MATH. SCAND. 51 (1982), 163–178

CONDITIONS FOR TWO SELF-ADJOINT OPERATORS TO COMMUTE OR TO SATISFY THE WEYL RELATION

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E. Nelson has given a striking example of two self-adjoint operators X and Y on a separable Hilbert space which commute on a common core without having commuting spectral measures [3, § 10]. Other examples serving the same purpose were given later in [2], [7], [6], [5]. (I take this opportunity to correct a minor error in [2], where $-i(f,g)$ should be replaced by $i(f,g)$ in Theorem 1 and elsewhere — an error of no consequence.)

I show that the simpler version of Nelson's example adopted in [8] has the further property that the common core in question is a core for the *product* XY as well. This answers a question posed to me by P. Masani.

Following a suggestion by K. Schmüdken, I show more generally that if p denotes a polynomial of degree $\leqq 2$ in two variables and with real coefficients, then the above core is a core for the symmetric operator $p(X, Y)$ if and only if p is *non-elliptic* (Theorems 2 and 1).

Next I obtain a similar result for the Heisenberg commutation relation $PQ - QP = -i$ (Theorems 3 and 1).

For the polynomial $p(\xi, \eta) = \xi^2 + \eta^2$, Theorem 1 is due to Nelson [3, p. 603] in the commuting case, see also A. E. Nussbaum [4] for a simpler proof; and to J. Dixmier [1] in the case of the Heisenberg commutation relation.

I wish to thank Palle T. Jørgensen for useful comments.

1. Extension of theorems of Dixmier and Nelson.

Throughout the present paper, p will denote a polynomial of degree $\leqq 2$ in two variables and with real coefficients. If X and Y are symmetric operators on a Hilbert space having the same domain of definition, then $p(X, Y)$ denotes the symmetric operator obtained from $p(\xi, \eta)$ by replacing the variables ξ, η by the operators X, Y, respectively, with the understanding that the product $\xi\eta$, if it occurs in $p(\xi, \eta)$, shall give rise to the Jordan product $\frac{1}{2}(XY + YX)$, which is a symmetric operator.

Received February 23, 1981.

The polynomial p is called *elliptic* if the terms of degree 2 form a definite quadratic form, say (strictly) positive definite.

As mentioned in the introduction the case $p(\xi,\eta)=\xi^2+\eta^2$ of the following theorem is due to Nelson [3] (cf. Nussbaum [4]) in the case $\varepsilon=0$, and to Dixmier [1] in the case $\varepsilon=1$. Our proof will consist in a reduction to these known results.

THEOREM 1. *Let X_0, Y_0 be symmetric operators on a complex Hilbert space $\mathscr{H}$, with a common dense domain $\mathscr{D}$ invariant under both. Given $\varepsilon=0$ or 1, suppose that*

$$(X_0Y_0-Y_0X_0)u = -i\varepsilon u, \quad \text{all } u\in\mathscr{D} .$$

Finally, let p denote a real, elliptic, second degree polynomial in two variables, and suppose that the symmetric operator

$$p(X_0, Y_0)$$

is essentially self-adjoint. Then the closures X, Y of X_0, Y_0, respectively, are self-adjoint, and

$$e^{isX}e^{itY} = e^{i\varepsilon st}e^{itY}e^{isX}, \quad \text{all } s,t\in \mathbf{R} .$$

REMARK. The invariance of $\mathscr{D}$ may be replaced, as in [3] and [4], by the weaker hypothesis that $\mathscr{D}$ be contained in the domain of X_0^2, X_0Y_0, Y_0X_0, and Y_0^2.

PROOF OF THEOREM 1. Changing ad lib the constant term, the elliptic polynomial p may be given the form

$$p(\xi,\eta) = (a\xi+b\eta+k)^2+(c\xi+d\eta+m)^2 ,$$

where a, b, c, d, k, and m are real constants, and $ad-bc\neq 0$. We may assume that

$$ad-bc = 1 .$$

The symmetric operators

$$P_0 = aX_0+bY_0+k, \quad Q_0 = cX_0+dY_0+m , \tag{1}$$

then satisfy $P_0\mathscr{D}\subset\mathscr{D}$, $Q_0\mathscr{D}\subset\mathscr{D}$, and

$$(P_0Q_0-Q_0P_0)u = -i\varepsilon u, \quad \text{all } u\in\mathscr{D} ,$$

$$P_0^2+Q_0^2 = p(X_0, Y_0) .$$

By assumption this latter operator is essentially self-adjoint. It therefore follows from the quoted results by Dixmier and Nelson that P_0 and Q_0 have self-adjoint closures (=adjoints) P and Q such that

$$e^{isP}e^{itQ} = e^{i\varepsilon st}e^{itQ}e^{isP}, \quad \text{all } s,t \in \mathbf{R} .$$

Solving (1) for X_0, Y_0, we obtain

$$X_0 = \alpha P_0 + \beta Q_0 + \varkappa, \qquad Y_0 = \gamma P_0 + \delta Q_0 + \mu ,$$

where α, β, γ, δ, $\varkappa$, and μ are real constants, and

$$\alpha\delta - \beta\gamma = 1 .$$

Writing

$$Z_0 = -\beta P_0 + \alpha Q_0 ,$$

we obtain

$$(X_0 - \varkappa)^2 + Z_0^2 = (\alpha^2 + \beta^2)(P_0^2 + Q_0^2) ,$$

$$(X_0 - \varkappa)Z_0 - Z_0(X_0 - \varkappa) = -i\varepsilon(\alpha^2 + \beta^2) .$$

From the known case of Theorem 1 it therefore follows that the symmetric operator $(\alpha^2+\beta^2)^{-\frac{1}{2}}(X_0-\varkappa)$ is essentially self-adjoint, and hence X_0 and similarly Y_0 have self-adjoint closures X and Y, respectively.

