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ASYMPTOTIC COMMUTANTS AND ZEROS OF VON NEUMANN ALGEBRAS

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1. Introduction.

Let H be a Hilbert space, S a subset of $B(H)$, the algebra of all bounded linear operators on H. An operator $T \in B(H)$ *asymptotically commutes* with S if for any bounded net $\{s_n\} \subseteqq S$ which converges to zero in the weak operator topology (WOT), we have $\|Ts_n - s_nT\| \to 0$. T is a *left* (*right*) *asymptotic zero* of S if for any bounded net $\{s_n\} \subseteqq S$ for which $s_n \to 0$ (WOT), we have $\|Ts_n\| \to 0$ ($\|s_nT\| \to 0$). We say that S has the *asymptotic commutant property* (ACP) if every operator which asymptotically commutes with S in fact commutes with S, and we say that S has the *asymptotic zero property* (AZP), if every left and right asymptotic zero is zero. We determine all von Neumann subalgebras of $B(H)$ with the ACP and AZP (Theorem 2.5); they are essentially the von Neumann subalgebras of $B(H)$ with no finite-dimensional direct summands.

An operator $T \in B(H)$ is in the asymptotic commutant of a WOT-closed subspace S of $B(H)$ if and only if the mapping $s \to Ts - sT$, $s \in S$, is compact, and T is a left (right) asymptotic zero of S if and only if the mapping $s \to Ts$ ($s \to sT$), $s \in S$, is compact. This is the point of view that will predominate in our study of asymptotic commutants and zeros. In fact, we will apply our main results (Theorems 2.8 and 2.12) to give a complete description of all compact derivations of a C*-subalgebra of $B(H)$ into $B(H)$ (Theorem 3.1). The present paper may be thought of as a continuation of work by C. A. Akemann and the second author ([1], [2]), and the first author ([12]). We might also mention the papers [8], [9], [11], and [13], where other types of asymptotic phenomena are studied.

Most of our notation and terminology is standard and requires no explaination. Throughout the paper H denotes a Hilbert space, WOT the weak operator topology on $B(H)$, SOT the strong operator topology on $B(H)$. $\mathbf{Z}_+$ and $\mathbf{C}$ will denote the positive integers and complex numbers, respectively. If X and Y are Banach spaces, then $B(X, Y)$ denotes the space of all bounded linear transformations of X into Y, and we set $B(X) = B(X, X)$. If $T \in B(X, Y)$, we

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define the mappings l_T: $B(X) \to B(X, Y)$ (respectively, r_T: $B(Y) \to B(X, Y)$) by l_T: $A \to TA$, $A \in B(X)$ (respectively, r_T: $A \to AT$, $A \in B(Y)$). If $T \in B(X)$, then ad_T: $B(X) \to B(X)$ is the mapping $A \to TA - AT$, $A \in B(X)$. We set Ball $X = \{x \in X : \|x\| \leqq 1\}$. All subspaces and subalgebras of $B(H)$ are assumed to contain the identity operator I on H.

2. The main results.

Our first task is to determine those von Neumann subalgebras of $B(H)$ with the AZP and ACP. The following lemma, for all practical purposes, reduces consideration to only the AZP.

2.1. LEMMA. *Let R be a von Neumann subalgebra of $B(H)$. Suppose R has the* AZP. *Then R has the* ACP.

PROOF. Assume R has the AZP, and suppose $T \in B(H)$ asymptotically commutes with R. We may take T to be self-adjoint, and it then suffices to show that $(I-P)TP = 0$ for each projection P of R. Fix a projection $P \in R$, and let $\{a_\alpha\}$ be a bounded net in R which converges to zero (WOT). Since

$$a_\alpha(I-P)TP = -(\mathrm{ad}_T(a_\alpha(I-P))P ,$$

we conclude that

$$\|a_\alpha(I-P)TP\| \leqq \|\mathrm{ad}_T(a_\alpha(I-P))\| \to 0 .$$

Thus $(I-P)TP$ is a left asymptotic zero of R, and so $(I-P)TP = 0$.

