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ON THE TRANSLATION FUNCTORS FOR A SEMISIMPLE ALGEBRAIC GROUP

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Introduction.

Let G be a simply connected semisimple algebraic group over an algebraically closed field of characteristic $p \neq 0$. Assume that the root system of G is irreducible. In [11] J. C. Jantzen introduced the functors T_λ^μ, called the translation functors in [2], between certain categories of rational G modules. Here λ and μ are weights in the closure of the bottom alcove. In [11] and [2], the effect of T_λ^μ on the simple modules was given under the assumption that λ is inside the bottom alcove; this result was then used, for example, to prove the translation principle ([11, § 3], [2, 2.5]).

In this paper we extend the result on the simple modules to the cases, where λ is in a facette and μ in its closure (Theorem 2.5). As corollaries we obtain generalizations of (a part of) the translation principle and [8, Theorem 2]; we also find some composition multiplicities of Weyl modules. Moreover, we derive analogous results for $\boldsymbol{u}_n$-T modules (cf. [12]).

As an application of the translation functors we generalize [10, Satz 5] as follows. Let C_n be the alcove with $(p^n-1)\varrho$ in its upper closure. Here ϱ is the sum of the fundamental dominant weights. We show that if λ lies in the closure of C_n or in the alcove immediately below it, then the indecomposable projective $\boldsymbol{u}_n$ module $Q(n,\lambda)$ can be lifted to a G module.

1. Preliminaries.

Let G be a simply connected semisimple algebraic group over an algebraically closed field of characteristic $p \neq 0$. Let T be a fixed maximal torus of G and let $\boldsymbol{u}_n$ be the hyperalgebra of the nth infinitesimal subgroup of G (cf. [4, 3.2], [14, 2.1]). We shall freely use the well-known properties of the categories of (rational) G modules, $\boldsymbol{u}_n$ modules and $\boldsymbol{u}_n$-T modules (cf. [4], [6], [7], [12], and [14]). Moreover, the reader may consult for instance [3], [5], [6], and [8] for preliminaries concerning the root system R of G, the character group $X = X(T)$ of T, the hyperplanes, facettes and alcoves in the Euclidean space

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$\mathsf{R} \otimes X$, as well as the Weyl group W and the affine Weyl group W_p operating in $\mathsf{R} \otimes X$. We assume that R is irreducible. Let X^+ be the set of dominant weights with respect to a fixed basis of R.

For a weight ν we let $\mathscr{M}_\nu$ be the category of the finite dimensional G modules for which the highest weights of their composition factors lie in $W_p.\nu$. Let C be an alcove. Each finite dimensional G module V has for any $\nu \in \bar{C}$ a unique maximal submodule V_ν contained in $\mathscr{M}_\nu$, and V is the direct sum of these submodules. Fix two weights λ, μ in $\bar{C}$. We define a functor $T_\lambda^\mu : \mathscr{M}_\lambda \to \mathscr{M}_\mu$ as follows: if $V \in \mathscr{M}_\lambda$, then $T_\lambda^\mu V = (V \otimes E)_\mu$, where E is the simple G module with the highest weight in $W(\mu - \lambda)$. The difference between this definition and the one in [2, § 2], [11, § 3], and [14, 5.2] is only notational; we allow C to be any alcove. Clearly $T_\lambda^\mu = T_{w.\lambda}^{w.\mu}$ for $w \in W_p$.

Analogous functors T_λ^μ for $\boldsymbol{u}_n - T$ modules were defined in [14, 5.2]. We extend the notation as above to encompass weights λ, μ in the closure of any alcove.

Finally, we mention two important formulas. For a weight ν set

$$S_\nu = \{ w \in W_p \mid w.\nu = \nu \}$$

as in [2]. Let C be an alcove and $\lambda, \mu \in \bar{C} \cap X$. If $w \in W_p$ and $w.\lambda \in X^+$, then

$$\text{(1)} \qquad \operatorname{ch} T_\lambda^\mu V(w.\lambda) = \sum_{w_1 \in W_1} \chi(ww_1.\mu)\,,$$

and for any $w \in W_p$

$$\text{(2)} \qquad \operatorname{ch} T_\lambda^\mu \hat{Z}(n, w.\lambda) = \sum_{w_1 \in W_1} \operatorname{ch} \hat{Z}(n, ww_1.\mu)\,,$$

where W_1 is a system of representatives for $S_\lambda / S_\lambda \cap S_\mu$. These are the formulas [14; 5.2(6), (7)], when C is the bottom alcove, and the general case follows easily from this.