In the case $\varepsilon=0$, P and Q are self-adjoint and commute. It follows by the operational calculus that $\alpha P + \beta Q + \varkappa$ ($\supset X_0$) and $\gamma P + \delta Q + \mu$ ($\supset Y_0$) have self-adjoint and commuting closures which extend, hence are equal to X and Y, respectively.

In the case $\varepsilon=1$, $\mathscr{H}$ decomposes into the Hilbert sum of minimal subspaces $\mathscr{H}^{(j)}$ reducing P and Q, hence reducing $\alpha P + \beta Q + \varkappa$ and $\gamma P + \delta Q + \mu$ as well. We proceed to show that these latter operators (extending X_0 and Y_0, respectively) have self-adjoint extensions which satisfy the Weyl relation, and which obviously extend, and hence are equal to X and Y, respectively.

For the stated purpose it suffices to work in each of the minimal reducing subspaces $\mathscr{H}^{(j)}$. We may therefore assume from the beginning that the solution P,Q to the Weyl relation is irreducible (in $\mathscr{H}$). By unitary equivalence we may further suppose that $\mathscr{H} = L^2(\mathbf{R})$, and that

$$P = -i\frac{d}{dx}, \qquad Q = x ,$$

both acting in the distribution sense on $L^2(\mathbf{R})$, the operator Q being multiplication by the independent variable x. If $\alpha \neq 0$ it follows that

$$\alpha P + \beta Q \subset \alpha U^{-1} P U ,$$

where U denotes the following unitary multiplication operator on $L^2(\mathsf{R})$:

$$U = \exp\left(\frac{i\beta}{2\alpha}x^2\right).$$

Similarly, if $\gamma \neq 0$,

$$\gamma P + \delta Q \subset \gamma V^{-1}PV,$$

where

$$V = \exp\left(\frac{i\delta}{2\gamma}x^2\right).$$

Summing up, if $\alpha \neq 0$ and $\gamma \neq 0$, then the closures of X_0 and Y_0 are the following self-adjoint operators X and Y, respectively:

$$X = \alpha U^{-1}PU + \varkappa, \qquad Y = \gamma V^{-1}PV + \mu .$$

Consequently,

$$e^{isX} = e^{is\varkappa}U^{-1}e^{is\alpha P}U, \qquad e^{itY} = e^{it\mu}V^{-1}e^{it\gamma P}V,$$

where $e^{is\alpha P}f(x) = f(x+s\alpha)$, etc. It is now straightforward to verify the Weyl relation for X, Y.

If, e.g., $\gamma = 0$, then $\alpha \neq 0$, and $Y = \delta Q + \mu$ is already self-adjoint with $e^{itY} = e^{it\mu}e^{it\delta x}$. Again the Weyl relation for X, Y is easily verified.

We proceed to show that the hypothesis in the above theorem that $p(X_0, Y_0)$ be essentially self-adjoint for at least one *elliptic* second degree polynomial p cannot be replaced by the essential self-adjointness of $p(X_0, Y_0)$ for *all non-elliptic* polynomials p of degree $\leqq 2$. For this purpose we shall use the version of Nelson's example adopted in [8].

2. The commuting case for non-elliptic p.

Nelson's example, as in [8, p. 273], uses the Riemann surface of the square root. In other words, let M denote the (non-trivial) two-fold cover of $\mathsf{R}^2 \setminus \{0\}$ endowed with the measure corresponding to Lebesgue measure, so that the projection pr: $M \to \mathsf{R}^2 \setminus \{0\}$ becomes locally measure preserving. By (x, y) we denote the generic point of $\mathsf{R}^2 \setminus \{0\}$, and also the standard local coordinates on M (instead of writing ($x \circ$pr, $y \circ$pr)). On the complex Hilbert space $\mathscr{H} = L^2(M)$ we have the self-adjoint operators

$$X = -i\partial/\partial x, \qquad Y = -i\partial/\partial y,$$

acting in the sense of distributions on M. They commute on the dense linear subspace $\mathscr{D}=C_0^\infty(M)$:

$$XYu = YXu \quad \text{for every } u \in \mathscr{D} .$$

Nelson showed that the restrictions X_0 and Y_0 of X and Y to $\mathscr{D}$ are essentially self-adjoint, but their closures X and Y do *not* have commuting spectral measures.

