varphi)$, $a \leq_\varphi a'$ means that $a \leq a'$ and that for any function $f \in A^C$ with $f(C)=\{a, a'\}$, $\bar{f}$ is constant. Recall the conditions (φ, k), (φ^{-1}, k), $k=1,2,3,4$. Collectively these assert that

$$E = (R(\varphi), \leq_\varphi) \quad \text{and} \quad F = (R(\varphi^{-1}), \leq_{\varphi^{-1}})$$

are sub-posets of A and B, respectively, that φ is an isomorphism from E to F, and that the maps $a \to \varphi(\langle a\rangle)$ and $b \to \varphi^{-1}(\langle b\rangle)$ are isomorphisms from A to F^D and from B to E^C, respectively.

As in Section 7 of [JM], the isomorphism φ in (2.1) induces an isomorphism

$$\psi\colon J(A)\cdot C^\delta \cong J(B)\cdot D^\delta\,,$$

which in turn induces isomorphisms

$$\psi_i\colon A_i\cdot C^\delta \cong B_i\cdot D^\delta\,,$$

where A_i and B_i $(i \in I)$ are the components of $J(A)$ and $J(B)$, respectively. This means that, for $u \in A_i$, $x \in C$, $v \in B_i$, and $y \in D$,

$$\psi_i(u,x) = (v,y) \quad \text{iff} \quad \varphi j(u,x) = j(v,y)\,.$$

Applying the strict refinement property to the isomorphisms ψ_i, and making use of the fact that C and D are directly indecomposable, we have for each $i \in I$ one of two possibilities: either there exist isomorphisms

$$\alpha_i\colon\ A_i \cong B_i, \qquad \gamma_i\colon\ C \cong D \tag{2.4}$$

such that, for all $u \in A_i$ and $x \in C$,

$$\psi_i(u,x) = (\alpha_i(u), \gamma_i(x))\,, \tag{2.5}$$

or else there exist a poset W_i and isomorphisms

$$\alpha_i' : W_i \cdot D^\delta \cong A_i, \quad \gamma_i : C \cong C ,$$

$$\beta_i' : W_i \cdot C^\delta \cong B_i, \quad \delta_i : D \cong D$$

such that, for all $w \in W_i$, $x \in C$, and $y \in D$,

$$\psi_i(\alpha_i'(w, y), \gamma(x)) = (\beta_i'(w, x), \delta_i(y)) .$$

We let I_0 be the set of all $i \in I$ for which the first case applies, and $I_1 = I - I_0$, and for $k = 0, 1$ we let

$$J_k(A) = \bigcup \{A_i : i \in I_k\}, \quad J_k(B) = \bigcup \{B_i : i \in I_k\} .$$

For $i \in I_1$ the notation can be somewhat simplified. Replacing x by $\gamma_i^{-1}(x)$ and y by $\delta_i^{-1}(y)$, we obtain

$$\psi_i(\alpha_i'(w, \delta_i^{-1}(y)), x) = (\beta_i'(w, \gamma_i^{-1}(x)), y) .$$

Defining

$$\alpha_i(w, y) = \alpha_i'(w, \delta_i^{-1}(y)), \quad \beta_i(w, x) = \beta_i'(w, \gamma_i^{-1}(x)) ,$$

we therefore obtain isomorphisms

$$\alpha_i : W_i \cdot D^\delta \cong A_i, \quad \beta_i : W_i \cdot C^\delta \cong B_i , \tag{2.6}$$

such that, for all $w \in W_i$, $x \in C$ and $y \in D$,

$$\psi_i(\alpha_i(w, y), x) = (\beta_i(w, x), y) . \tag{2.7}$$

Note, finally, that the two formulas (2.5) and (2.7) can also be written

$$\varphi j(u, x) = j(\alpha_i(u), \gamma_i(x)) , \tag{2.8}$$

$$\varphi j(\alpha_i(w, y), x) = j(\beta_i(w, x), y) . \tag{2.9}$$

Our first lemma is largely a translation into the present notation of various results from Section 7 of [JM].

LEMMA 1. (i) *For all* $f \in A^C$, $i \in I_0$, $u \in A_i$, *and* $x \in C$,

$$u \le f(x) \quad \textit{iff } \alpha_i(u) \le \bar{f}(\gamma_i(x)) .$$

(ii) *For all* $f \in A^C$, $i \in I_1$, $w \in W_i$, $x \in C$, *and* $y \in D$,

$$\alpha_i(w, y) \le f(x) \quad \textit{iff } \beta_i(w, x) \le \bar{f}(y) .$$

(iii) *An element* $a \in A$ *belongs to* $R(\varphi)$ *iff, for all* $i \in I_1$ *and* $w \in W_i$, *the inclusion* $\alpha_i(w, y) \le a$ *either holds for all* $y \in D$, *or else for none.*

(iv) *For all* $a, a' \in R(\varphi)$, $a \le_\varphi a'$ *iff* $a \le a'$ *and, for all* $u \in J_0(A)$, $u \le a'$ *implies* $u \le a$.

(v) $J_0(A) \subseteq R(\varphi)$.
(vi) *For all* $i \in I_0$ *and* $u \in A_i$, $\alpha_i(u) = \mathring{\varphi}(u)$.

PROOF. Observe that $u \leq f(x)$ iff $j(u,x) \leq f$, and that $\alpha_i(u) \leq \bar{f}(\gamma_i(x))$ iff $j(\alpha_i(u), \gamma_i(x)) \leq \bar{f}$. From this (i) follows by (2.8). Similarly,

$$\alpha_i(w,y) \leq f(x) \quad \text{iff } j(\alpha_i(w,y),x) \leq f, \quad \text{and}$$
$$\beta_i(w,x) \leq \bar{f}(y) \quad \text{iff } j(\beta_i(w,x),y) \leq \bar{f},$$

whence (ii) follows by (2.9).