2. T_λ^μ for $L(\lambda)$ and $\hat{L}(n, \lambda)$.

First we generalize the character formula (1) of the previous section.

LEMMA 2.1. *Let $\lambda, \mu \in X$ be in the closure of the same alcove and let W_1 be a system of representatives for $S_\lambda / S_\lambda \cap S_\mu$. If V is a finite dimensional G module and*

$$\operatorname{ch} V = \sum_{w \in W_p} a_w \chi(w \cdot \lambda)$$

with $a_w \in \mathbf{Z}$, then

$$\operatorname{ch} T_\lambda^\mu V = \sum_{w \in W_p} \sum_{w_1 \in W_1} a_w \chi(ww_1 \cdot \mu)\,.$$

PROOF. Suppose $\chi(w.\lambda)=0$. Then $s_\alpha w.\lambda = w.\lambda$ for some $\alpha \in R$. Hence $w^{-1}s_\alpha w \in S_\lambda$. For each w_1 in W_1, there is a unique w_2 in W_1 with $w^{-1}s_\alpha w w_1.\mu = w_2.\mu$. Putting $\zeta(w_1)=w_2$, we get a map $\zeta: W_1 \to W_1$ that is injective and hence bijective. Now

$$\sum_{w_1 \in W_1} \chi(ww_1.\mu) = \sum_{w_1 \in W_1} \chi(w\zeta(w_1).\mu)$$
$$= \sum_{w_1 \in W_1} \chi(s_\alpha w w_1.\mu) = - \sum_{w_1 \in W_1} \chi(ww_1.\mu)\,.$$

Hence this sum is zero. Therefore we may assume that $\chi(w.\lambda) \neq 0$, whenever $a_w \neq 0$. Then for each w with $a_w \neq 0$, there is a $\sigma_w \in W$ with $\sigma_w w.\lambda \in X^+$. Now we have

$$\operatorname{ch} V = \sum_{w \in W_p} a_w \det(\sigma_w) \operatorname{ch} V(\sigma_w w.\lambda)\,.$$

Let E be the simple module with the highest weight in $W(\mu-\lambda)$. Then

$$\sum_{v \in \bar{C}} \operatorname{ch}(V \otimes E)_v = \sum_{v \in \bar{C}} \sum_{w \in W_p} a_w \det(\sigma_w) \operatorname{ch}(V(\sigma_w w.\lambda) \otimes E)_v\,.$$

Clearly the characters of modules belonging to different categories $\mathcal{M}_v$. $v \in \bar{C}$, are linearly independent. Hence

$$\operatorname{ch} T_\lambda^\mu V = \sum_{w \in W_p} a_w \det(\sigma_w) \operatorname{ch} T_\lambda^\mu V(\sigma_w w.\lambda)$$
$$= \sum_{w \in W_p} \sum_{w_1 \in W_1} a_w \det(\sigma_w) \chi(\sigma_w w w_1.\mu)$$
$$= \sum_{w \in W_p} \sum_{w_1 \in W_1} a_w \chi(ww_1.\mu)$$

by (1) of section 1.

Let $\lambda, \mu \in X$ be in the closure of the same alcove. In [14, 5.3], Jantzen showed that if a G module $V \in \mathcal{M}_\lambda$ has a filtration by Weyl modules, then so has $T_\lambda^\mu V$. Moreover, [14, 5.1] gives the corresponding result for $\boldsymbol{u}_n - T$ modules and Z filtrations (see [12, 3.3]). Hence the formulas (1) and (2) of section 1 give immediately the following two lemmas (cf. [11, § 3]). Lemma 2.2 is contained in [2, 2.1c], too.