THEOREM 2. *The symmetric operator $S_0=p(X_0, Y_0)$ is essentially self-adjoint if (and only if) p is non-elliptic.*

PROOF. A linear bijection of $\mathbf{R}^2$ onto itself with determinant 1 induces a measure preserving bijection of M onto itself and hence in turn a unitary transformation of $\mathscr{H}=L^2(M)$ onto itself, leaving $\mathscr{D}$ invariant. In this way one easily reduces to the case where the principal part of the non-elliptic polynomial p has one of the forms $\xi\eta$, ξ^2, or ξ. In the last case we have $p(\xi)=\xi+c$ (c constant), and hence $S_0=X_0+c$, which is essentially self-adjoint because X_0 is so, as mentioned above. In the cases of degree 2 we apply a unitary transformation W of the form

$$Wu(x,y) = e^{i(ax+by)}u(x,y), \qquad u \in \mathscr{H} ,$$

where a and b are real constants. Then W maps $\mathscr{D}$ onto itself, and $WX_0W^{-1}=X_0-a$, $WY_0W^{-1}=Y_0-b$, whence

$$Wp(X_0, Y_0)W^{-1} = p(X_0-a,\ Y_0-b) .$$

This allows us to reduce $S_0=p(X_0, Y_0)$ to one of the standard forms (after a change of the constant term)

$$X_0Y_0 \quad \text{or} \quad X_0^2+cY_0 ,$$

where c is a (real) constant.

A. *The hyperbolic case* $S_0=X_0Y_0$ $(=Y_0X_0)$. The adjoint of $S_0=X_0Y_0$ is the operator

$$S = -\frac{\partial^2}{\partial x\partial y}$$

acting in the sense of distributions on M. To prove that S_0 is essentially self-adjoint we consider any $u \in \operatorname{dom} S$ (the domain of S) such that $Su=iu$, that is:

$$-\frac{\partial^2 u}{\partial x\partial y} = iu , \tag{2}$$

or equivalently: u should be orthogonal to the range of S_0+i. Since S_0 is real, it suffices to prove that any such vector u must be 0.

We begin by working in the upper half-plane

$$\mathbf{R}^2_+ = \{(x,y) \in \mathbf{R}^2 \mid y>0\}$$

in one of the sheets of M. Let v denote the partial Fourier transform of u in the x-direction:

$$v(\xi,y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbf{R}} u(x,y)e^{-ix\xi}\,dx\ ,$$

that is,

$$v(\cdot,y) = F^{-1}u(\cdot,y)\ ,$$

where generally F and F^{-1} denote the Fourier transforms on the Schwartz space $\mathscr{S}'=\mathscr{S}'(\mathbf{R})$ of tempered distributions on $\mathbf{R}$. Here $y>0$ is supposed to be such that $u(\cdot,y) \in L^2(\mathbf{R})$ (this holds for a.e. y). By Fubini's theorem and the Parseval identity, v is of class $L^2(\mathbf{R}^2_+)$. The differential equation (2) transforms into

$$\xi\frac{\partial v(\xi,y)}{\partial y} = -v(\xi,y)\ . \tag{3}$$

Let us show that this relation even holds for a.e. $\xi \in \mathbf{R}$ when $\partial v(\xi,y)/\partial y$ is interpreted as the distribution derivative of $v(\xi,\cdot) \in L^2(\mathbf{R}_+)$.

Let $\varphi\otimes\psi \in C_0^\infty(\mathbf{R}^2_+)$, i.e., $\varphi \in C_0^\infty(\mathbf{R})$ and $\psi \in C_0^\infty(\mathbf{R}_+)$. By assumption,

$$(u \mid (S_0+i)(\varphi\otimes\psi)) = 0\ ,$$

that is,

$$\iint_{\mathbf{R}^2_+} u(x,y)\overline{(-\varphi'(x)\psi'(y)+i\varphi(x)\psi(y))}\,dx\,dy = 0\ .$$

Taking partial Fourier transforms F^{-1} in the x-direction, and writing $\hat{\varphi} = F^{-1}\varphi$, we obtain by Fubini and Parseval:

$$\iint_{\mathbf{R}^2_+} v(\xi,y)\overline{(-i\xi\hat{\varphi}(\xi)\psi'(y)+i\hat{\varphi}(\xi)\psi(y))}\,d\xi\,dy = 0\ . \tag{4}$$

The functions w_1 and w_2 defined a.e. on $\mathbf{R}$ by

$$w_1(\xi) = \int_{\mathbf{R}_+} v(\xi,y)\overline{\psi(y)}\,dy, \qquad w_2(\xi) = \int_{\mathbf{R}_+} v(\xi,y)\overline{\psi'(y)}\,dy$$

are in $L^2(\mathbf{R})$ because $v(\cdot,\cdot)$ is a Hilbert–Schmidt kernel. The locally integrable function

$$w(\xi) = w_1(\xi) - \xi w_2(\xi)$$

is therefore a tempered distribution on $\mathbf{R}$: $w \in \mathscr{S}'$, and we get from (4)

$$\int_{\mathbf{R}} w(\xi)\overline{\hat{\varphi}(\xi)}\,d\xi = \int_{\mathbf{R}} (w_1(\xi) - \xi w_2(\xi))\overline{\hat{\varphi}(\xi)}\,d\xi = 0\,,$$