Given $a \in A$, let $f = \langle a \rangle$. Then $a \in R(\varphi)$ iff, for all $v \in J(B)$, the inclusion $v \leq \bar{f}(y)$ is independent of y. If $v \in J_0(B)$, then this inclusion never depends on y. Indeed, we have $v = \alpha_i(u)$ and $y = \gamma_i(x)$ for some $i \in I_0$, $u \in A_i$, and $x \in C$, and by (i) the inclusions $v \leq \bar{f}(y)$ and $u \leq f(x)$ are equivalent. The latter inclusion does not depends on x, since f is constant, and the former is therefore independent of y. If $v \in J_1(B)$, then $v = \beta_i(w,x)$ for some $i \in I_1$, $w \in W_i$ and $x \in C$. By (ii), the inclusions $v \leq \bar{f}(y)$ and $\alpha_i(w,y) \leq a$ are equivalent. Hence $v \leq \bar{f}(y)$ is independent of y iff $\alpha_i(w,y) \leq a$ is. This proves (iii).

Suppose $a, a' \in R(\varphi)$ and $a \leq a'$. To say that $a \leq_\varphi a'$ means that, for any function $f \in A^C$ with $f(C) = \{a, a'\}$, $\bar{f}$ is constant. Equivalently, it means that, for any such function f, and for any $v \in J(B)$, the inclusion $v \leq \bar{f}(y)$ is independent of the element $y \in D$. For $v \in J_1(B)$ this is automatically the case. Indeed, we have $v = \beta_i(w,x)$ for some $i \in I$, $w \in W_i$, and $x \in C$, and by (ii) the inclusions $v \leq \bar{f}(y)$ and $\alpha_i(w,y) \leq f(x)$ are equivalent. By (iii), the latter inclusion does not depend on y, for $f(x)$ is either a or a', and therefore belongs to $R(\varphi)$. We therefore need only consider the case when $v \in J_0(B)$. In this case $v = \alpha_i(u)$ and $y = \gamma_i(x)$ for some $i \in I_0$, $u \in A_i$, and $x \in C$, and by (i) the inclusions $v \leq \bar{f}(y)$ and $u \leq f(x)$ are equivalent. Since $f(C) = \{a, a'\}$, this shows that the inclusion $v \leq \bar{f}(y)$ is independent of y iff the two inclusions $u \leq a$ and $u \leq a'$ either both hold or both fail. This proves (iv).

(v) is an immedutate consequence of (iii), for if $a \in J_0(A)$, and if $i \in I_1$, $w \in W_i$ and $y \in D$, then the two element a and $\alpha_i(w,y)$ belong to different components of $J(A)$, and are therefore not comparable.

Finally suppose $i \in I_0$ and $a \in A_i$. By (v), $a \in R(\varphi)$, and therefore $\varphi(\langle a \rangle) = \langle \mathring{\varphi}(a) \rangle$. Letting $f = \langle a \rangle$, and picking any $x \in C$, we have by (i), $\alpha_i(a) \leq \bar{f}(\gamma_i(x)) = \mathring{\varphi}(a)$. To prove the opposite inclusion we consider any $v \in J(B)$ with $v \leq \mathring{\varphi}(a)$, and show that $v \leq \alpha_i(a)$. First suppose $v \in J_1(b)$. Then $v = \beta_k(w,x)$ for some $k \in I_1$, $w \in W_k$ and $x \in C$. Picking any $y \in D$, we have $\beta_k(w,x) \leq \bar{f}(y)$ and therefore by (ii), $\alpha_k(w,y) \leq f(x) = a$, which is impossible because a and $\alpha_k(w,y)$ belong to different components of $J(A)$. We must therefore have $v \in J_0(B)$. Hence $v = \alpha_k(u)$ for some $k \in I_0$ and $u \in A_k$, and the inclusions $v \leq \mathring{\varphi}(a)$ and u

$\leq a$ are equivalent by (i). But the inclusion $u \leq a$ can only hold of $k=i$, and in this case it implies that $v=\alpha_i(u) \leq \alpha_i(a)$. This completes the proof of (vi).

3. The case $I_0 + \varnothing + I_0$.

It will be shown here that if I_0 and I_1 are both non-empty, then A has a non-trivial decomposition. More specifically, it will be shown that the sets $J_0(A)$ and $J_1(A)$ have least upper bounds c_0 and c_1, and that

$$A \cong [0, c_0] \cdot [0, c_1] .$$

Some additional notation is needed. For $a \in A$, $b \in B$, and $k=0,1$, we let

$$J(a) = \{u \in J(A) : u \leq a\}, \quad J_k(a) = J(a) \cap J_k(A) ,$$
$$J(b) = \{v \in J(B) : v \leq b\}, \quad J_k(b) = J(b) \cap J_k(B) .$$

We let M and N be the sets consisting of the maximal elements of C and of D, respectively. Actually the only properties of these sets that will be used are that they are proper filters that are invariant under all the automorphisms of C and of D, respectively, and that if γ is an isomorphism from C to D, then $\gamma(M)=N$. We let

$$J_{1,N}(A) = \{\alpha_i(w,y) : i \in I_1, \; w \in W_i, \; y \in N\} ,$$
$$J_{1,M}(B) = \{\beta_i(w,x) : i \in I, \; w \in W_i, \; x \in M\} .$$

Finally, we shall be working extensively with functions of the form $\langle a[M], a' \rangle$ and $\langle b[N], b' \rangle$, where $a' \leq a$ in A and $b' \leq b$ in B, and to simplify the notation, we denote these simply by $a'a$ and $b'b$, respectively. Notice that in this context the constant functions $\langle a \rangle$ and $\langle b \rangle$ are sometimes written aa and bb.

LEMMA 2. *There exist elements $c, c', c_0, c_1 \in A$ and $d, d', d_0, d_1 \in B$ such that*

$$\varphi(01) = dd', \quad \varphi(cc') = 01 ,$$
$$\varphi(0c_1) = dd, \quad \varphi(cc) = 0d_1 ,$$
$$\varphi(c_01) = d'd', \quad \varphi(c'c') = d_01 .$$

Furthermore,

$$J(c) = J_{1,N}(A), \quad J(c') = J_{1,N}(A) \cup J_0(A) ,$$
$$J(c_0) = J_0(A), \quad J(c_1) = J_1(A) ,$$
$$J(d) = J_{1,M}(B), \quad J(d') = J_{1,M}(B) \cup J_0(B) ,$$
$$J(d_0) = J_0(B), \quad J(d_1) = J_1(B) .$$

PROOF. Let $f=01$. We claim that

$$(1)\qquad J(\bar{f}(y)) = \begin{cases} J_{1,M}(B) & \text{if } y \in D-N \\ J_{1,M}(B) \cup J_0(B) & \text{if } y \in N\,. \end{cases}$$

To prove this, we show that an element $v \in J_0(B)$ belongs to $J(\bar{f}(y))$ iff $y \in N$, and that an element $v \in J_1(B)$ belongs to $J(\bar{f}(y))$ iff $v \in J_{1,M}(B)$.