2.2. LEMMA. *Suppose R is a von Neumann subalgebra of $B(H)$ with no non-zero minimal projections. Then R has the* AZP *in $B(H)$.*

PROOF. By a theorem of Dye ([7], Theorem 1), the unitary group of R is WOT-dense in Ball (R). Thus, we may choose a net $\{u_\alpha\}$ of unitaries in R converging to 0 (WOT). Let T be an asymptotic zero of R. Then $\|u_\alpha T\| \to 0$, and since $\|T\| = \|u_\alpha T\|$, for all α, it follows that $\|T\|$, and hence T, is zero.

The next lemma is undoubtedly well-known, but we prove it for completeness' sake.

2.3. LEMMA. *Let R be a von Neumann algebra whose center Z has no nonzero minimal projections. Then R has no nonzero minimal projections.*

PROOF. Let $\mathscr{P}$ and $\mathscr{P}_Z$ denote respectively the set of all projections and central projections in R. Suppose $P \in \mathscr{P}$ is minimal in $\mathscr{P}$. If $E \in \mathscr{P}_Z$, then

$$(*) \qquad \text{either } EP = 0 \quad \text{or} \quad P \leqq E .$$

Let $C(P)$ = central cover of P. If $C(P)=0$, then $P=0$ and we are done, so suppose $C(P) \neq 0$. By definition of $C(P)$ and (*), each $F \in \mathscr{P}_Z$ with $F < C(P)$ is orthogonal to P. Since Z has no nonzero minimal projections,

$$C(P) = \sup\{F \in \mathscr{P}_Z : \ F < C(P)\} ,$$

and so $P = PC(P) = 0$, whence $C(P) = 0$, a contradiction.

In every von Neumann algebra R, there is a unique central projection p which is the supremum of all central projections q of R with the property that qR is a direct sum of finite-dimensional algebras. We call this central projection p the *purely discrete support* of R, and note that pR is also a direct sum of finite-dimensional algebras.

2.4. LEMMA. *Let R be a von Neumann subalgebra of $B(H)$ whose purely discrete support is zero. Then R has the* AZP *and* ACP *in $B(H)$.*

PROOF. By Lemma 2.1, we need only verify that R has the AZP in $B(H)$. Now by the classical structure theory, R is a direct sum of algebras of type I with either no nonzero minimal projections or of infinite degree of homogeneity, and algebras of type II and type III (some of these summands may, of course, be absent). Since type II and type III algebras have no nonzero minimal projections, we conclude by Lemma 2.3 that the only direct summands of R with nonzero minimal projections are type I factors of infinite degree of homogeneity. But each one of these has a subalgebra without nonzero minimal projections, and so we conclude that R has a von Neumann subalgebra with no nonzero minimal projections. Thus by Lemma 2.2, this subalgebra, and hence R, has the AZP in $B(H)$.

Now, let R be an arbitrary von Neumann subalgebra of $B(H)$ with purely discrete support p. By Lemma 2.4, $(I-p)R$ has the ACP and AZP in $B((I-p)(H))$. Since pR is a direct sum of finite-dimensional algebras, it hence follows that R does not have the AZP if $pR \neq (0)$, and R does not have the ACP if $pR \neq \{\lambda p \colon \lambda \in \mathbf{C}\}$. All of this, therefore, implies the following theorem:

2.5. THEOREM. *Let R be a von Neumann subalgebra of $B(H)$, with purely discrete support p. Then R has the asymptotic commutant property* (*respectively, the asymptotic zero property*) *if and only if $pR = \{\lambda p \colon \lambda \in \mathbf{C}\}$* (*respectively, $p = 0$*).

2.6. COROLLARY. *Suppose R is a factor. Then R has the asymptotic commutant property (respectively, the asymptotic zero property) if and only if R is infinite dimensional or $R=\{\lambda I\colon \lambda \in \mathbf{C}\}$ (respectively, R is infinite dimensional).*

Thus the purely discrete support of a von Neumann subalgebra of $B(H)$ is its only obstruction to having the ACP or AZP.