and hence

$$\langle Fw, \bar{\varphi}\rangle = \langle w, F\bar{\varphi}\rangle = 0$$

in the duality between $\mathscr{S}'(\mathbf{R})$ and $\mathscr{S}(\mathbf{R})$. Varying $\varphi \in C_0^\infty(\mathbf{R})$, we infer that $Fw = 0$, and hence $w = 0$ a.e. For almost every $\xi \in \mathbf{R}$, we thus have

$$\int_0^\infty v(\xi,y)\overline{(\psi(y) - \xi\psi'(y))}\,dy = 0\,.$$

Finally, put $\psi = \psi_n$, where the sequence $(\psi_n) \subset C_0^\infty(\mathbf{R}_+)$ is chosen so that the pairs (ψ_n, ψ_n') form a dense sequence in the graph of the operator d/dy acting on $C_0^\infty(\mathbf{R}_+)$, this graph being viewed as a subset of the separable Hilbert space $L^2(\mathbf{R}_+) \times L^2(\mathbf{R}_+)$. This allows us to conclude that there is a single null set on $\mathbf{R}$ outside of which the above relation holds for *all* $\psi \in C_0^\infty(\mathbf{R})$. And that just means that the function $v(\xi,\cdot) \in L^2(\mathbf{R}_+) \subset L_{\mathrm{loc}}(\mathbf{R}_+)$ satisfies (3) in the distribution sense on $\mathbf{R}_+$ for almost every $\xi \in \mathbf{R}$.

From (3) follows then that $\partial v(\xi,y)/\partial y$ likewise is of class $L_{\mathrm{loc}}(\mathbf{R}_+)$, and hence that $v(\xi,\cdot)$ is locally absolutely continuous on $\mathbf{R}_+$ for a.e. $\xi \in \mathbf{R}$. Consequently, $v(\xi,\cdot)$ is a solution in the classical sense to the ordinary differential equation (3), and so

$$(5_+) \qquad v(\xi,y) = \frac{f(\xi)}{\xi}e^{-y/\xi}, \qquad y > 0\,,$$

for a.e. $\xi \in \mathbf{R}$. In our case the arbitrary function f must be Lebesgue measurable (being equal to $v(\xi,y)\xi e^{y/\xi}$ for any fixed $y > 0$). Moreover,

$$\int_{\mathbf{R}} |v(\xi,y)|^2\,d\xi = \int_{\mathbf{R}} |f(\xi)|^2\xi^{-2}e^{-2y/\xi}\,d\xi$$

is integrable over $\mathbf{R}_+$ with respect to y, because $v \in L^2(\mathbf{R}_+^2)$. Consequently, by Fubini,

$$(6_+) \qquad \begin{cases} f(\xi) = 0 \quad \text{for a.e. } \xi < 0\,, \\ \displaystyle\int_{\mathbf{R}} |\xi|^{-1}|f(\xi)|^2\,d\xi < \infty\,, \end{cases}$$

and so in particular $f \in \mathscr{S}'(\mathbf{R}) \cap L^2_{\mathrm{loc}}(\mathbf{R})$.

Similarly, when working instead in the lower half-plane $\mathbf{R}^2_-$ in one of the sheets of M, we obtain for the partial Fourier transform v of u with respect to x for $y<0$:

$$(5_-)\qquad v(\xi,y) = \frac{g(\xi)}{\xi}e^{-y/\xi}, \quad y < 0\,,$$

where $g \in \mathscr{S}'(\mathbf{R})\cap L^2_{\mathrm{loc}}(\mathbf{R})$ and

$$(6_-)\qquad \begin{cases} g(\xi) = 0 \quad \text{for a.e. } \xi>0\,, \\ \int_{\mathbf{R}} |\xi|^{-1}|g(\xi)|^2\,d\xi < \infty\,. \end{cases}$$

We may assume that the upper and lower half-planes considered above fit together along the positive x-axis so as to belong to one and the same sheet of M. Now consider the *right* half-plane $x>0$ of that sheet, and choose this time $\varphi \in C_0^\infty(\mathbf{R}_+)$, $\psi \in C_0^\infty(\mathbf{R})$. Inserting (5) in the analogue of (4) in which we now integrate over the whole (ξ,y)-plane, we get from (5), (6)

$$\int_0^\infty\int_0^\infty f(\xi)\overline{\hat\varphi(\xi)}\frac{\partial}{\partial y}(e^{-y/\xi}\overline{\psi}(y))\,d\xi\,dy$$
$$+\int_{-\infty}^0\int_{-\infty}^0 g(\xi)\overline{\hat\varphi(\xi)}\frac{\partial}{\partial y}(e^{-y/\xi}\overline{\psi}(y))\,d\xi\,dy = 0\,.$$

Choosing ψ so that $\psi(0)\neq 0$, we thus obtain

$$\int_{\mathbf{R}}(f(\xi)-g(\xi))F\bar\varphi(\xi)\,d\xi = 0 \quad \text{for all } \varphi \in C_0^\infty(\mathbf{R}_+)\,.$$

The tempered distributions Ff and Fg on $\mathbf{R}$ thus satisfy

$$(7_+)\qquad Ff = Fg \quad \text{on } \mathbf{R}_+\,.$$

Let $\tau\colon M \to M$ denote the shift between sheets of M (i.e., for any $m \in M$, $\tau(m)$ and m are distinct, but have the same projection on $\mathbf{R}^2\setminus\{0\}$). Along with any solution $u \in L^2(M)$ to (2), $u\circ\tau$ is likewise a solution to (2). Since τ is involutory we have the decomposition

$$u = \tfrac{1}{2}(u+u\circ\tau)+\tfrac{1}{2}(u-u\circ\tau)$$

of an arbitrary solution u into the sum of two solutions, one invariant under τ, the other invariant except for a change of sign. It suffices therefore to show that any solution $u \in L^2(M)$ to (2) such that either $u\circ\tau=u$ or $u\circ\tau=-u$ must be 0.