First, if $v \in J_0(B)$, then $v=\alpha_i(u)$ and $y=\gamma_i(x)$ for some $i \in I_0$, $u \in A_i$ and $x \in C$, and the conditions

$$v \le \bar{f}(y), \quad u \le f(x), \quad x \in M, \quad y \in N$$

are equivalent. On the other hand, if $v \in J_1(B)$, then $v=\beta_i(w,x)$ for some $i \in I_1$, $w \in W_i$ and $x \in C$, and the conditions

$$v \le \bar{f}(y), \quad \alpha_i(w,y) \le f(x), \quad x \in M, \quad v \in J_{1,M}(B)$$

are equivalent.

From (1) it follows that $\bar{f}$ is constant on each of the sets N and $D-N$, that is $\bar{f}$ is of the form dd', and (1) also shows that the formulas for $J(d)$ and $J(d')$ hold. By symmetry, $\varphi^{-1}(01)$ is of the form cc', and the formulas for $J(c)$ and $J(c')$ hold.

Next, letting $g=cc$ and $h=c'c'$, we claim that

$$(2)\qquad J(\bar{g}(y)) = \begin{cases} \varnothing & \text{for } y \in D-N \\ J_1(B) & \text{for } y \in N \end{cases}$$

$$(3)\qquad J(\bar{h}(y)) = \begin{cases} J_0(B) & \text{for } y \in D-N \\ J(B) & \text{for } y \in N \end{cases}$$

To prove this it suffices to show that for $v \in J_0(B)$, $v \le \bar{g}(y)$ never holds, but $v \le \bar{h}(y)$ always holds, and that for $v \in J_1(B)$ each of the inclusions $v \le \bar{g}(y)$, $v \le \bar{h}(y)$ holds just in case $y \in N$. First, for $v \in J_0(B)$ we have $v=\alpha_i(u)$ and $y=\gamma_i(x)$, where $i \in I_0$, $u \in A_i$ and $x \in C$, and the inclusions $v \le \bar{g}(y)$ and $v \le \bar{h}(y)$ are equivalent to $u \le c$ and $u \le c'$, respectively, but from the formulas for $J(c)$ and $J(c')$ we see that $u \le c$ holds for no $u \in J_0(A)$, while $u \le c'$ always holds. On the other hand, if $v \in J_1(B)$, then $v=\beta_i(w,x)$ for some $i \in I$, $w \in W_i$ and $x \in C$, and again consulting the formulas for $J(c)$ and $J(c')$, we see that the conditions

$$v \le \bar{g}(y), \quad \alpha_i(w,y) \le c, \quad y \in N$$

are equivalent, and also the conditions

$$v \le \bar{h}(y), \quad \alpha_i(w,y) \le c', \quad y \in N\,.$$

From (2) and (3) we infer that $\varphi(cc)$ and $\varphi(c'c')$ are of the form $0d_1$ and $d_0 1$, respectively, and that the indicated formulas for $J(d_0)$ and $J(d_1)$ hold. By symmetry, $\varphi^{-1}(dd)$ and $\varphi^{-1}(d'd')$ are of the form $0c_1$ and c_01, respectively, and the indicated formulas for $J(c_0)$ and $J(c_1)$ hold. This completes the proof of the lemma.

The elements $c, c', \ldots, d_0, d_1$ in Lemma 2 are obviously unique. The notation introduced there will be in effect throughout this section.

COROLLARY 3. *The elements* $0, c, c', c_0, c_1, 1$ *in* A *and* $0, d, d', d_0, d_1, 1$ *in* B *form lattices that are homomorphic image of the lattices in Figure* 1.

PROOF. It is clear that, for any elements $a, a', a'' \in A$, $a = a' \wedge a''$ iff $J(a) = J(a') \cap J(a'')$, and that $J(a) = J(a') \cup J(a')$ implies $a = a' \vee a''$. From this and the corresponding statement about B we see that all the joins and meets indicated in the figure are correct.

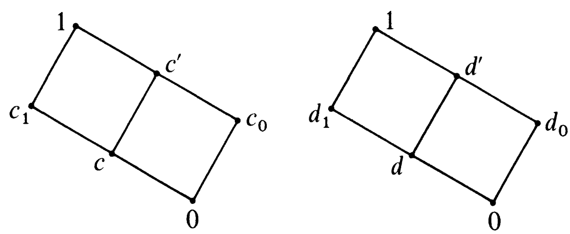

Fig. 1

LEMMA 4. *The functions listed below are mapped as indicated*:

f	$\bar{f}$
11	11
$c_1 1$	$d_1 1$
$c'1$	$d'1$
$c_1 c_1$	$d_1 d_1$
$c1$	$d1$
$c'c'$	$d_0 1$

f	$\bar{f}$
$c_0 1$	$d'd'$
cc_1	dd_1
cc'	01
01	dd'
$c_0 c'$	$d_0 d'$
cc	$0d_1$

f	$\bar{f}$
$0c_1$	dd
$0c'$	$0d'$
$c_0 c_0$	$d_0 d_0$
$0c$	$0d$
$0c_0$	$0d_0$
00	00

PROOF. We first observe that, according to Lemma 2, φ maps the functions

$$01, \quad cc', \quad 0c_1, \quad cc, \quad c_01, \quad c'c'$$

onto

$$dd', \quad 01, \quad dd, \quad 0d_1, \quad d'd', \quad d_01\ ,$$

respectively. Letting $f = c_0 c_0$ and $g = c_1 c_1$, we easily check that $J(\bar{f}(y)) = J_0(B)$ and $J(\bar{g}(y)) = J_1(B)$, whence $\bar{f} = d_0 d_0$ and $\bar{g} = d_1 d_1$. Of course φ maps 00 onto 00 and 11 onto 11. The remaining eight entries in the table can be easily verified by expressing the functions involved as joins and meets:

$$\begin{aligned} c_1 1 &= c_1 c_1 \vee 01, & d_1 1 &= d_1 d_1 \vee dd', \\ c'1 &= c'c' \vee c_0 1, & d'1 &= d_0 1 \vee d'd', \\ c1 &= cc' \vee 01, & d1 &= 01 \vee dd', \\ cc_1 &= cc \vee 0c_1, & dd_1 &= 0d_1 \vee dd, \\ c_0 c' &= c'c' \wedge c_0 1, & d_0 d' &= d_0 1 \wedge d'd', \\ 0c' &= 0c \vee 0c_0, & 0d' &= 0d \vee 0d_0, \\ 0c &= 0c_1 \wedge 0c', & 0d &= dd \wedge 0d', \\ 0c_0 &= 0c' \wedge c_0 c_0, & 0d_0 &= 0d' \wedge d_0 d_0. \end{aligned}$$

These joins and meets, which are suggested by Figure 2, are easily verified with the aid of Corollary 3.