We turn next to the problem of determining the aymptotic commutant and asymptotic zeros of a given von Neumann algebra R. Theorem 2.5 shows that this essentially involves studying the problem only for purely discrete R.

Suppose then that the von Neumann algebra R on H has a decomposition of the form $\oplus_{k=1}^{\infty} M_{n_k}(Z_k)$, where Z_k is a purely atomic abelian von Neumann algebra. By compressing R to each of the minimal projections in the center of R, we see that $R=\oplus\{M_{n_\alpha}\colon \alpha \in \mathscr{A}\}$, where $\alpha \to n_\alpha$ is positive integer-valued and M_{n_α} is isomorphic to the $n_\alpha \times n_\alpha$ complex matrices. For each $\alpha \in \mathscr{A}$, set P_α $=$minimal central projection in R supporting M_{n_α}. Let Σ denote the family of all finite subsets of $\mathscr{A}$, and set

$$P_\sigma = \oplus\{P_\alpha\colon \alpha \in \sigma\} \qquad \text{for } \sigma \in \Sigma\,.$$

If Σ is directed by inclusion then $\{P_\sigma\colon \sigma \in \Sigma\}$ becomes an increasing net of central projections converging in the strong operator topology to the identity operator on H. We call this net the *central supporting system* of R, and denote it by $S(R)$. We say that an operator $T \in B(H)$ is *refined by $S(R)$ from the right* (*respectively, left*) if $\lim_\sigma \|TP_\sigma - T\| = 0$ (respectively, $\lim_\sigma \|P_\sigma T - T\| = 0$).

2.7. LEMMA. *Let R be a purely discrete von Neumann subalgebra of $B(H)$, with central supporting system $S(R)$. Let $T \in B(H)$. Then T is a left (right) asymptotic zero of R if and only if T is refined by $S(R)$ from the right (left).*

PROOF. ($\Rightarrow$). Let $F_\sigma = 1 - P_\sigma$, $\sigma \in \Sigma$. Suppose $T \in B(H)$ is a left asymptotic zero of R. We assert first that $\lim_\sigma \|l_T|_{RF_\sigma}\| = 0$. For suppose not. Then there exists $\delta > 0$ such that for each $\sigma \in \Sigma$, there is a $\sigma_1 \in \Sigma$ with $\sigma \subsetneqq \sigma_1$ and $a = a_{\sigma_1} \in \operatorname{Ball}(RF_{\sigma_1})$ for which $\|Ta\| \geqq \delta$. Now for each $a \in R$, $Ta = \text{SOT-}\lim_\sigma TaP_\sigma$. It follows that we may select a sequence $\{a_k\} \subseteqq \operatorname{Ball} R$ and a sequence $\{\sigma_k\} \subseteqq \Sigma$ such that

(a) $\sigma_k \subsetneqq \sigma_{k+1}$, $\quad \forall k \in \mathbf{Z}_+$,
(b) $a_k \in RF_{\sigma_k} \cap RP_{\sigma_{k+1}}$, $\quad \forall k \in \mathbf{Z}_+$, and
(c) $\|Ta_k\| \geqq \delta/2$, $\quad \forall k \in \mathbf{Z}_+$.

But by (a) and (b), $a_k \to 0$ (WOT), and so we must have $\|Ta_k\| \to 0$, which contradicts (c). This verifies our assertion, and since $\|TP_\sigma - T\| \leqq \|l_T|_{RF_\sigma}\|$, T is hence refined by $S(R)$ from the right.

($\Leftarrow$). Assume now that $\lim_\sigma \|TP_\sigma - T\| = 0$. Let $\{a_m\} \subseteqq \operatorname{Ball} R$ with $a_m \to 0$ (WOT). We must show that $\|Ta_m\| \to 0$.