LEMMA 2.2. *Let F be a facette, $\lambda \in F \cap X^+$ and $\mu \in \bar{F} \cap X$. Then $T_\lambda^\mu V(\lambda) \cong V(\mu)$ for $\mu \in X^+$, while $T_\lambda^\mu V(\lambda)=0$ for $\mu \notin X^+$.*

LEMMA 2.2′. *Let F be a facette, $\lambda \in F \cap X$ and $\mu \in \bar{F} \cap X$. Then $T_\lambda^\mu \hat{Z}(n,\lambda) \cong \hat{Z}(n,\mu)$.*

We use the sign $\uparrow$ for the strong linkage relation as in [9, § 6]. From [15] we get the following result.

LEMMA 2.3. *Let F, F' be facettes, $\lambda \in F \cap X^+$ and $\xi \in F' \cap X^+$. Assume that $\bar{F} \cap \overline{F'} \neq \emptyset$, $\xi \uparrow \lambda$ and that $\xi \uparrow \tau \uparrow \lambda$ implies $\tau \in X^+$. Then* $[V(\lambda):L(\xi)] \neq 0$.

Set $\mathfrak{C}_0 = \{x \in \mathbf{R} \otimes X \mid \langle x+\varrho, \alpha^\vee \rangle > 0 \ \forall \alpha \in R^+\}$. For a facette F let $\hat{F}$ be its upper closure (cf. [8]).

COROLLARY 2.4. *If F is a facette, $w \in W_p$, $\lambda \in F \cap X^+$ and $\bar{F} \cap (w.F)\hat{} \cap \mathfrak{C}_0 \neq \emptyset$, then* $[V(\lambda): L(w.\lambda)] \neq 0$.

PROOF. Let F_0 be a facette in $\bar{F} \cap (w.F)\hat{} \cap \mathfrak{C}_0$ and C an alcove with $w.F \subseteqq \hat{C}$ ([8, Satz 4]). Clearly $F_0 \subseteqq \hat{C}$. In the notation of [9, p. 137], C and $w^{-1}.C$ belong to $\mathfrak{K}(F_0)$. By [9, Lemma 6] $C \uparrow w^{-1}.C$; hence $w.\lambda \uparrow \lambda$.

Now let $w.\lambda \uparrow \tau \uparrow \lambda$. There is a chain $w.\lambda < s_1 w.\lambda < \ldots < s_k \ldots s_1 w.\lambda = \tau$, where each s_i is a reflection, $k \geqq 0$. The chain gives $C \uparrow C'$, where $C' = s_k \ldots s_1.C$ and $\tau \in \overline{C'}$. Similarly, the relation $\tau \uparrow \lambda$ gives an alcove C'' with $C' \uparrow C''$ and $\lambda \in \overline{C''}$. Then $C, C'' \in \mathfrak{K}(F_0)$. By [9, Lemma 6], $C' \in \mathfrak{K}(F_0)$. Hence $F_0 \subseteqq \overline{C'}$ and $s_k \ldots s_1.F_0 \subseteqq \overline{C'}$. Therefore $F_0 = s_k \ldots s_1.F_0 \subseteqq \overline{F'}$, where $F' = s_k \ldots s_1 w.F$. Hence $\tau \in F' \subseteqq \mathfrak{C}_0$. Now the assertion follows from lemma 2.3.

We prove analogous results for $\boldsymbol{u}_n - T$ modules.

LEMMA 2.3'. *Let F, F' be facettes and $\lambda \in F \cap X$, $\xi \in F' \cap X$. If $\bar{F} \cap \overline{F'} \neq \emptyset$ and $\xi \uparrow \lambda$, then* $[\hat{Z}(n,\lambda): \hat{L}(n,\xi)] \neq 0$.