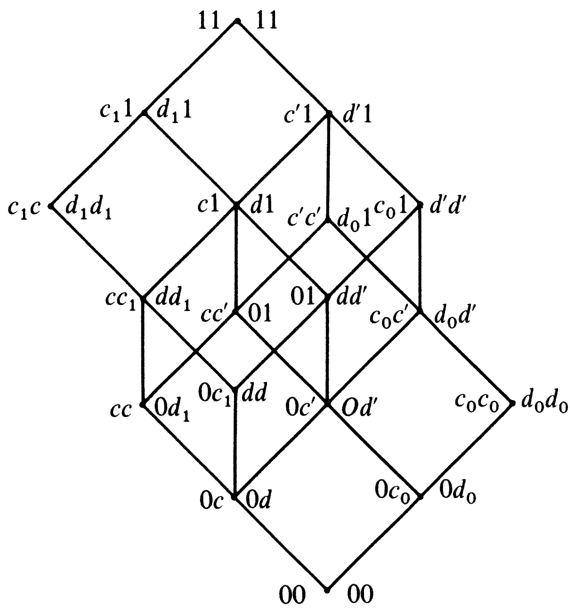

Fig. 2

LEMMA 5. *For any $a \in A$, and for $u = c, c', c_0, c_1$, the join $a \vee u$ and the meet $a \wedge u$ exist, and*

$$J(a \vee u) = J(a) \cup J(u).$$

PROOF. For any $f \in A^C$, the following conditions are easily seen to be equivalent:

$$f \vee g \quad \text{exists for all } g \in A^C ,$$
$$f(x) \vee a \quad \text{exists for all } a \in A \text{ and } x \in C .$$

Using this, the corresponding fact about B^D, and the fact that φ preserves joins, we see that the following joins exist for all $g \in A^C$, $a \in A$ and $b \in B$:

$$01 \vee \bar{g}, \quad cc' \vee g, \quad c \vee a, \quad c' \vee a ,$$
$$cc \vee g, \quad 0d_1 \vee \bar{g}, \quad d_1 \vee b ,$$
$$c'c' \vee g, \quad d_0 1 \vee \bar{g}, \quad d_0 \vee b .$$

By symmetry, $d \vee b$, $d' \vee b$, $c_1 \vee a$ and $c_0 \vee a$ exist. To prove the existence of the meets $a \wedge u$, we simply replace all joins by meets in the above argument.

To prove that

$$J(a \vee c) = J(a) \cup J(c) , \tag{1}$$

it suffices to show that, for any $u \in J(a) - J_{1,N}(A)$,

$$u \leq a \vee c \text{ implies } u \leq a .$$

For this purpose consider the functions

$$f = aa, \; g = cc', \; h = f \vee g = (a \vee c)(a \vee c') .$$

Then $\bar{g} = 01$, and therefore

$$\bar{h}(y) = \begin{cases} 1 & \text{for } y \in N \\ \bar{f}(y) & \text{for } y \in D - N . \end{cases}$$

First suppose $u \in J_0(A)$. Then $u \in A_i$ for some $i \in I_0$, and picking any $x \in C - M$, we see that the following conditions are equivalent:

$$u \leq a \vee c = h(x), \quad \alpha_i(u) \leq \bar{h}(\gamma_i(x)) = \bar{f}(\gamma_i(x)), \quad u \leq f(x) = a .$$

On the other hand, if $u \in J_1(A)$ (and of course $u \notin J_{1,N}(A)$), then $u = \alpha_i(w, y)$ for some $i \in I_1$, $w \in W_i$ and $y \in D - N$. Again picking $x \in C - M$, we see that the conditions

$$u \leq a \vee c = h(x), \quad \beta_i(w, x) \leq \bar{h}(y) = \bar{f}(y), \quad u \leq f(x) = a$$

are equivalent.

To prove that

$$J(a \vee c') = J(a) \cup J(c') , \tag{2}$$

it suffices to show that, for any $u \in J_1(A) - J_{1,N}(A)$,

$$u \le a \vee c' \text{ implies } u \le a .$$

Consider the same functions f, g and h as before. We have $u = \alpha_i(w, y)$ with $i \in I_1$, $w \in W_i$ and $y \in D - N$, and picking in this case $x \in M$, we see that the conditions

$$u \le a \vee c' = h(x), \quad \beta_i(w, x) \le \bar{h}(y) = \bar{f}(y), \quad u \le f(x) = a$$

are equivalent.

From (1) and (2) we obtain by symmetry

$$J(b \vee d) = J(b) \cup J(d) , \tag{3}$$

$$J(b \vee d') = J(b) \cup J(d') . \tag{4}$$

To prove that

$$J(a \vee c_0) = J(a) \cup J(c_0) , \tag{5}$$

it suffices to show that, for any $u \in J_1(A)$,

$$u \le a \vee c_0 \text{ implies } u \le a .$$

We have $u = \alpha_i(w, y)$ with $i \in I_1$, $w \in W_i$ and $y \in D$. Consider the functions

$$f = aa, \quad g = c_0 1, \quad h = f \vee g = (a \vee c_0)1 .$$

Then $\bar{g} = d'd'$, and therefore $\bar{h}(y) = \bar{f}(y) \vee d'$. Picking any $x \in C - M$, we see that the conditions

$$u \le a \vee c_0 = h(x), \quad \beta_i(w, x) \le \bar{h}(y) = \bar{f}(y) \vee d' ,$$
$$\beta_i(w, x) \le \bar{f}(y), \quad u \le f(x) = a$$

are equivalent. Here we have made use of (4) and the fact that $\beta_i(w, x)$ is not a member of $J(d')$, since $x \notin M$.

