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ON THE COMPOSITION FACTORS OF WEYL MODULES

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Introduction.

Let G be a connected reductive algebraic group over an algebraically closed field k of characteristic $p \neq 0$. Let B be a Borel subgroup of G and, T a maximal torus in B with character group $X(T)$. Assume that half the sum of the roots of B lies in $X(T)$. Fix the set of dominant weights in $X(T)$ with respect to the Borel subgroup opposite to B. In this paper we prove a result that generalizes [5, Satz 11] and is also closely related to [2, Theorem 3.1]. Let χ, λ be two dominant weights in facettes F, F', respectively. Assume that $\bar{F} \cap \overline{F'} \neq \emptyset$, $\lambda \uparrow \chi$ (cf. section 1) and that any weight ξ satisfying $\lambda \uparrow \xi \uparrow \chi$ is dominant. Let w be an element of the Weyl group W. Then the composition multiplicity of the simple module M_λ in $H^i(G/B, L(w \cdot \chi))$ is non-zero if and only if $i = \ell(w)$, and in this case it is independent of w. We derive this result by imitating the proof of [2, 3.1] and using certain facts from alcove geometry established by Jantzen [5].

1. Preliminaries.

Let G, B, and T be as above. Let R be the root system of G and R_+ the set of positive roots so that $-R_+$ is the root system for B. The corresponding set of simple roots is denoted by S and the set of dominant weights in $X = X(T)$ by X_+. In general, we follow the notation of [1] and [2] with one essential exception: the sign $\uparrow$ is used for the strong linkage relation as in [5, § 6]. For preliminaries concerning alcoves, facettes etc., we refer to [3], [4], and [5].

Let w_0 be the longest element in W and $w_0 = s_{\beta_1} \ldots s_{\beta_N}$ a reduced expression for it, $\beta_i \in S$. We keep this expression fixed for the rest of the paper unless otherwise stated. For a dominant weight χ write $\chi_j = s_{\beta_j} \ldots s_{\beta_1} \cdot \chi$, $0 \leqq j \leqq N$. The following two long exact sequences were derived in [1] and [2] for $1 \leqq j \leqq N$:

$$\text{(i)} \quad \begin{array}{l} 0 \to H^0(\chi_j) \to 0 \to H^0(\bar{V}_{\chi,j}) \to \\ \ldots \to H^{i+1}(\chi_j) \to H^i(\chi_{j-1}) \to H^{i+1}(\bar{V}_{\chi,j}) \to \ldots \end{array}$$

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and

(ii)
$$0 \to H^0(C_{\chi,j}) \to H^0(\bar{V}_{\chi,j}) \to 0 \to$$
$$\dots \to H^{i+1}(C_{\chi,j}) \to H^{i+1}(\bar{V}_{\chi,j}) \to H^i(Q_{\chi,j}) \to \dots .$$

Here we have written $\bar{V}_{\chi,j}$, $C_{\chi,j}$, and $Q_{\chi,j}$ short for the B modules $\bar{V}^{\beta_j}_{\chi_{j-1}+\varrho}(-\varrho)$, $C^{\beta_j}_{\chi_{j-1}+\varrho}(-\varrho)$, and $Q^{\beta_j}_{\chi_{j-1}+\varrho}(-\varrho)$, respectively. Moreover, the weights of $C_{\chi,j}$ and $Q_{\chi,j}$ are

$$\{\chi_j + mp\beta_j \mid 0 < mp < \langle \beta_j^\vee, \chi_{j-1} + \varrho \rangle\};$$

their multiplicities equal 1.

2. The theorem.

We shall consider the following condition for pairs of weights (χ, λ):

(*) *$\chi \in F \cap X_+$, $\lambda \in F' \cap X_+$, F and F' are facettes with $\bar{F} \cap \overline{F'} \neq \varnothing$, $\xi \in X_+$ whenever $\lambda \uparrow \xi \uparrow \chi$.*

