MATH. SCAND. 51 (1982), 151–157

COMPARISON OF THE LENGTH OF INFINITE GEODESICS IN MANIFOLDS WITH NONPOSITIVE CURVATURE

THÓRDUR JÓNSSON

Abstract.

Let M be a simply connected complete Riemannian manifold of nonpositive curvature. Let M^∞ be the points at infinity. Suppose there is a unique geodesic in M joining any two points in M^∞. An even number of distinct points in M^∞, $\{\alpha_1,\ldots,\alpha_{2m}\}$, can be joined pairwise by geodesics in $(2m-1)!!$ different ways. We show that there is a canonical way of defining the difference in length of two such geodesic configurations. This difference is a continuous function of $\alpha_1,\ldots,\alpha_{2m}$ in the cone topology and is invariant under asymptotic extensions to M^∞ of isometries on M. We compute this difference explicitly for the Poincaré disc.

1. Preliminaries.

Let M be a simply connected complete Riemannian manifold of nonpositive sectional curvature K. We denote the geodesic distance between two points $x,y \in M$ by $d(x,y)$. All geodesics in M are parametrized by arclength. If γ_1 and γ_2 are two geodesics in M, we say they are asymptotic, written $\gamma_1 \sim \gamma_2$, if there is a number $c>0$ such that $d(\gamma_1(t),\gamma_2(t))<c$ for all $t>0$. Let M^∞ be the collection of all geodesics in M modulo the equivalence relation $\sim$. We call M^∞ the points at infinity and denote $M \cup M^\infty$ by $\bar{M}$. If γ is a geodesic in M, the equivalence class of γ in M^∞ is denoted $[\gamma]$. We say that γ begins at $\alpha \in M^\infty$ and ends at $\omega \in M^\infty$ (or joins α with ω), if $[\gamma]=\omega$ and $[\bar{\gamma}]=\alpha$, where $\bar{\gamma}$ denotes γ with reversed orientation.

Let $x \in M$ and let γ be a geodesic in M. We shall need the following two facts:

1) There is a unique geodesic through x asymptotic to γ.

2) The limit

$$\lim_{t\to\infty} [d(x,\gamma(t))-t] \equiv b_\gamma(x) \tag{1}$$

Received June 1, 1981.

exists and is continuous in x. For a proof see e.g. Busemann [1 III, § 4].

We shall assume that our manifold M satisfies Axioms 1 and 2 [3], i.e. if α and ω are two distinct points in M^∞, there is a geodesic γ, unique up to parametrization, joining α with ω. A sufficient condition for the existence of γ is $K<C$, where C is a negative constant [2]. Uniqueness holds under weaker conditions, e.g. $K<0$ [3].

2. Comparison of Geodesic Configurations.

Let $A=\{\alpha_1,\alpha_2,\ldots,\alpha_{2m}\}$ be a collection of an even number of distinct points in M^∞. The points in A can be joined pairwise by geoedesics in precisely $(2m-1)!!$ ways. We are interested in comparing the lengths of the geodesics, albeit infinite, in two such configurations.

We denote the family of all pairings of $\{1,2,\ldots,2m\}$ by P_{2m} and regard each element of P_{2m} as a collection of m pairs. If α_p and α_q are two distinct points in A, let γ_{pq} be a geodesic beginning at α_p and ending at α_q, with the convention $\gamma_{pq}(0)=\gamma_{qp}(0)$.

Let γ be a geodesic in M. From (1) we see that it is natural to regard $b_\gamma(x)$ as the difference of the distance from x to $[\gamma]$ in M^∞ and the distance from $\gamma(0)$ to $[\gamma]$, both of which are infinite. This motivates the following definition. For $\tau,\eta \in P_{2m}$, let

$$D(\tau,\eta)(\alpha_1,\ldots,\alpha_{2m}) \equiv \sum_{j=1}^{2m} b_{\gamma_{kj}}(\gamma_{ij}(0)) , \tag{2}$$

where τ pairs i with j and η pairs j with $k, j=1,2,\ldots,2m$. The quantity $D(\tau,\eta)(\alpha_1,\ldots,\alpha_{2m})$ has an interpretation as the difference in length of all the γ_{ij} geodesics and all the γ_{kj} geodesics.