Finally, to prove that

$$J(a \vee c_1) = J(a) \cup J(c_1) ,$$

it suffices to show that, for all $u \in J_0(A)$,

$$u \le a \vee c_1 \text{ implies } u \le a .$$

In this case we consider the functions

$$f = 0a, \quad g = 0c_1, \quad h = f \vee g = 0(a \vee c_1) ,$$

noting that $\bar{g} = dd$, and therefore $\bar{h}(y) = \bar{f}(y) \vee d$ for all $y \in D$. We have $u \in A_i$ for some $i \in I_0$, and picking any $x \in M$, we make use of (3) and the fact that $\alpha_i(u) \notin J(d)$ to infer that the conditions

$$u \leq a \vee c_1 = h(x), \qquad \alpha_i(u) \leq \bar{h}(\gamma_i(x)) = \bar{f}(\gamma_i(x)) \vee d\,,$$
$$\alpha_i(u) \leq \bar{f}(\gamma_i(x)), \qquad u \leq f(x) = a$$

are equivalent. This completes the proof of the lemma.

LEMMA 6. *For any $a_0, a_1 \in A$, if $a_0 \leq c_0$ and $a_1 \leq c_1$, then $a_0 \vee a_1$ exists, and*

$$(a_0 \vee a_1) \wedge c_0 = a_0, \qquad (a_0 \vee a_1) \wedge c_1 = a_1\,.$$

PROOF. We first consider the case when $a_1 \leq c$, and we let $f = a_0 c_0$, $g = a_1 c$. Then

$$0c_0 \leq f \leq c_0 c_0 \quad \text{and} \quad g \leq cc\,,$$

and therefore

$$0d_0 \leq \bar{f} \leq d_0 d_0 \quad \text{and} \quad \bar{g} \leq 0d_1\,.$$

For $y \in D$, we therefore have $\bar{f}(y) = d_0$ if $y \in N$, but $\bar{g}(y) = 0$ if $y \notin N$, so that in either case $\bar{f}(y) \vee \bar{g}(y)$ exists. Consequently $\bar{f} \vee \bar{g}$ exists, and hence so does $f \vee g$. Since $c_0 \vee c = c'$, we infer that a_0 and a_1 have a least upper bound in the interval $[0, c']$. By symmetry, if $b_0 \leq d_0$ and $b_1 \leq d$, then b_0 and b_1 have a least upper bound in $[0, d']$.

Considering now arbitrary elements $a_0 \leq c_0$ and $a_1 \leq c_1$, we let $f = 0a_0$ and $g = 0a_1$. Then $f \leq 0c_0$ and $g \leq 0c_1$, and consequently $\bar{f} \leq 0d_0$ and $\bar{g} \leq dd$. Thus, for all $y \in D$, the elements $\bar{f}(y)$ and $\bar{g}(y)$ have a least upper bound in $[0, d']$. From this it follows that $\bar{f}$ and $\bar{g}$ have a least upper bound in $[00, d'd']$, and therefore f and g have a least upper bound in $[00, c_0 1]$. Obviously this implies that $a_0 \vee a_1$ exists.

By Lemmas 2 and 5,

$$J(a_0 \vee c_1) \cap J(c_0) = J(a_0)\,,$$

therefore $(a_0 \vee c_1) \wedge c_0 = a_0$, and consequently $(a_0 \vee a_1) \wedge c_0 = a_0$. Similarly, $(a_0 \vee a_1) \wedge c_1 = a_1$.

THEOREM 7. $A \cong [0, c_0] \cdot [0, c_1]$.

PROOF. The maps

$$\chi: \; a \to (a \wedge c_0, a \wedge c_1), \qquad \mu: \; (a_0, a_1) \to a_0 \vee a_1$$

from A to $[0, c_0] \cdot [0, c_1]$ and from $[0, c_0] \cdot [0, c_1]$ to A are well defined, and they are obviously isotone. Furthermore, $\lambda\mu$ is the identity map by Lemma 6, and to see that $\mu\lambda$ is the identity map we need only observe that

$$J(a \wedge c_0) \cup J(a \wedge c_1) = J(a)\,,$$

and hence $(a \wedge c_0) \vee (a \wedge c_1) = a$. The two maps are therefore isomorphisms.

4. The case $I_1 = \varnothing$.

If A is directly indecomposable, then it follows from Theorem 7 that either $c_0 = 0$ or $c_1 = 0$, and hence either $I_0 = \varnothing$ or $I_1 = \varnothing$. We show here that in the latter case the isomorphisms φ is of the form $f \to \xi \circ f \circ \gamma^{-1}$, where $\xi: A \cong B$ and $\gamma: C \cong D$.

LEMMA 8. *If $I_1 = \varnothing$, then $R(\varphi) = A$ and $R(\varphi^{-1}) = B$, and hence $\mathring{\varphi}: A \cong B$.*

PROOF. If $I_1 = \varnothing$, then $J(A) = J_0(A)$, and hence by Lemma 1 (v), $J(A) \subseteq R(\varphi)$. Since every member of A is the least upper bound of elements in $J(A)$, it follows that $R(\varphi) = A$. Similarly, $R(\varphi^{-1}) = B$, and it trivially follows that $\mathring{\varphi}: A \cong B$.

LEMMA 9. *If $I_1 = \varnothing$, and if A is directly imdecomposable, then all the γ_i's are equal.*

PROOF. Assuming that $I_1 = \varnothing$, and that the γ_i's are not all equal, we shall show that A has a non-trivial direct decomposition. Let S be the set of all elements $x \in C$ such that the elements $\gamma_i(x)$ are not all equal, and choose a maximal member s of S. Then all the isomorphisms γ_i agree on the filter $G = \{x \in C : x > s\}$ of C, and hence they all send G into the same filter $H = \gamma_i(G)$ of D. The set $T = \{\gamma_i(s) : i \in I\}$ has at least two elements, and $H \cup T$ is a filter of D. Letting

$$f = \langle 1[G], 0\rangle, \qquad g = \langle 1[G \cup \{s\}], 0\rangle\,,$$

we have $A \cong [f, g] \cong [\bar{f}, \bar{g}]$. It is easy to check that $\bar{f} = \langle 1[H], 0\rangle$, and that $\bar{g}(y) = 1$ for $y \in H$, and $\bar{g}(y) = 0$ for $y \in D - (H \cup T)$. Furthermore, the elements $b_t = \bar{g}(t)$ with $t \in T$ are all distinct from 0, for we have $t = \gamma_i(s)$ for some $i \in I$, and choosing $u \in A_i$, we see that $\alpha_i(u) \leq \bar{g}(t)$. Observe that the members of T are pairwise noncomparable, since they are the images of s under isomorphisms from C to D, and since C and D are assumed to satisfy the ascending chain condition. It follows that the interval $[\bar{f}, \bar{g}]$ in B^D consists of all functions h from D to B such that $h(y) = 1$ for $y \in H$, $h(y) = 0$ for $y \in D - (H \cup T)$, and $h(t) \leq b_t$ for $t \in T$. Thus A is isomorphic to the direct product of the intervals $[0, b_t]$ with $t \in T$.