PROOF. For $x \in X$, we let $T(x)$ be the translation in $\mathbf{R} \otimes X$ by x. Choose an integer m. According to [12, 2.8]

$$[\hat{Z}(n,\lambda): \hat{L}(n,\xi)] = [\hat{Z}(n,\lambda + mp^n\varrho): \hat{L}(n,\xi + mp^n\varrho)] .$$

Hence we may replace λ, ξ, F and F' by $\lambda + mp^n\varrho$, $\xi + mp^n\varrho$, $T(mp^n\varrho)(F)$ and $T(mp^n\varrho)(F')$, respectively. Taking m large enough, we can therefore assume that if $\lambda \uparrow \nu \uparrow \xi$ or $[\hat{Z}(n,\lambda): \hat{L}(n,\nu)] \neq 0$, then $\nu \in X^+$. Now 2.3 implies $[V(\lambda): L(\xi)] \neq 0$. On the other hand, by [14, 3.1(5)] we have

$$\operatorname{ch} V(\lambda) = \sum_{\nu \in X^+} \sum_{\tau \in X_n} [\hat{Z}(n,\lambda): \hat{L}(n, p^n\nu + \tau)] \operatorname{ch} V(\nu)^{Fr^n} \operatorname{ch} L(\tau) .$$

Here $X_n = X_n(T)$ (cf. [12, 1.4]). Put $\xi = p^n\eta + \tau$, $\tau \in X_n$. Then

$$0 \neq [V(\lambda):L(\xi)] = \sum_{\nu \in X^+} [\hat{Z}(n,\lambda):\hat{L}(n,p^n\nu+\tau)][V(\nu):L(\eta)] .$$

Let ν be a weight giving a non-zero term on the right hand side. Then $\eta \uparrow \nu$ and $(p^n\nu+\tau) \uparrow \lambda$ by the strong linkage principle [1, cor. 3] and its counterpart [14, 3.3]. In particular $\nu \geqq \eta$. This implies easily $(p^n\eta+\tau) \uparrow (p^n\nu+\tau)$. Hence $\xi \uparrow (p^n\nu+\tau) \uparrow \lambda$.

Let C be an alcove with $\lambda \in \bar{C}$. As in the proof of 2.4 we can find alcoves C' and C'' with $C' \uparrow C'' \uparrow C$, $p^n\nu+\tau \in \overline{C''}$, and $\xi \in \overline{C'}$. Let F_0 be a facette in $\bar{F} \cap \overline{F'}$. Then C, C' and C'' are in $\mathfrak{R}(F_0)$ by [9, Lemma 6]. Set $F''=w.F'$, where w is the element of W_p with $C''=w.C'$. Now $F_0 \subseteqq \overline{C'} \cap \overline{C''}$. Therefore $w.F_0=F_0$. On the other hand $p^n\nu+\tau \in \overline{C''} \cap (W_p.\xi)$ and $w.\xi \in \overline{C''}$; so

$$T(p^n(\nu-\eta)).\xi = p^n\nu+\tau = w.\xi .$$

This implies that $T(p^n(\nu-\eta))$ and w coincide on F', and therefore on F_0. Hence

$$T(p^n(\nu-\eta)).F_0 = w.F_0 = F_0 .$$

So we actually have $\nu=\eta$. This proves the lemma.

Corollary 2.4′. *If F is a facette, $w \in W_p$, $\lambda \in F \cap X$ and $\bar{F} \cap (w.F)\hat{} \neq \emptyset$, then $[\hat{Z}(n,\lambda)$: $\hat{L}(n,w.\lambda)] \neq 0$.*

Now we are ready to prove the main results of the paper.

Theorem 2.5. *Let F be a facette, $\lambda \in F \cap X^+$ and $\mu \in \bar{F} \cap X$. If $\mu \in \hat{F}$, then $T_\lambda^\mu L(\lambda) \cong L(\mu)$; otherwise $T_\lambda^\mu L(\lambda)=0$.*

Theorem 2.5′. *Let F be a facette, $\lambda \in F \cap X$ and $\mu \in \bar{F} \cap X$. If $\mu \in \hat{F}$, then $T_\lambda^\mu \hat{L}(n,\lambda) \cong \hat{L}(n,\mu)$; otherwise $T_\lambda^\mu \hat{L}(n,\lambda)=0$.*

Proof. Let F be a facette, $\lambda \in F \cap X^+$ and $\mu \in \hat{F} \cap X$. We get the first assertion of 2.5 as in [11]: Write

$$\chi_p(\lambda) = \sum_{w \in W_p} a(w,\lambda)\chi(w.\lambda)$$

as in [8, p. 130]. Then [8, Theorem 1] and Lemma 2.1 imply

$$\operatorname{ch} L(\mu) = \sum_{w \in W_p} a(w,\lambda)\chi(w.\mu) = \operatorname{ch} T_\lambda^\mu L(\lambda) .$$

Hence $T_\lambda^\mu L(\lambda) \cong L(\mu)$.