Consider first the case $u\circ\tau=u$. Then u may be regarded as an element of $L^2(\mathbf{R}^2)$ solving (2) in $\mathbf{R}^2\setminus\{0\}$. The argument serving to establish (7_+) then shows that $Ff=Fg$ also on $\mathbf{R}_-$. (Alternatively, just replace $u(x,y)$ by $u(-x,$

$-y$), which likewise satisfies (2).) We have thus found that $F(f-g)=0$ in $\mathbf{R}\setminus\{0\}$, showing that $f-g$ is a polynomial, which however must be 0 because $|\xi|^{-\frac{1}{2}}(f(\xi)-g(\xi))$ is of class $L^2(\mathbf{R})$ by (6). It follows that $f=g$, and indeed that $f=g=0$ because $f=0$ on $\mathbf{R}_-$ and $g=0$ on $\mathbf{R}_+$, by (6).

In the remaining case $u\circ\tau=-u$ we get a change of sign in (5_-) when we now pass from the upper half-plane $\mathbf{R}^2_+$ (in the same sheet as before) to the lower half-plane $\mathbf{R}^2_-$ in the *opposite* sheet from before, while crossing this time the x-axis on its *negative* part. In other words, g should be replaced now by $-g$ in (5_-). The computation which led to (7_+) carries over in every other respect, when we work instead in the half-plane $x<0$ in the sheet just described, and we thus obtain:

$$(7_-) \qquad Ff = -Fg \quad \text{on } \mathbf{R}_- .$$

Since $f\in L^2_{\mathrm{loc}}(\mathbf{R})$ and $f=0$ on $\mathbf{R}_-$ by (6_+), the convolution $f*f$ is a well-defined, continuous, finite-valued function on $\mathbf{R}$, vanishing on $\mathbf{R}_-\cup\{0\}$:

$$(f*f)(x) = \int_0^x f(x-y)f(y)\,dy .$$

By the Cauchy–Schwarz inequality we have for $x>0$

$$\frac{1}{x}|(f*f)(x)| \leqq \frac{1}{x}\int_0^x |f(y)|^2\,dy = \int_0^x \frac{y}{x}h(y)\,dy$$

where $h(y):=y^{-1}|f(y)|^2$ is integrable over $\mathbf{R}$ by (6_+). It follows by the dominated convergence theorem that

$$(8) \qquad \frac{1}{x}(f*f)(x)\to 0, \quad \frac{1}{x}(g*g)(x)\to 0 \quad \text{as } |x|\to\infty$$

by a similar argument applied to g.

In order to exploit (7) we perform a regularization so as to overcome the difficulty that Ff and Fg are not known to be functions, but only tempered distributions. Choose $k\in C_0^\infty(\mathbf{R})$ so that $k\geqq 0$, $\operatorname{supp} k\subset[-1,1]$, and $\int k(x)\,dx=1$, and write $\sqrt{2\pi}F^{-1}k=\hat{k}$. Also put

$$k_n(x) = nk(nx), \qquad \hat{k}_n(\xi)=\hat{k}(\xi/n)$$

for $n\in\mathbf{N}$. Writing

$$f_n = \hat{k}_n f, \qquad g_n = \hat{k}_n g ,$$

we thus have

$$(9) \qquad Ff_n = k_n * Ff, \qquad Fg_n = k_n * Fg .$$

The finite and continuous functions $f*f$ and $g*g$ on R are tempered on account of (8), and

$$f_n*f_n \to f*f, \quad g_n*g_n \to g*g \quad \text{weakly in } \mathscr{S}' \tag{10}$$

by the dominated convergence theorem. From (7) and (9) we obtain

$$F(f_n*f_n) = \sqrt{2\pi}(Ff_n)^2 = \sqrt{2\pi}(Fg_n)^2 = F(g_n*g_n) \quad \text{in } \mathsf{R}\setminus[-1/n,1/n]\,,$$

since supp $k_n \subset [-1/n, 1/n]$. Inserting this in (10) we get

$$F(f*f) = F(g*g) \quad \text{in } \mathsf{R}\setminus\{0\}\,.$$

It follows that the continuous function $f*f-g*g$ on R is a polynomial. This polynomial must be a constant, on account of (8), and the constant must be 0 because $(f*f)(0)=0=(g*g)(0)$. Since $f*f=0$ on R_- and $g*g=0$ on R_+, we conclude that $f*f=g*g=0$ on R. It follows that $f=g=0$ by the Titchmarsh theorem as extended by Schwartz [9, theorem XIV]. Consequently, $v=0$, and finally $u=0$.