THEOREM 10. *If $I_1 = \varnothing$ and A is directly indecomposable, then there exists an isomorphism γ from C to D such that*

$$\bar{f} = \mathring{\varphi} \circ f \circ \gamma^{-1} \quad \text{for all } f \in A^C .$$

PROOF. By the preceding two lemmes, $\mathring{\varphi}$ is an isomorphism from A to B, and all the γ_i's are equal to the same isomorphism γ from C to D. The map

$$\varphi'(f) = \mathring{\varphi} \circ f \circ \gamma^{-1}$$

is therefore an isomorphism from A^C to B^D. To show that φ and φ' are equal, we merely observe that they agree on the set

$$J(A^C) = \{j(a,x) : \ a \in J(A),\ x \in C\} ,$$

since for $i \in I$, $a \in A_i$, and $x \in C$ we have by (2.8) and Lemma 1 (vi),

$$\begin{aligned} \varphi j(a,x) &= j(\alpha_i(a), \gamma_i(x)) = j(\mathring{\varphi}(a), \gamma(x)) \\ &= \mathring{\varphi} \circ j(a,x) \circ \gamma^{-1} . \end{aligned}$$

5. The case $I_0 = \varnothing$.

In the notation of Section 7 of [JM], this is the case in which all the sets Z_i are trivial. By [JM, Corollary 7.8], the properties (φ, k) and (φ^{-1}, k), $k = 1,2,3,4$, therefore hold. The first part of the next theorem is just a restatement of these properties, and the second part is a more detailed version of Theorem 3.3 in [JM].

THEOREM 11. *Suppose $I_0 = \varnothing$. Then*

$$E = (R(\varphi), \leq_\varphi) \quad \text{and} \quad F = (R(\varphi^{-1}), \leq_{\varphi^{-1}})$$

are subposets of A and B, respectively, $\mathring{\varphi}$: $E \cong F$, and

$$\begin{aligned} \lambda : \ A \cong F^D, &\quad \text{where } \lambda(a) = \varphi(\langle a \rangle) , \\ \mu : \ B \cong E^C, &\quad \text{where } \mu(b) = \varphi^{-1}(\langle b \rangle) . \end{aligned}$$

Furthermore, the diagram

$$\begin{array}{ccc} A^C & \xrightarrow{\varphi} & B^D \\ {\scriptstyle \lambda^C}\downarrow & & \downarrow{\scriptstyle \mu^C} \\ (F^D)^C & \xrightarrow{\nu} & (E^C)^D \end{array}$$

commutes, where λ^C and μ^C are the isomorphisms induced by λ and μ, and for $h \in (F^D)^C$, $x \in C$ and $y \in D$,

$$\nu(h)(y)(x) = \mathring{\varphi}^{-1}(h(x)(y)) .$$

PROOF. By $(\varphi,4)$ and $(\varphi^{-1},4)$, E and F are subposets of A and B, respectively, by $(\varphi,2)$, $\mathring{\varphi}$ is an isomorphism from E to F, and by $(\varphi^{-1},3)$ and $(\varphi,3)$ we have λ: $A\cong F^D$ and μ: $B\cong E^C$.

The commutativity of the diagram is verified by direct calculations. Given $f\in A^C$, $x\in C$ and $y\in D$, we want to show that

$$\nu(\lambda^C(f))(y)(x) = \mu^C(\bar{f})(y)(x)$$

or, equivalently, that

$$\varphi(\langle f(x)\rangle)(y) = \mathring{\varphi}(\varphi^{-1}(\langle \bar{f}(y)\rangle)(x)) .$$

Letting

$$g = \langle f(x)\rangle, \quad \bar{h} = \langle \bar{f}(y)\rangle ,$$

we can write this as

$$\bar{g}(y) = \mathring{\varphi}(h(x)) .$$

Since the functions g and $\bar{h}$ are constant, the elements $\bar{g}(y)$ and $h(x)$ belong to $R(\varphi^{-1})$ and $R(\varphi)$, respectively. In general, for $a\in R(\varphi)$ and $b\in R(\varphi^{-1})$, the assertion that $\varphi(a)=b$ means that, for all $i\in I$, $w\in W_i$, $s\in C$, and $t\in D$, the inclusions $\alpha_i(w,t)\leq a$ and $\beta_i(w,s)\leq b$ are equivalent. Since, by Lemma 1 (iii), these two inclusions do not depend on s and t, we can fix these elements. Thus $\mathring{\varphi}(a)=b$ iff, for all $i\in I$ and $w\in W_i$, the inclusions $\alpha_i(w,y)\leq a$ and $\beta_i(w,x)\leq b$ are equivalent.

To apply this with $a=h(x)$ and $b=\bar{g}(y)$, we simply observe that the six inclusions

$$\alpha_i(w,y) \leq h(x), \quad \beta_i(w,x) \leq \bar{h}(y) ,$$

$$\beta_i(w,x) \leq \bar{f}(y), \quad \alpha_i(w,y) \leq f(x) ,$$

$$\alpha_i(w,y) \leq g(x), \quad \beta_i(w,x) \leq \bar{g}(y)$$

are equivalent. This completes the proof of the theorem.

6. The first induction.

The assumptions introduced at the beginning of Section 2 are no longer in effect.

THEOREM 12. *Suppose A is a bounded, directly indecomposable poset with $\nabla J(A)=A$, and C is a connected, directly indecomposable poset that satisfies the* *ascending chain condition, and suppose A is not isomorphic to X^C for any poset X. Then, for any natural number n, the natural map*

$$\mathrm{Aut}\,(A) \times \mathrm{Aut}\,(C^n) \to \mathrm{Aut}\,(A^{C^n})$$

is an isomorphism.