We have $a_m = \oplus_\alpha a_{\alpha m} \in \oplus_\alpha M_{n_\alpha}$. Now $a_m \to 0$ (WOT), if and only if WOT-$\lim_m a_{\alpha m} = 0$, for each α. For each fixed α, $a_{\alpha m}$ is an $n_\alpha \times n_\alpha$ scaler matrix with $n_\alpha < \infty$, and so WOT-$\lim_m a_{\alpha m} = 0$ if and only if $\lim_m \|a_{\alpha m}\| = 0$.

Let $\varepsilon > 0$. Choose $\sigma \in \Sigma$ such that $\|TP_\sigma - T\| < \varepsilon/2$. We have

$$P_\sigma a_m = \oplus \{a_{\alpha m} : \alpha \in \sigma\},$$

and thus from the previous paragraph we can find $M \in \mathbf{Z}_+$ such that $\forall m \geqq M$, $\|P_\sigma a_m\| < \varepsilon/2\|T\|$, whence $\forall m \geqq M$, $\|Ta_m\| < \varepsilon$.

The following theorem, our first main result, completely determines the structure of the asymptotic zeros of a von Neumann subalgebra of $B(H)$.

2.8. THEOREM. *Let R be a von Neumann subalgebra of $B(H)$, with purely discrete support p. Let $T \in B(H)$. Then T is a left (respectively, right) asymptotic zero of R if and only if T has an operator matrix relative to the decomposition $H = p(H) \oplus (I-p)(H)$ of the form*

$$\begin{pmatrix} T_{11} & 0 \\ T_{21} & 0 \end{pmatrix}$$

$$\left(\textit{respectively, } \begin{pmatrix} T_{11} & T_{12} \\ 0 & 0 \end{pmatrix}\right),$$

where T_{11} and $T_{21}^* T_{21}$ (respectively, T_{11} and $T_{12} T_{12}^*$) are both refined from the right (respectively, left) by the central supporting system of pR.

PROOF. This follows straightforwardly from Theorem 2.5 and Lemma 2.7, upon noticing that if M is any von Neumann subalgebra of $B(H)$ on any Hilbert space H, if K is another Hilbert space, and if $T \in B(H, K)$, then $l_T|_M$: $M \to B(H, K)$ is compact if and only if T^*T is a left asymptotic zero of M. The "only if" implication is clear, so suppose T^*T is a left asymptotic zero of M. Let $V: K \to H$ be a partial isometry whose initial space contains $T(H)$. Let $S = VT$. Then $S \in B(H)$, and $S^*S = T^*V^*VT = T^*T$, and so $l_{S^*S}|_M$: $M \to B(H)$ is compact. Thus be the spectral theorem and polar decomposition of S, $l_S|_M$: $M \to B(H)$ is compact. Since

$$\|Sa\| = \|VTa\| = \|Ta\|, \quad \forall a \in M,$$

it follows that $l_T|_M$: $M \to B(H, K)$ is compact.

2.9. COROLLARY. *Let R be as in Theorem 2.8 with $T \in B(H)$. Then T is both a* *left and right asymptotic zero of R if and only if T has an operator matrix relative to $H = p(H) \oplus ((I-p)(H))$ of the form*

$$\begin{pmatrix} T_{11} & 0 \\ 0 & 0 \end{pmatrix},$$

where T_{11} is refined from both the left and right by the central supporting system of pR.

We turn next to the description of the asymptotic commutant of a purely discrete algebra R. Let $\{P_\sigma: \sigma \in \Sigma\}$ be the central supporting system of R as previously defined, and set $F_\sigma = 1 - P_\sigma$, $\sigma \in \Sigma$.

2.10. DEFINITION. An operator $T \in B(H)$ *almost commutes with* R if T commutes with RF_σ, for some $\sigma \in \Sigma$. An operator $T \in B(H)$ *approximately commutes with* R if T is the norm limit of operators which almost commute with R.