B. *The parabolic case* $S_0=X_0^2+cY_0$ ($c\in\mathsf{R}$). The adjoint of S_0 is

$$S = -\frac{\partial^2}{\partial x^2}-ic\frac{\partial}{\partial y}\,,$$

acting in the distribution sense on $\mathscr{H}=L^2(M)$. Now (2) is replaced by

$$-\frac{\partial^2 u}{\partial x^2}-ic\frac{\partial u}{\partial y} = iu\,. \tag{11}$$

(The case of the eigenvalue $-i$ for S is reduced to that of $+i$ when we replace y by $-y$ and u by $\bar{u}$.) The partial Fourier transform $v(\cdot,y)=F^{-1}u(\cdot,y)$ now satisfies

$$\xi^2 v(\xi,y)-ic\frac{\partial v(\xi,y)}{\partial y} = iv(\xi,y)\,. \tag{12}$$

If $c=0$ this gives at once $v=0$ in either half-plane $y>0$ or $y<0$, and hence $u=0$. We may therefore assume that $c=1$. The L^2-solutions of (12) for $y\neq 0$ have then the form

$$v(\xi,y) = f(\xi)\exp(-y-i\xi^2 y) \quad \text{for } y>0\,,$$

whereby $f\in L^2(\mathsf{R})$; and $v=0$ for $y<0$. Proceeding as in the hyperbolic case we easily get $Ff=0$ on $\mathsf{R}\setminus\{0\}$, hence $f=0$, $v=0$, $u=0$.

3. The Heisenberg case for non-elliptic p.

As to the Heisenberg commutation relation we take, following [8, p. 275],

$$P = -i\frac{\partial}{\partial x}, \quad Q = -i\frac{\partial}{\partial y}+x,$$

acting as operators on $\mathscr{H}=L^2(M)$ in the distribution sense (with M as in section 2). Here x stands for the operator of multiplication by the first coordinate x. Thus P and Q are self-adjoint, and we have, in terms of the operators X, Y from section 2,

$$P = X, \quad Q \supset Y+x,$$

noting that Y and x commute (in the strict sense). Clearly

$$(PQ-QP)u = -iu, \quad \text{all } u \in C_0^\infty(M).$$

The restrictions P_0, Q_0 of P, Q to $\mathscr{D}=C_0^\infty(M)$ are essentially self-adjoint, but P and Q do not satisfy the Weyl relation, see [8].

THEOREM 3. *The symmetric operator $T_0=p(P_0,Q_0)$ is essentially self-adjoint if (and only if) p is non-elliptic.*

A. *The hyperbolic case*:

$$p(\xi,\eta) = (a\xi+b\eta+k)(c\xi+d\eta+m)$$

with real constants a, b, c, d, k, and m such that $ad-bc\neq 0$, or equally well

$$ad-bc = 1.$$

Leaving out, for simplicity, the subscript 0 indicating restriction to $\mathscr{D}$, we thus have (on $\mathscr{D}$):

$$\begin{aligned}2T = {} & (aX+b(Y+x)+k)(cX+d(Y+x)+m) \\ & +(cX+d(Y+x)+m)(aX+b(Y+x)+k).\end{aligned}$$

The linear bijection $M \to M$, defined by

$$\begin{pmatrix}x\\y\end{pmatrix} \mapsto \begin{pmatrix}a & c\\ b & d\end{pmatrix}\begin{pmatrix}x\\y\end{pmatrix}, \tag{13}$$

induces a unitary transformation A on $\mathscr{H}=L^2(M)$ leaving $\mathscr{D}$ invariant and transforming $aX+bY$ into X $(=-i\partial/\partial x)$ and $cX+dY$ into Y $(=-i\partial/\partial y)$. Thus T is transformed into $\tilde{T}=ATA^{-1}$, given on $\mathscr{D}$ by

$$2\tilde{T} = (X+b(ax+cy)+k)(Y+d(ax+cy)+m) + (Y+d(ax+cy)+m)(X+b(ax+cy)+k) .$$

Finally we transform $\tilde{T}$ by a unitary multiplication operator of the form

$$U = e^{iq(x,y)} ,$$

where q is a real, second degree polynomial in two variables. Again $\mathscr{D}$ is left invariant, and

$$UXU^{-1} = X-\frac{\partial q}{\partial x}, \quad UYU^{-1} = Y-\frac{\partial q}{\partial y}$$

on $\mathscr{D}$. Taking

$$q(x,y) = \tfrac{1}{2}bax^2+bcxy+\tfrac{1}{2}dcy^2+kx+my ,$$

we obtain on $\mathscr{D}$, since $ad-bc=1$,

$$2U\tilde{T}U^{-1} = X(Y+x)+(Y+x)X = PQ+QP$$

The hyperbolic case has thus been reduced to the case $p(\xi,\eta)=\xi\eta$.