PROOF. The case $n=0$ is trivial. For $n=1$, the results in the preceding sections apply with $B=A$ and $D=C$. Given an automorphism φ of A^C, it follows from Theorem 7 that either $c_0=0$ or $c_1=0$, i.e., that either $I_0=\varnothing$ or $I_1=\varnothing$. Since A is not isomorphic to any power X^C, Theorem 11 rules out the possibility $I_0=\varnothing$. Hence $I_1=\varnothing$, and reference to Theorem 10 completes the proof for this case.

We now assume that the theorem holds for a given value $n \geq 1$, and for all smaller values, and show that it also holds with n replaced by $n+1$. To simplify the notation, we use the following convention: For $x \in C$ and $y \in C^n$, (x,y) is the juxtaposition of the sequences (x) and y, that is, if $y=(y_0, y_1, \ldots, y_{n-1})$, then $(x,y)=(x, y_0, y_1, \ldots, y_{n-1})$. Similarly, for $x, y \in C$ and $z \in C^{n-1}$, (x,y,z) is the juxtaposition of (x,y) and z.

Let $\varkappa$ be the isomorphism from $A^{C^{n+1}}$ to $(A^{C^n})^C$ such that, for $f \in A^{C^{n+1}}$, $x \in C$ and $y \in C^n$.

$$\varkappa(f)(x)(y) = f(x,y)\,.$$

Consider an automorphism φ of $A^{C^{n+1}}$, and let φ' be the induced automorphism of $(A^{C^n})^C$, i.e., the automorphism such that the diagram

$$\begin{array}{ccc} A^{C^{n+1}} & \xrightarrow{\varphi} & A^{C^{n+1}} \\ \varkappa\downarrow & & \downarrow\varkappa \\ (A^{C^n})^C & \xrightarrow{\varphi'} & (A^{C^n})^C \end{array}$$

commutes. We are going to apply the results from the preceding sections with φ replaced by φ', with A and B replaced by A^{C^n}, and with C and D replaced by C. By [JM, Corollary 6.6], $\nabla J(A^{C^n}) = A^{C^n}$, and the conditions (2.2) and (2.3) are therefore satisfied. Of course the auxiliary notions introduced in Sections 2 and 3 are now to be interpreted relative to φ' rather than to φ.

Since A is bounded and directly indecomposable, and since C^n is connected, it follows from [JM, Theorem 9.1] that A^{C^n} is directly indecomposable. Consequently, the direct decomposition of A^{C^n} given by Theorem 7 must be trivial, i.e., we must have either $I_0=\varnothing$ or $I_1=\varnothing$.

CASE 1. $I_1=\varnothing$. By Theorem 10 there exist automorphisms η of A^{C^n} and δ of C such that

$$\varphi'(g) = \eta \circ g \circ \delta^{-1} \quad \text{for all } g \in (A^{C^n})^C$$

and by the inductive hypothesis there exist automorphisms ξ of A and σ of C^n such that $\eta(h) = \xi \circ h \circ \sigma^{-1}$ for all $h \in A^{C^n}$. For $f \in A^{C^{n+1}}$, $x \in C$ and $y \in C^n$, we therefore have

$$\begin{aligned} \varphi(f)(x,y) &= \varkappa(\varphi(f))(x)(y) \\ &= \varphi'(\varkappa(f))(x)(y) \\ &= (\eta \circ \varkappa(f) \circ \delta^{-1})(x)(y) \\ &= \eta(\varkappa(f)(\delta^{-1}(x)))(y) \\ &= (\xi \circ \varkappa(f)(\delta^{-1}(x)) \circ \sigma^{-1})(y) \\ &= \xi(\varkappa(f)(\delta^{-1}(x))(\sigma^{-1}(y))) \\ &= \xi(f(\delta^{-1}(x), \sigma^{-1}(y))) . \end{aligned}$$

Thus $\varphi(f) = \xi \circ f \circ \gamma^{-1}$, where γ is the automorphism of C^{n+1} that takes (x,y) into $(\delta(x), \sigma(y))$ for all $x \in C$ and $y \in C^n$.

CASE 2. $I_0 = \varnothing$. We now apply Theorem 11. By [JM, Theorem 8.2], the posets E and F are isomorphic to $A^{C^{n-1}}$. Hence we obtain a commutative diagram

$$\begin{array}{ccc} (A^{C^n})^C & \xrightarrow{\varphi} & (A^{C^n})^C \\ {\scriptstyle \lambda^C}\downarrow & & \downarrow{\scriptstyle \mu^C} \\ ((A^{C^{n-1}})^C)^C & \xrightarrow{\nu} & ((A^{C^{n-1}})^C)^C \end{array}$$

where λ and μ are isomorphisms from A^{C^n} to $(A^{C^{n-1}})^C$, and ν is obtained from an automorphism ϱ of $A^{C^{n-1}}$ by letting

$$\nu(h)(x)(y) = \varrho(h(y)(x)) \quad \text{for } h \in ((A^{C^{n-1}})^C)^C,\ x,y \in C .$$

Let τ be the isomorphism from A^{C^n} to $(A^{C^{n-1}})^C$ such that

$$\tau(g)(x)(y) = g(x,y) \quad \text{for } x \in C,\ y \in C^{n-1} .$$

Then

$$\lambda = \tau \circ \lambda', \quad \mu = \tau \circ \mu' ,$$

where λ' and μ' are automorphisms of A^{C^n}. By the inductive hypothesis, there exist automorphisms ξ_λ, ξ_μ and ξ_ϱ of A, $\gamma_\lambda, \gamma_\mu$ of C^n, and γ_ϱ of C^{n-1} such that, for all $g \in A^{C^n}$ and $h \in A^{C^{n-1}}$,

$$\lambda'(g) = \xi_\lambda \circ g \circ \gamma_\lambda^{-1}, \quad \mu'(g) = \xi_\mu \circ g \circ \gamma_\mu^{-1}, \quad \varrho(h) = \xi_\varrho \circ h \circ \gamma_\varrho^{-1} .$$