LEMMA 2.1. *Assume that (χ, λ) satisfies (*). Let $1 \leqq j \leqq N$, $0 < np < \langle \beta_j^\vee, \chi_{j-1} + \varrho \rangle$ and $w \in W$. Put $\xi = w.(\chi_j + np\beta_j)$. Then $\xi \uparrow \chi$ and $\xi \neq \chi$. If in addition $\lambda \uparrow \xi$, then $w = s_{\beta_1} \dots s_{\beta_{j-1}}$ and $\xi = s_\alpha.\chi + np\alpha$, where $\alpha = s_{\beta_1} \dots s_{\beta_{j-1}}(\beta_j)$; moreover, $np < \langle \alpha^\vee, \chi + \varrho \rangle = \langle \beta_j^\vee, \chi_{j-1} + \varrho \rangle < (n+1)p$.*

PROOF. Set $w' = ws_{\beta_{j-1}} \dots s_{\beta_1} \in W$ and $\alpha = s_{\beta_1} \dots s_{\beta_{j-1}}(\beta_j)$. Then $\alpha \in R_+$ by [3, p. 158]. A direct calculation gives $\xi = w'.(s_\alpha.\chi + np\alpha)$ and $\langle \alpha^\vee, \chi + \varrho \rangle = \langle \beta_j^\vee, \chi_{j-1} + \varrho \rangle$. As in the proof of [2, 3.1], we see that if $w'(\alpha) > 0$, then

$$\xi = s_{w'(\alpha)}.(w'.\chi) + npw'(\alpha) \uparrow w'.\chi \uparrow \chi ,$$

where $\xi \neq w'.\chi$, and if $w'(\alpha) < 0$, then

$$\xi = w's_\alpha.\chi + npw'(\alpha) \uparrow w's_\alpha.\chi \uparrow \chi ,$$

where $\xi \neq w's_\alpha.\chi$.

Now assume that $\lambda \uparrow \xi$. Let F_0 and F'' be facettes with $F_0 \subseteq \bar{F} \cap \overline{F'}$ and $\xi \in F''$. From [5, Lemma 6] it follows easily that $F_0 \subseteq \overline{F''}$. Now [5, Lemma 7] implies that $w' = 1$ and that $s_\alpha.x + np\alpha = x$ for any x in F_0. Hence $w = s_{\beta_1} \dots s_{\beta_{j-1}}$. Since $\chi \in F$, $F_0 \subseteq \bar{F}$, and $\xi = s_\alpha.\chi + np\alpha \prec \chi$, we also have $np < \langle \alpha^\vee, \chi + \varrho \rangle < (n+1)p$.

From the proof we get

COROLLARY 2.2. *If (χ, λ) satisfies (*), $1 \leqq j \leqq N$, $0 < np < \langle \beta_j^\vee, \chi_{j-1} + \varrho \rangle$, $w \in W$ and $\lambda \uparrow \xi = w.(\chi_j + np\beta_j)$, then (ξ, λ) satisfies (*), too.*

The following lemma combined with [2, 2.3] gives an improvement of [2, 2.5].

LEMMA 2.3. *Assume that* (χ, λ) *satisfies* (*). *If* $0 \leqq i \leqq N$, $w \in W$ *and* $i \neq \ell(w)$, *then* $[H^i(w.\chi): M_\lambda] = 0$.

PROOF. We consider first the case $i > \ell(w)$. Let λ be fixed and assume that there exists χ such that (χ, λ) satisfies (*) and $[H^i(w.\chi)\colon M_\lambda] \neq 0$ for some i, w with $i > \ell(w)$. Assume further that χ is minimal with respect to $\uparrow$ with this property. Set $n = \ell(w)$. Since $\ell(ww_0) = N - n$ by [3, p. 158], we can find for w_0 a reduced expression $s_{\beta_1} \dots s_{\beta_N}$ such that $w = s_{\beta_n} \dots s_{\beta_1}$. We may assume that this is the expression used above. Then $w.\chi = \chi_n$. Now $H^{N+i-n}(\chi_N) = 0$, since $N + i - n > N = \dim G/B$. Hence [2, 1.3] implies that $[H^q(\chi_{j+1} + mp\beta_{j+1})\colon M_\lambda] \neq 0$ for some q, j, m with $q \geqq i + j - n \geqq i$ and $0 < mp < \langle \check{\beta}_{j+1}, \chi_{j+1} + \varrho \rangle$.