In order to treat this standard case in a manner parallel to that of Case A in Theorem 2, we interchange the coordinates x and y, and thus consider instead the operator T_0 given by

$$2T_0 = Y_0(X_0+y)+(X_0+y)Y_0$$

with the adjoint

$$T = T_0^* = -\frac{\partial^2}{\partial x\partial y}-iy\frac{\partial}{\partial y}-i/2$$

acting in the distribution sense on $L^2(M)$. To prove that T_0 is essentially self-adjoint, we consider for a given $\lambda \in \mathbf{R}\setminus\{0\}$, any $u \in \operatorname{dom} T$ such that $Tu=i\lambda u$, that is

$$-\frac{\partial^2 u}{\partial x\partial y}-iy\frac{\partial u}{\partial y} = i(\lambda+\tfrac{1}{2})u .$$

We shall prove that any such vector u must be 0.

Proceeding as in the proof of Case A in Theorem 2 we consider separately the upper and lower half-planes $\mathbf{R}^2_+$ and $\mathbf{R}^2_-$ in one of the sheets of M. The partial Fourier transform v of u in the x-direction satisfies

(14) $$(\xi+y)\frac{\partial v}{\partial y} = -(\lambda+\tfrac{1}{2})v .$$

The solutions $v \in L^2(\mathbf{R}^2_+)$ to (14) for $y>0$ may be written

$$(15_+) \qquad v(\xi,y) = \frac{f(\xi)}{\xi}\left|1+\frac{y}{\xi}\right|^{-\lambda-\frac{1}{2}} \quad \text{for } 0 < y < \begin{cases} \infty & \text{if } \xi>0\,, \\ -\xi & \text{if } \xi<0; \end{cases}$$

and $v=0$ elsewhere (for $y>0$). Here

$$(16_+) \qquad \begin{cases} f(\xi) = 0 \quad \text{for } \lambda\xi<0 \\ \int_{\mathbf{R}} |\xi|^{-1}|f(\xi)|^2\,d\xi < \infty\,. \end{cases}$$

The solutions $v \in L^2(\mathbf{R}^2_-)$ to (14) for $y<0$ may be written

$$(15_-) \qquad v(\xi,y) = \frac{g(\xi)}{\xi}\left|1+\frac{y}{\xi}\right|^{-\lambda-\frac{1}{2}} \quad \text{for } 0 > y > \begin{cases} -\infty & \text{if } \xi<0\,, \\ -\xi & \text{if } \xi>0; \end{cases}$$

and $v=0$ elsewhere (for $y<0$). Here

$$(16_-) \qquad \begin{cases} g(\xi) = 0 \quad \text{for } \lambda\xi > 0 \\ \int_{\mathbf{R}} |\xi|^{-1}|g(\xi)|^2\,d\xi < \infty\,. \end{cases}$$

Assuming that the upper and lower half-planes fit together along the positive x-axis so as to belong to one sheet of M, we proceed as in the proof of Theorem 2. Again we arrive at (7_+): $Ff=Fg$ on $\mathbf{R}_+$ (in either case $\lambda>0$ or $\lambda<0$). As before, we reduce to the case where $u\circ\tau=-u$, and in that case we have (7_-): $Ff=-Fg$ on $\mathbf{R}_-$. In view of (16) we conclude as before that $f=g=0$, and hence $v=0$, $u=0$.

B. *The parabolic case and the linear case*:

$$p(\xi,\eta) = (a\xi+b\eta+k)^2+c\xi+d\eta\,,$$
$$T_0 = (aX_0+bY_0+bx+k)^2+cX_0+dY_0+dx\,.$$

1°) $ad-bc\neq 0$, say $ad-bc=1$. The unitary operator induced by (13) as in Case A transforms T_0 into

$$\tilde{T}_0 = (X_0+b(ax+cy)+k)^2+Y_0+d(ax+cy)\,.$$

Next the unitary operator $U=e^{iq}$ with q as in Case A (now with $m=0$) transforms $\tilde{T}_0$ into the final form

$$U\tilde{T}_0U^{-1} = X_0^2+Y_0+x = P_0^2+Q_0\,,$$

corresponding to the standard parabolic polynomial $\xi^2+\eta$. We shall prove that this final operator $T_0=X_0^2+Y_0+x$ is essentially self-adjoint. Its adjoint is

$$T = T_0^* = -\frac{\partial^2}{\partial x^2} - i\frac{\partial}{\partial y} + x$$

acting in the distribution sense on $L^2(M)$. Let $u \in \operatorname{dom} T$ satisfy $Tu = iu$ or $Tu = -iu$, that is,

$$-\frac{\partial^2 u}{\partial x^2} - i\frac{\partial u}{\partial y} + xu = iu$$

or $-iu$. We shall deduce that $u = 0$. For this purpose it suffices to consider the case of the eigenvalue $+i$ for T, since this case is transformed into that of $-i$, when we replace y by $-y$ and u by $\bar{u}$.