PROPOSITION 1. *$D(\tau,\eta)(\alpha_1,\ldots,\alpha_{2m})$ is independent of the choice of geodesics γ_{pq}.*

PROOF. The geodesics γ_{pq} are unique up to parametrization, and since they have unit speed, they are unique up to a translation. If γ a geodesic, let δ be its translate by t_0, i.e. $\delta(t)=\gamma(t+t_0)$. Then

$$b_\delta(x) = b_\gamma(x)+t_0 \tag{3}$$

for any $x \in M$. Similarly, if γ_1 and γ_2 are asymptotic,

$$b_{\gamma_1}(\gamma_2(t_0)) = b_{\gamma_1}(\gamma_2(0))-t_0 \tag{4}$$

for any $t_0 \in \mathrm{R}$. Let δ_{pq} be the translate of γ_{pq} by t_{qp} and $\delta_{pq}(0)=\delta_{qp}(0)$.

Then $t_{pq}=-t_{qp}$ and

(5) $$b_{\delta_{ij}}(\delta_{kj}(0)) = b_{\gamma_{ij}}(\gamma_{kj}(0))+t_{ij}-t_{kj} .$$

Summing (5) over j yields the desired result.

3. Properties of $D(\tau,\eta)$.

For any $x \in M$, $y \in \bar{M}$, $x \neq y$, let γ_{xy} be the unique geodesic in M joining x with y, such that $\gamma_{xy}(0)=x$. Let $x \in M$, $y,z \in \bar{M}$ be three distinct points. We define the angle subtended by y and z at x as the angle between the tangent vectors $\gamma'_{xy}(0)$ and $\gamma'_{xz}(0)$ in M_x, the tangent space to M at x. We denote this angle by $\sphericalangle_x(y,z)$. If v is a unit vector in M_x and γ a geodesic such that $\gamma'(0)=v$, let

(6) $$C(v,\varepsilon) = \{y \in \bar{M} \mid \sphericalangle_x([\gamma],y)<\varepsilon\} .$$

The set $C(v,\varepsilon)$ is called a cone with vertex x, axis v and angle ε. The following result is due to Eberlein and O'Neill [3].

PROPOSITION 2. *There is a unique topology T on $\bar{M}$ such that*

1) *The topology induced on M by T is the Riemannian topology, and M is a dense open subset of $\bar{M}$.*

2) *If γ is a geodesic in M, then its asymptotic extension is continuous,* i.e. $\lim_{t\to\infty}\gamma(t)=[\gamma]$.

3) *If φ is an isometry of M, then its asymptotic extension is a homeomorphism.*

4) *For each $\alpha \in M^\infty$, the family of cones containing α form a local basis for T at α.*

Let Λ_{2m} denote the set

$$\{(\alpha_1,\ldots,\alpha_{2m}) \mid \alpha_i \in M^\infty, \quad i=1,\ldots,2m,\ \alpha_i\neq\alpha_j \text{ if } i\neq j\}$$

equipped with the topology induced by the cone topology.

PROPOSITION 3. *Let $\tau,\eta \in P_{2m}$. The function*

(7) $$D(\tau,\eta) : \Lambda_{2m} \to \mathbf{R}$$

is continuous and antisymmetric in τ and η. If φ is an isometry of M and $\bar{\varphi}$ its asymptotic extension, then

(8) $$D(\tau,\eta)(\bar{\varphi}(\alpha_1),\ldots,\bar{\varphi}(\alpha_{2m})) = D(\tau,\eta)(\alpha_1,\ldots,\alpha_{2m})$$

for any $(\alpha_1,\ldots,\alpha_{2m}) \in \Lambda_{2m}$.

PROOF. Let SM be the unit tangent bundle of M and let $\pi: SM \to M$ be the projection. If $v \in SM$, let γ_v be the geodesic defined by $\gamma_v(0) = \pi(v)$, $\gamma_v'(0) = v$. The function

$$B: SM \times M \to \mathbf{R}$$

given by $B(v, x) = b_{\gamma_v}(x)$ is continuous [4]. We say that a sequence of geodesics γ_n converges to γ if γ_n have a reparametrization such that $\gamma_n'(0)$ converges to $\gamma'(0)$ in SM.

Let now $\alpha, \omega \in M^\infty$, $\alpha \neq \omega$, and suppose $(\alpha_n, \omega_n) \to (\alpha, \omega)$. Let γ_n and γ be the geodesics joining α_n with ω_n and α with ω, respectively. Then γ_n converges to γ, see [3]. If now α_i^n is a sequence in M^∞ converging to α_i, $i = 1, 2, \ldots, 2m$, where $(\alpha_1, \ldots, \alpha_{2m}) \in \Lambda_{2m}$, then the geodesic joining α_i^n with α_j^n, $i \neq j$, converges to γ_{ij}. The continuity of $D(\tau, \eta)$ follows.