Next let F and λ be as above and $\mu \in (\bar{F} \setminus \hat{F}) \cap X$. If $\mu \notin X^+$, then $T_\lambda^\mu L(\lambda)$ is zero by 2.2, since it is a quotient of $T_\lambda^\mu V(\lambda)$. Therefore we may assume that $\mu \in X^+$. From [8, Satz 4] one easily sees that μ is in the upper closure of $w.F$ for some $w \in W_p$. Put $x = [V(\lambda): L(w.\lambda)]$. By Corollary 2.4, $x \neq 0$. Then $w.\lambda \neq \lambda$, and $V(\lambda)$ has $L(\lambda)$ once and $L(w.\lambda)$ x times as a composition factor. Operating with T_λ^μ to a composition series of $V(\lambda)$, we get a filtration for $T_\lambda^\mu V(\lambda) \cong V(\mu)$ in which $T_\lambda^\mu L(\lambda)$ occurs once, and $T_\lambda^\mu L(w.\lambda)$ x times as a quotient. By the first part of the proof $T_\lambda^\mu L(w.\lambda) \cong L(\mu)$. Now $T_\lambda^\mu L(\lambda)$ is a quotient of $T_\lambda^\mu V(\lambda) \cong V(\mu)$; so if it were not zero, it would have $L(\mu)$ as a quotient. This would imply $[V(\mu): L(\mu)] \geqq 1 + x \geqq 2$. Hence $T_\lambda^\mu L(\lambda) = 0$. Note too that we actually get $x = 1$. This gives a part of Corollary 2.6.

Now let us consider Theorem 2.5′. We prove only the first assertion; the rest of 2.5′ can be derived in a way entirely analogous to the proof of 2.5. So let F be a facette, $\lambda \in F \cap X$ and $\mu \in \hat{F} \cap X$. If $\lambda \in X_n$, then $\mu \in X_n$, and by 2.5 we get (see [14, 5.2])

$$T_\lambda^\mu \hat{L}(n, \lambda) \cong T_\lambda^\mu (L(\lambda)|_{u_n - T}) \cong (T_\lambda^\mu L(\lambda))_{u_n - T} \cong \hat{L}(n, \mu) .$$

Next let $\lambda = \lambda' - p^n v$ with $\lambda' \in X_n$. Put $\mu' = \mu - p^n v$ and $F' = T(-p^n v)(F)$. Then F' is a facette, $\lambda' \in F' \cap X$ and $\mu' \in \hat{F}' \cap X$. Using a result analogous to [14, 5.2(8)] we have

$$\begin{aligned} T_\lambda^\mu \hat{L}(n, \lambda) &\cong T_\lambda^\mu (\hat{L}(n, p^n v) \otimes \hat{L}(n, \lambda')) \\ &\cong \hat{L}(n, p^n v) \otimes T_{\lambda'}^{\mu'} \hat{L}(n, \lambda') \cong \hat{L}(n, \mu) . \end{aligned}$$

COROLLARY 2.6. *Let F, λ and w be as in* 2.4 (*respectively* 2.4′). *Then* $[V(\lambda): L(w.\lambda)]$ (*respectively* $[\hat{Z}(n, \lambda): \hat{L}(n, w.\lambda)]$) *equals* 1.