For $f \in A^{C^{n+1}}$, $x,y \in C$ and $z \in C^{n-1}$ we obtain by direct calculations

$$(\nu\circ\lambda^C\circ\varkappa)(f)(x)(y)(z) = \xi_\varrho\big(\xi_\lambda(f(y,\gamma_\lambda^{-1}(x,\gamma_\varrho^{-1}(z))))\big),$$

$$(\mu^C\circ\varkappa\circ\varphi)(f)(x)(y)(z) = \xi_\mu(\bar{f}(x,\gamma_\mu^{-1}(y,z))).$$

Letting

$$\xi = \xi_\mu^{-1}\circ\xi_\varrho\circ\xi_\lambda,$$

we therefore have

$$\bar{f}(x,\gamma_\mu^{-1}(y,z)) = \xi(f(y,\gamma_\lambda^{-1}(x,\gamma_\varrho^{-1}(z)))).$$

Thus, if we define the automorphisms δ and σ of C^{n+1} by

$$\delta(x,y,z) = (x,\gamma_\mu^{-1}(y,z)), \quad \sigma(x,y,z) = (y,\gamma_\lambda^{-1}(x,\gamma_\varrho^{-1}(z))),$$

then $\bar{f}(\delta(x,y,z))=\xi(f(\sigma(x,y,z)))$, and letting $\gamma=\delta\circ\sigma^{-1}$, we conclude that $\bar{f}=\xi\circ f\circ\gamma^{-1}$.

The theorem follows by induction on n.

7. The second induction.

The following simple lemma enables us to extend the preceding theorem to a much larger class of exponents.

LEMMA 13. *For any posets A, C and D, if the natural maps*

$$\mathrm{Aut}\,(A)\times\mathrm{Aut}\,(C)\to\mathrm{Aut}\,(A^C),$$

$$\mathrm{Aut}\,(A^C)\times\mathrm{Aut}\,(D)\to\mathrm{Aut}\,((A^C)^D)$$

are isomorphisms, then so is the natural map

$$\mathrm{Aut}\,(A)\times\mathrm{Aut}\,(C\cdot D)\to\mathrm{Aut}\,(A^{C\cdot D}).$$

PROOF. Given an automorphism φ of $A^{C\cdot D}$, the induced automorphism φ' of $(A^C)^D$ is, by hypothesis, of the form $\varphi'(g)=\eta\circ g\circ\delta^{-1}$, where η and δ are automorphisms of A^C and of D, respectively. Also by hypothesis, η is of the form $\eta(h)=\xi\circ h\circ\sigma^{-1}$, where ξ and σ are automorphisms of A and C, respectively. Easy calculations show that $\varphi(f)=\xi\circ f\circ\gamma$, where $\gamma(x,y)=(\sigma(x),\delta(y))$.

THEOREM 14. *Suppose A is a bounded, directly and exponentially indecomposable poset with $\nabla J(A)=A$, and C is a connected, finitely factorable poset that satisfies the ascending chain condition. Then the natural map*

$$\mathrm{Aut}\,(A) \times \mathrm{Aut}\,(C) \to \mathrm{Aut}\,(A^C)$$

is an isomorphisms.

PROOF. We have

$$C \cong C_1^{n_1} C_2^{n_2} \dots C_k^{n_k},$$

where $C_1, C_2, \dots, C_k$ are pairwise non-isomorphic, directly indecomposable posets. For $k=1$, the conclusion holds by Theorem 12. We therefore assume that $k>1$, and that the theorem holds for all smaller values.

Let

$$C' = C_1^{n_1} C_2^{n_2} \dots C_{k-1}^{n_{k-1}}.$$

By the inductive hypothesis, the natural map

$$\mathrm{Aut}\,(A) \times \mathrm{Aut}\,(C') \to \mathrm{Aut}\,(A^{C'})$$

is an isomorphism. By [JM, Theorem 9.1 and Corollary 6.6], the poset $A^{C'}$ is directly indecomposable and satisfies the condition $\nabla J(A^{C'}) = A^{C'}$, and from [JM, Theorem 8.1] we see that $A^{C'}$ is not isomorphic to a poset of the form X^{C_k}. Hence, by Theorem 12, the natural map

$$\mathrm{Aut}\,(A^{C'}) \times \mathrm{Aut}\,(C_k^{n_k}) \to \mathrm{Aut}\,((A^{C'})^{C_k^{n_k}})$$

is an isomorphism. Reference to the preceding lemma completes the proof.

8. The proof of the Principal Theorem.

Under the hypothesis of the Principal Theorem, A is obviously bounded and directly and exponentially indecomposable, and C is connected. Also, $\nabla J(A) = A$ by [JM, Corollary 6.3. (v)]. Furthermore, from the fact that A^C satisfies the descending chain condition it follows that C satisfies the ascending chain condition, for the map $x \to j(1,x)$ is an embedding of C^δ into A^C. In fact, every strictly decreasing sequence of filters in C must be finite, for if the sequence of filters G_n is strictly decreasing, then the sequence of functions $\langle 1[G_n], 0\rangle$ is strictly decreasing. Using this, we will show that C is finitely factorable, thereby completing the check of the hypotheses of Theorem 14.

Suppose C is not finitely factorable. Let $C_0 = C$. Then C_0 has a non-trivial factorization λ_0: $C \cong D_0 \cdot C_1$, and one of the factors, say C_1, is not finitely factorable. Therefore C_1 has a non-trivial factorization λ_1: $C_1 \cong D_1 \cdot C_2$, with C_2 not finitely factoriable. Continuing in this manner, we obtain isomorphisms λ_n: $C_n \cong D_n \cdot C_{n+1}$ for $n = 2, 3, \dots$ such that D_n is non-trivial and C_{n+1} is not finitely factorable. Choosing $d_n, d'_n \in D_n$ with $d_n \not\leq d'_n$, associate with each filter G in C_{n+1} the filter

$$G' = \{(x,y) \in D_n \cdot C_{n+1} : \ x \geq d_n, \ y \in C\} ,$$

in $D_n \cdot C_{n+1}$, and the filter $\mu_n(G) = \lambda_n^{-1}(G')$ in C_n. It is clear that μ_n is injective and order preserving, and that $\mu_n(C_{n+1})$ is a proper filter in C_n. Letting

$$H_n = \mu_0 \mu_1 \ldots \mu_n(C_{n+1}) ,$$

we conclude that the filters $H_0, H_1, \ldots$ of C form a strictly decreasing sequence.

Reference to Theorem 14 completes the proof.

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