The antisymmetry of $D(\tau, \eta)$ is clear since

$$b_{\gamma_{ij}}(\gamma_{kj}(0)) = -b_{\gamma_{kj}}(\gamma_{ij}(0)) .$$

Let φ be an isometry. Then $\varphi \circ \gamma_{ij}$ is geodesic joining $\bar{\varphi}(\alpha_i)$ with $\bar{\varphi}(\alpha_j)$ and

$$b_{\varphi \circ \gamma}(\varphi(x)) = b_\gamma(x)$$

for any geodesic γ and $x \in M$. This proves (8).

4. Regular families.

We would like to give another definition of D which is more convenient in applications [5] and exhibits $D(\tau, \eta)(\alpha_1, \ldots, \alpha_{2m})$ as the limit of differences in length of finite segments of fixed geoedesics. Let $\{\Sigma_n\}_{n=1}^\infty$ be an increasing family of compact subsets of M such that

$$\bigcup_{n=1}^\infty \Sigma_n = M . \tag{9}$$

We say that such a family is *regular* if the following two conditions are met:

(i) For any geodesic γ in M, the intersection $\gamma \cap \Sigma_n$ is connected for all n sufficiently large.

(ii) If γ_1 and γ_2 are two asymptotic geodesics in M, then

$$\lim_{n \to \infty} d(\gamma_1(t_1^{(n)}), \gamma_2(t_2^{(n)})) = 0 , \tag{10}$$

where $t_i^{(n)}$ is the largest value of t for which

$$\gamma_i(t) \in \Sigma_n, \quad i = 1, 2 .$$

The existence of a regular family is not guaranteed without a further assumption. We say that the manifold M satisfies the zero axiom if the distance between any two asymptotic geodesics is zero. The conditions under which the zero axiom holds are discussed in detail in [3]. A sufficient condition for the zero axiom to hold is $K \leqq c < 0$.

LEMMA. *Suppose M satisfies the zero axiom. Let $p \in M$ and let Σ_n be the closed ball centered at p with radius n. Then $\{\Sigma_n\}_{n=1}^{\infty}$ is regular.*

PROOF. It is clear that $\{\Sigma_n\}$ satisfies (9), and $\Sigma_n \cap \gamma$ is connected for any geodesic γ since Σ_n is convex. Let γ_1 and γ_2 be two asymptotic geodesics, and let $t_i^{(n)}$ be as in (10), $i=1,2$. We denote $\gamma_1(t_1^{(n)})$ by x_n and $\gamma_2(t_2^{(n)})$ by y_n. Note that $d(\gamma_1, \gamma_2(t))$ is a convex [2] bounded function of $t \geqq 0$ and takes values arbitrarily close to 0. It follows that $d(\gamma_1, \gamma_2(t))$ decreases monotonically to zero. Let z_n be the unique point on γ_1 closest to y_n. By a suitable reparametrization of γ_1, we have $b_{\gamma_1}(p)=0$. Given $\varepsilon>0$, we can therefore choose N so large that for all $n \geqq N$

$$\begin{aligned} d(y_n, z_n) &< \varepsilon \\ |d(p, x_n) - d(x_n, \gamma_1(0))| &< \varepsilon \\ |d(p, z_n) - d(z_n, \gamma_1(0))| &< \varepsilon\,. \end{aligned}$$

It follows that $d(x_n, y_n) < 4\varepsilon$ and the proof is complete.

Let us now assume that there exists a regular family $\Phi = \{\Sigma_n\}$ in M. Let $(\alpha_1, \ldots, \alpha_{2m})$ be a fixed point in Λ_{2m}, $\tau, \eta \in P_{2m}$ and define

$$D_n(\Phi; \tau, \eta) = \sum_{(j,k)\in\tau} |\gamma_{jk} \cap \Sigma_n| - \sum_{(k,l)\in\eta} |\gamma_{kl} \cap \Sigma_n|\,, \tag{11}$$

where $|\gamma \cap \Sigma_n|$ denotes the length of the geodesic segment $\gamma \cap \Sigma_n$.