Now 2.2 and 2.5 imply the following generalization of (a part of) the translation principle (cf. [2, 2.5], [10, Satz 7], [11, §3]):

COROLLARY 2.7. *Let F be a facette, $\lambda \in F \cap X^+$, $\mu \in \hat{F} \cap X^+$, $w \in W_p$ and assume that $w.\lambda \in X^+$. Then $[V(w.\lambda): L(\lambda)]$ equals $[V(w.\mu): L(\mu)]$ for $w.\mu \in X^+$ and* 0 *for $w.\mu \notin X^+$.*

COROLLARY 2.7′. *Let F be a facette, $\lambda \in F \cap X$, $\mu \in \hat{F} \cap X$, and $w \in W_p$. Then $[\hat{Z}(n, w.\lambda): \hat{L}(n, \lambda)]$ equals $[\hat{Z}(n, w.\mu): \hat{L}(n, \mu)]$.*

Theorem 2.5 can also be used to generalize [8, Theorem 2] in the following way (see also [2, 2.4]).

COROLLARY 2.8. *Let F be a facette, $\lambda \in F \cap X^+$ and $\mu \in (\bar{F} \setminus \hat{F}) \cap X^+$. Write*

$$\chi_p(\lambda) = \sum_{w \in W_p} a(w, \lambda)\chi(w.\lambda),$$

where $w.\lambda \in X^+$ *if* $a(w, \lambda) \neq 0$. *Then*

$$\sum_{w \in W_p} a(w, \lambda)\chi(w.\mu) = 0 .$$

If $w \in W_p$ *with* $\chi(w.\mu) \neq 0$, *then*

$$\sum_{w' \in S_\mu} a(ww', \lambda) = 0 .$$

PROOF. The first claim follows from 2.1 and 2.5.

If $w \in W_p$ with $a(w, \lambda) \neq 0$, then $w.\mu \in X^+$ or $\chi(w.\mu)=0$. Omitting the terms with $\chi(w.\mu)=0$ in the sum $\sum_w a(w, \lambda)\chi(w.\mu)$ and combining equal characters, we get a linear combination of linearly independent characters. Hence the first part of 2.8 implies the second one.

3. T_λ^μ for $\hat{Q}(n, \lambda)$.

In this section we use the translation functors to generalize [10, Satz 5]. The following lemma is analogous to [13, Satz 2.24].

LEMMA 3.1. *Let* F *be a facette,* $\lambda \in \hat{F} \cap X$ *and* $\mu \in F \cap X$. *Then* $T_\lambda^\mu \hat{Q}(n, \lambda) \cong \hat{Q}(n, \mu)$.

PROOF. The $\mathbf{u}_n - T$ module $T_\lambda^\mu \hat{Q}(n, \lambda)$ is projective (see [14, 5.4]), and hence it is a direct sum of modules of the form $\hat{Q}(n, w.\mu)$, $w \in W_p$. The multiplicity of $\hat{Q}(n, w.\mu)$ in the sum equals the dimension of

$$\mathrm{Hom}_{\mathbf{u}_n, T}(\hat{L}(n, w.\mu), T_\lambda^\mu \hat{Q}(n, \lambda)) \cong \mathrm{Hom}_{\mathbf{u}_n, T}(T_\mu^\lambda \hat{L}(n, w.\mu), \hat{Q}(n, \lambda))$$

(cf. [2, §2]) By 2.5′ this dimension is 1 if $w.\mu=\mu$, and 0 otherwise.

COROLLARY 3.2. *Let* F *be a facette,* $\lambda \in \hat{F} \cap X$ *and* $\mu \in F \cap X$. *If* $\hat{Q}(n, \lambda)$ *extends to a* G *module, then* $\hat{Q}(n, \mu)$ *extends, too.*

PROOF. If Q is a G module with $Q|_{\mathbf{u}_n - T} \cong \hat{Q}(n, \lambda)$, then $T_\lambda^\mu Q$ is a G module with $(T_\lambda^\mu Q)|_{\mathbf{u}_n - T} \cong \hat{Q}(n, \mu)$.

The following theorem shows that the restriction $p \nmid f$ in [10, Satz 5] is superfluous (see also [12; 4.2, 6.1]). For $\nu \in X$ set

$$S(\nu) = \sum_{\tau \in W(\nu)} e(\tau) \quad \text{and} \quad \nu^{(n)} = w_0(\nu) + (p^n - 1)\varrho .$$

Here w_0 is the longest element of W. Let C_n be the alcove with $(p^n-1)\varrho$ in its upper closure. Finally, we denote the Steinberg module $V((p^n-1)\varrho)$ by St_n.