Set $\xi = w_1 . (\chi_{j+1} + mp\beta_{j+1}) \in X_+ - \varrho$, $w_1 \in W$. Then by [1, Theorem 1], (*) and Lemma 2.1, we have $\lambda \uparrow \xi \uparrow \chi$, $\xi \neq \chi$ and $w_1 = s_{\beta_1} \dots s_{\beta_j}$. By Corollary 2.2, (ξ, λ) satisfies (*); hence $[H^r(w_1^{-1}.\xi)\colon M_\lambda] = 0$ for any $r > \ell(w_1) = j$. However, this is a contradiction since $q \geqq i + j - n > j$.

Now let $i < \ell(w)$. It is easily checked that also the pair $(-w_0(\chi), -w_0(\lambda))$ satisfies (*). Using Serre duality as in the proof of [1, Lemma 4] we get

$$H^i(w.\chi) \cong H^{N-i}(-w.\chi - 2\varrho)^* \cong H^{N-i}(ww_0.(-w_0(\chi)))^* .$$

Since $N - i > N - \ell(w) = \ell(ww_0)$ the first part of the proof gives

$$0 = [H^{N-i}(ww_0.(-w_0(\chi)))\colon M_{-w_0(\lambda)}] = [H^i(w.\chi)\colon M_\lambda] .$$

Now we are ready to give the main result of the paper.

THEOREM 2.4. *If* (χ, λ) *satisfies* (*) *and* $\lambda \uparrow \chi$, *then*

$$[H^0(\chi)\colon M_\lambda] = [H^1(\chi_1)\colon M_\lambda] = \dots = [H^N(\chi_N)\colon M_\lambda] \neq 0 .$$

PROOF. For $\lambda = \chi$ the assertion follows from [2, 2.6]. We fix λ and use induction on χ with respect to $\uparrow$.

Let $I \subseteq \{1, \dots, N\}$ be the set of those i's for which there exist $w_i \in W$ and $n_i \in \mathbf{Z}$ such that $0 < n_i p < \langle \check{\beta}_i, \chi_{i-1} + \varrho \rangle$ and $\lambda \uparrow w_i.(\chi_i + n_i p\beta_i)$. By lemma 2.1 the elements w_i and n_i are unique for each i in I; in fact $w_i = s_{\beta_1} \dots s_{\beta_{i-1}}$ and $n_i p < \langle \check{\beta}_i, \chi_{i-1} + \varrho \rangle < (n_i + 1)p$.

We check first that I is not empty. As $\lambda \neq \chi$ we can find a weight ξ of the form $s_\alpha.\chi + np\alpha$, $\alpha \in R_+$, $n \in \mathbf{Z}$, with $\lambda \uparrow \xi \uparrow \chi$ and $\xi \neq \chi$. Then $\xi \in X_+$; hence $\langle \check{\alpha}, \chi + \varrho \rangle > np > 0$. By [3, p. 158], $\alpha = s_{\beta_1} \dots s_{\beta_{i-1}}(\beta_i)$ for some i. Setting $w = s_{\beta_1} \dots s_{\beta_{i-1}}$, we get

$$\xi = s_\alpha . \chi + np\alpha = w . (\chi_i + np\beta_i)$$

and

$$\langle \beta_i^\vee, \chi_{i-1} + \varrho \rangle = \langle \alpha^\vee, \chi + \varrho \rangle > np > 0 .$$

Hence $i \in I$.

Consider the chain

$$H^N(\chi_N) \xrightarrow{\psi_N} H^{N-1}(\chi_{N-1}) \xrightarrow{\psi_{N-1}} \dots \xrightarrow{\psi_1} H^0(\chi) ,$$

where the homomorphisms ψ_j are as in (i). This chain played a crucial role in the proof of [2, 3.1].

We show that $[\mathrm{Ker}\,\psi_j: M_\lambda] = [\mathrm{Coker}\,\psi_j: M_\lambda] = 0$ for $j \in \{1, \dots, N\} \setminus I$. Assume on the contrary that one of these multiplicities is non-zero. By the sequence (i), $[H^q(\bar{V}_{\chi,j}) : M_\lambda] \neq 0$ for some $q \geqq 0$. The sequence (ii) implies that the same is true with $\bar{V}_{\chi,j}$ replaced by $C_{\chi,j}$ or $Q_{\chi,j}$. As we know the weights, and hence the composition factors, of the modules $C_{\chi,j}$ and $Q_{\chi,j}$ we find that $[H^q(\chi_j + mp\beta_j) : M_\lambda] \neq 0$ for some $q \geqq 0$, $0 < mp < \langle \beta_j^\vee, \chi_{j-1} + \varrho \rangle$. Then $\lambda \uparrow w . (\chi_j + mp\beta_j)$ for some $w \in W$. Hence $j \in I$, which is a contradiction.