PROPOSITION 4. *Let $\Phi = \{\Sigma_n\}$ be a regular family of compact subsets of M. Let τ and η be two pairings of $\{1, \ldots, 2m\}$. Then*

$$\lim_{n\to\infty} D_n(\Phi; \tau, \eta) = D(\tau, \eta)(\alpha_1, \ldots, \alpha_{2m})\,.$$

PROOF. Let n be so large that $\gamma_{pq}(0) \in \Sigma_n$ and $\gamma_{pq} \cap \Sigma_n$ is connected for any p, q. Let $t_{pq}^{(n)}$ be the largest value of t for which $\gamma_{pq}(t) \in \Sigma_n$. Let τ pair j with i and let η pair j with k. Then

$$D_n(\Phi; \tau, \eta) = \sum_{j=1}^{2m} [t_{ij}^{(n)} - t_{kj}^{(n)}]\,. \tag{12}$$

Note that

(13) $$|d(\gamma_{ij}(0),\gamma_{ij}(t_{ij}^{(n)}))-d(\gamma_{ij}(0),\gamma_{kj}(t_{kj}^{(n)}))| \leqq d(\gamma_{ij}(t_{ij}^{(n)}),\gamma_{kj}(t_{kj}^{(n)}))$$

and the right hand side of (13) tends to 0 as $n \to \infty$ by condition (ii) for the regular family Φ. Combining (12) and (13) gives the desired result.

5. Example. The Poincaré disc.

Let M be the Poincaré disc, i.e. the open unit disc in $\mathbf{R}^2$ with the metric

$$d^2s = (1-x^2-y^2)^{-2}(dx^2+dy^2).$$

The Riemannian manifold M has constant negative curvature and thus satisfies all our hypotheses above, i.e. axioms 1,2 and the zero axiom.

Let

$$\{\alpha_1,\alpha_2,\ldots,\alpha_{2m}\} \subset M^\infty, \qquad \alpha_i \neq \alpha_j$$

if $i \neq j$. To compute $D(\tau,\eta)$ we take a regular family e.g. $\{\Sigma_n\}$, where Σ_n is the closed disc of radius $(n-1)/n$. Designating points $(x,y) \in M$ by complex numbers $z=x+iy$, the distance from z_1 to z_2 is given by

(14) $$d(z_1,z_2) = \tfrac{1}{2}\log\frac{1+f(z_1,z_2)}{1-f(z_1,z_2)},$$

where

$$f(z_1,z_2) = \left|\frac{z_1-z_2}{1-\bar{z}_1z_2}\right|, \quad \text{see e.g. [6]}.$$

We can identify M^∞ with the complex numbers of modulus 1. A short calculation and Proposition 4 now give

(15) $$D(\tau,\eta)(\alpha_1,\alpha_2,\ldots,\alpha_{2m}) =$$
$$\sum_{(i,j)\in\tau} \log|\sin[\tfrac{1}{2}(\theta_i-\theta_j)]| - \sum_{(k,l)\in\eta} \log|\sin[\tfrac{1}{2}(\theta_k-\theta_l)]|,$$

where $\alpha_j=e^{i\theta_j}$, $j=1,2,\ldots,2m$. It is easy to see directly that (15) is invariant under the isometries of M, which are fractional linear transformations.

The problem of comparing the length of infinite geodesics arose in the study of solutions to a nonlinear elliptic partial differential equation in the Poincaré disc [5]. The corresponding action functional was estimated in terms of the length of geodesic segments and the lowest action solution corresponds to the 'shortest" geodesic configuation. This will be further discussed elsewhere.

REFERENCES

1. H. Busemann, *Metric methods in Finsler spaces and in the foundations of geometry*, Princeton University Press, Princeton, N.J., 1942.
2. R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Amer. Math. Soc. 145 (1969), 1–49.
3. P. Eberlein and B. O'Neill, *Visibility manifolds*, Pacific J. Math. 46 (1973), 45–109.
4. P. Eberlin, *Geodesic flows on negatively curved manifolds* II, Trans. Amer. Math. Soc. 178 (1973), 57–82.
5. T. Jónsson, *Merons and elliptic equations with infinite action*, Ph.D. thesis, Harvard University, 1980.
6. C. Carathéodory, *Conformal representation*, Cambridge University Press, Cambridge, 1963.

DIVISION OF MATHEMATICS
THE SCIENCE INSTITUTE
UNIVERSITY OF ICELAND
107 REYKJAVÍK
ICELAND