THEOREM 3.3. *If* $\lambda \in \overline{C_n} \cap X^+$, *then* $\hat{Q}(n,\lambda)$ *can be lifted to a* G *module. Moreover*, $\operatorname{ch} \hat{Q}(n,\lambda) = S(\lambda^{(n)}) \operatorname{ch} St_n$.

PROOF. We leave it to the reader to verify that

$$\operatorname{ch} T^{\lambda}_{(p^n-1)\varrho} St_n = S(\lambda^{(n)}) \operatorname{ch} St_n$$

(use formula (1) of section 1 and [10, p. 447(1)]). For $\mu \in X$, we have

$$\operatorname{ch} \hat{Z}(n,\mu) = e(\mu-(p^n-1)\varrho) \operatorname{ch} St_n$$

by [12, p. 285]. Hence by [12; 3.7, 3.8] the lowest weight of $\hat{Q}(n,\tau)$, $\tau \in X$, is the lowest weight of $\hat{Z}(n,\tau)$, i.e. $\tau-2(p^n-1)\varrho$. Since the projective $\boldsymbol{u}_n-T$ module $T^{\lambda}_{(p^n-1)\varrho} St_n$ has $w_0(\lambda^{(n)})-(p^n-1)\varrho = \lambda-2(p^n-1)\varrho$ as its lowest weight, it has $\hat{Q}(n,\lambda)$ as a direct $\boldsymbol{u}_n-T$ summand. Hence

$$\operatorname{ch} \hat{Q}(n,\lambda) = S(\lambda^{(n)}) \operatorname{ch} St_n - \sum_{\nu \in X} a_\nu e(\nu)$$

for some non-negative integers a_ν. On the other hand, by [12, 3.8]

$$\operatorname{ch} \hat{Q}(n,\lambda) = \Phi \operatorname{ch} St_n ,$$

where

$$\Phi = \sum_{\mu \in X} [\hat{Z}(n,\mu) : \hat{L}(n,\lambda)]\, e(\mu-(p^n-1)\varrho) .$$

The coefficients in this sum are non-negative; in particular, the coefficient of $e(\lambda-(p^n-1)\varrho)$ is 1. Now $\operatorname{ch} \hat{Q}(n,\lambda)$ is invariant under W as $\lambda \in X_n$ ([14, 5.8]). Then also Φ is invariant under W, since $\mathbf{Z}[X]$ is an integral domain. Hence in the expression above for Φ, some terms form $S(\lambda-(p^n-1)\varrho) = S(\lambda^{(n)})$. Thus we have

$$\operatorname{ch} \hat{Q}(n,\lambda) = S(\lambda^{(n)}) \operatorname{ch} St_n + \sum_{\nu \in X} b_\nu e(\nu) ,$$

where the numbers b_ν are non-negative. This implies the asserted character formula. Moreover,

$$(T^{\lambda}_{(p^n-1)\varrho} St_n)|_{\mathbf{u}_n-T} \cong \hat{Q}(n,\lambda) .$$

Hence $\hat{Q}(n,\lambda)$ extends to a G module.

Let C_n be as above and let C'_n be the alcove immediately below it. That is, $C'_n = s(C_n)$, where s is the reflection in the hyperplane

$$\{x \in \mathbf{R} \otimes X \mid \langle x+\varrho, \check{\alpha}_0 \rangle = p(p^{n-1}(h-1)-1)\},$$

α_0 the maximal short root.

COROLLARY 3.4. *If λ is a weight in $\overline{C_n} \cup C'_n$, then $\hat{Q}(n,\lambda)$ can be lifted to a G module.*

PROOF. By 3.3 we may assume that $\lambda \in C'_n$. Then $p \geqq h$. Hence there is a weight in the common wall of C_n and C'_n. Now 3.2 and 3.3 imply the result.

COROLLARY 3.5. *If G is of type A_2, then each $\hat{Q}(n,\lambda)$, $\lambda \in X_n$, can be lifted to a G module.*

PROOF. For $n=1$ one can use 3.4. Then the general case follows from [16].

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