Now let $i \in I$. We claim that $[\mathrm{Ker}\,\psi_i: M_\lambda] = [\mathrm{Coker}\,\psi_i: M_\lambda] \neq 0$. Set $\xi = w_i . (\chi_i + n_i p \beta_i)$. Then $\lambda \uparrow \xi \uparrow \chi$ and $\xi \neq \chi$. By Corollary 2.2 we can use the induction hypothesis to get $[H^j(\xi_j): M_\lambda] \neq 0$ for each j. Hence it is enough to prove

$$[\mathrm{Coker}\,\psi_i: M_\lambda] = [H^{i-1}(\xi_{i-1}): M_\lambda] = [\mathrm{Ker}\,\psi_i: M_\lambda] .$$

We only show the first equality; the other one can be treated analogously. Lemma 2.3 gives $[H^{i+1}(\chi_i): M_\lambda] = 0$. Hence by the sequences (i) and (ii), it is enough to prove

(a) $$[H^{i-1}(Q_{\chi,i}): M_\lambda] = [H^{i-1}(\xi_{i-1}): M_\lambda] ,$$

(b) $$[H^j(C_{\chi,i}): M_\lambda] = 0 \quad \text{for } j = i, i+1 .$$

Since $n_i p < \langle \beta_i^\vee, \chi_{i-1} + \varrho \rangle < (n_i + 1)p$, the weights of $Q_{\chi,i}$ are $\{\chi_i + mp\beta_i \mid 0 < m \leqq n_i\}$. In particular, $\chi_i + n_i p \beta_i$ is the highest weight and its multiplicity is 1. We get an exact sequence

$$0 \to Q' \to Q_{\chi,i} \to k_{\chi_i + n_i p \beta_i} \to 0 ,$$

where the weights of Q' are $\{\chi_i + mp\beta_i \mid 0 < m < n_i\}$. If M_λ were a composition factor of $H^q(Q')$ for some $q \geqq 0$, then it would also be a composition factor of $H^q(\chi_i + mp\beta_i)$ with $0 < m < n_i$. This would imply $\lambda \uparrow w . (\chi_i + mp\beta_i)$ for some $w \in W$, and hence $m = n_i$. Therefore $[H^q(Q'): M_\lambda] = 0$ for $q = i-1, i$ and

$$[H^{i-1}(Q_{\chi,i}): M_\lambda] = [H^{i-1}(\chi_i + n_i p \beta_i): M_\lambda] .$$

Now (a) follows since $\chi_i + n_i p\beta_i = s_{\beta_{i-1}} \ldots s_{\beta_1} . \xi = \xi_{i-1}$. To prove (b) we proceed as above and get

$$[H^j(C_{\chi,i}) : M_\lambda] = [H^j(\xi_{i-1}) : M_\lambda] .$$

By Lemma 2.3 the right hand side is zero for $j = i, i+1$.

Now we have $[\operatorname{Coker} \psi_j : M_\lambda] = [\operatorname{Ker} \psi_j : M_\lambda]$ for each $j = 1, \ldots, N$. This implies

$$[H^N(\chi_N) : M_\lambda] = \ldots = [H^0(\chi) : M_\lambda] .$$

Finally, for $i \in I$ we get

$$[H^{i-1}(\chi_{i-1}) : M_\lambda] \geqq [\operatorname{Coker} \psi_i : M_\lambda] \neq 0 .$$

The theorem follows.

In particular, the theorem gives information of the composition factors of Weyl modules (see [1, p. 55]).

Combining Lemma 2.3 and Theorem 2.4 we get

COROLLARY 2.5. *Assume that* (χ, λ) *satisfies* (*), $\lambda \uparrow \chi$ *and* $w \in W$. *Then* $[H^i(w . \chi) : M_\lambda] \neq 0$ *if and only if* $i = \ell(w)$. *Moreover,* $[H^{\ell(w)}(w . \chi) : M_\lambda]$ *is independent of* w.

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