Proceeding as in the proof of Theorem 2 we obtain for the partial Fourier transform v of u in the upper or lower halfplane in one of the sheets of M:

$$\xi^2 v - i\frac{\partial v}{\partial y} + i\frac{\partial v}{\partial \xi} = iv\ ,$$

the L^2-solutions of which are

$$v(\xi, y) = f(\xi + y)\exp\left(\xi + \frac{i}{3}\xi^3\right) \quad \text{for } y > 0\ , \tag{17}$$

and $v = 0$ for $y < 0$. Here f is subject to

$$\int_{\mathsf{R}} e^{2s}|f(s)|^2\, ds < \infty\ . \tag{18}$$

Working in the right or the left half-plane in the sheet continuing the upper half-plane just considered, we find, by the method leading to (7), for any $\varphi \in C_0^\infty(\mathsf{R}_+)$, respectively $\varphi \in C_0^\infty(\mathsf{R}_-)$, and for any $\psi \in C_0^\infty(\mathsf{R})$

$$\begin{aligned} 0 &= (u \mid (T+i)(\varphi \otimes \psi)) \\ &= (u(x,y) \mid -\varphi''(x)\psi(y) - i\varphi(x)\psi'(y) + (x+i)\varphi(x)\psi(y)) \\ &= (v(\xi,y) \mid \xi^2\hat{\varphi}(\xi)\psi(y) - i\hat{\varphi}(\xi)\psi'(y) + i\hat{\varphi}'(\xi)\psi(y) + i\hat{\varphi}(\xi)\psi(y))\ . \end{aligned}$$

Inserting (17) for $y > 0$, and $v = 0$ for $y < 0$, and substituting $y = s - t$, $\xi = t$, we obtain after a reduction

$$\begin{aligned} 0 &= -i\iint_{s>t} f(s)\frac{\partial}{\partial t}\left[\overline{\hat{\varphi}(t)\psi(s-t)\exp\left(t + \frac{i}{3}t^3\right)}\right] ds\, dt \\ &= -i\overline{\psi(0)}\int_{\mathsf{R}} f(s)\overline{\hat{\varphi}(s)}\exp\left(s + \frac{i}{3}s^3\right) ds\ . \end{aligned}$$

Choosing ψ so that $\psi(0) \neq 0$, we infer that

$$F\left(f(s)\exp\left(s+\frac{i}{3}s^3\right)\right) = 0 \quad \text{on } \mathbf{R}\setminus\{0\},$$

showing that the L^2-function $s \mapsto f(s)\exp(s+(i/3)s^3)$ (cf. (18)) is zero, and so $f=0$, $v=0$, $u=0$.

2°) $ad-bc=0$, $a\neq 0$; say $a=1$, hence $d=bc$. In place of (13) we now use the bijection $(x,y)\mapsto(x,bx+y)$ to obtain

$$\begin{aligned}\tilde{T}_0 &= (X_0+bx+k)^2+c(X_0+bx)\\ &= (X_0+bx+k+\tfrac{1}{2}c)^2+\text{constant}.\end{aligned}$$

Taking $U=e^{iq}$ with $q=\frac{1}{2}bx^2+(k+\frac{1}{2}c)x$, we obtain $UT_0U^{-1}=X_0^2+\text{constant}$. The real, symmetric operator X_0^2 is essentially self-adjoint because 0 is the only L^2 solution to $-\partial^2u/\partial x^2=iu$ in either half-plane $y>0$ or $y<0$.

3°) $ad-bc=0$, $a=0$, $b\neq 0$; say $b=1$. Thus $c=0$, and

$$T_0 = (Y_0+x+k)^2+d(Y_0+x).$$

Taking $U=e^{iq}$ with $q=xy+(k+(d/2))y$, we obtain $UT_0U^{-1}=Y_0^2+\text{constant}$, which is essentially self-adjoint just like X_0^2.

4°) $(a,b)=(0,0)$. Leaving out the constant k^2, we have

$$T_0 = cX_0+d(Y_0+x).$$

Since X_0 is known to be essentially self-adjoint, we may suppose that $d\neq 0$, say $d=1$. The linear bijection $(x,y)\mapsto(x+cy,y)$ transforms T_0 into

$$\tilde{T}_0 = Y_0+x+cy.$$

Taking $U=e^{iq}$ with $q=xy+\frac{1}{2}cy^2$, we obtain $U\tilde{T}_0U^{-1}=Y_0$, which is likewise known to be essentially self-adjoint.

4. The case of a polynomial p with complex coefficients.

One might ask what happens in the case of a second degree polynomial $p(\xi,\eta)$ with *complex* coefficients. Instead of asking for essential self-adjointness of $S_0=p(X_0,Y_0)$ (in Theorem 2) and of $T_0=p(P_0,Q_0)$ (in Theorem 3), one should now ask whether the closure of this minimal operator S_0, respectively T_0, is the corresponding maximal operator

$$S = p\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial y}\right), \qquad T = p\left(\frac{1}{i}\frac{\partial}{\partial x},\frac{1}{i}\frac{\partial}{\partial y}+x\right).$$

Unlike the situation with p real, the answer to this question cannot be expressed solely in terms of the principal part of p when p is complex. A simple example, pointed out to me by Palle T. Jørgensen, is the maximal operator $S = -(\partial/\partial x - 1)(\partial/\partial y - 1)$ on $L^2(M)$, corresponding to the non-elliptic polynomial $p(\xi, \eta) = (\xi + i)(\eta + i)$. This operator S is *not* the closure of its restriction to $\mathscr{D} = C_0^\infty(M)$ because $(X + i)(Y + i)\mathscr{D}$ is not dense in $L^2(M)$. Thus p behaves in the present context like an elliptic polynomial, although the principal part $\xi\eta$ of p is hyperbolic and covered by Theorem 2, being real.

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