The following lemma now characterizes the asymptotic commutant of R in terms of Definition 2.10:

2.11. LEMMA. *An operator $T \in B(H)$ asymptotically commutes with R if and only if T approximately commutes with R.*

PROOF. ($\Leftarrow$). If $T \in B(H)$ almost commutes with R, then $\mathrm{ad}_T|_R: R \to B(H)$ has finite rank, and hence if $T \in B(H)$ approximately commutes with R, then $\mathrm{ad}_T|_R: R \to B(H)$ is compact.

($\Rightarrow$). By replacing $\|l_T|_{RF_\sigma}\|$ with $\|\mathrm{ad}_T|_{RF_\sigma}\|$ in the first half of the proof of Lemma 2.7, we deduce that

$$\lim_\sigma \|\mathrm{ad}_T|_{RF_\sigma}\| = 0 .$$

For each $\sigma \in \Sigma$, RF_σ is a direct sum of finite-dimensional algebras, and is hence hyperfinite. Thus for each $\sigma \in \Sigma$, RF_σ has property P of Schwartz ([10], Definition 1 and Lemma 2). Thus by Theorem 2.3 of [4], the distance of T from the commutant of RF_σ does not exceed $\|\mathrm{ad}_T|_{RF_\sigma}\|$, for each $\sigma \in \Sigma$. Since

$$\lim_\sigma \|\mathrm{ad}_T|_{RF_\sigma}\| = 0 ,$$

it hence follows that T approximately commutes with R.

Putting Lemma 2.11 together with our previous information, we obtain our second main result, which characterizes the operators in the asymptotic commutant:

2.12. THEOREM. *Let R be a von Neumann subalgebra of $B(H)$, with purely discrete support p. Let $T \in B(H)$. Then T asymptotically commutes with R if and only if T has an operator matrix relative to the decomposition $H = p(H) \oplus (I-p)(H)$ of the form*

$$\begin{pmatrix} T_{11} & 0 \\ 0 & T_{22} \end{pmatrix},$$

where T_{11} approximately commutes with pR in $B(p(H))$ and T_{22} commutes with $(I-p)R$ in $B((I-p)(H))$.

3. An application to derivations.

One of our original motivations for studying the problems in this paper was the structure of derivations of a C*-subalgebra A of $B(H)$ into $B(H)$. Much interesting recent progress has been made in this subject, most notably in the work of Erik Christensen [5], [6]. We will apply the results of Section 2 to the study of *compact* derivations of A into $B(H)$. Indeed, the following result in concert with Theorem 2.12 completely determines the structure of such derivations. We express a grateful acknowledgement to Erik Christensen for some remarks which led to its proof.

3.1. THEOREM. *Let A be a* C**-subalgebra of $B(H)$, $\delta\colon A \to B(H)$ a derivation. Let A^- denote the* WOT*-closure of A in $B(H)$. Then δ is compact if and only if there exists a $T \in B(H)$ asymptotically commuting with A^- such that $\delta = \mathrm{ad}_T|_A$.*

PROOF. ($\Leftarrow$). This is clear.

($\Rightarrow$). We claim first that δ extends to a compact derivation of A^- into $B(H)$. To see this, notice first that $\delta^{**}\colon A^{**} \to B(H)$ is a $\sigma(A^{**}, A^*)$-ultraweakly continuous compact linear extension of δ to the enveloping W*-algebra A^{**} of A. Let π denote the $\sigma(A^{**}, A^*)$-ultraweakly continuous extension of the identity representation of A on H to a representation of A^{**} onto A^-. From the $\sigma(A^{**}, A^*)$-density of A in A^{**} and the fact that δ is a derivation, we conclude that

$$(3.1) \qquad \delta^{**}(ab) = \pi(a)\delta^{**}(b) + \delta^{**}(a)\pi(b), \qquad \forall\, a,\ b \in A^{**}.$$

Let $1-z$ = central support projection of $\ker \pi$ in A^{**}. There is an ultraweak $\sigma(A^{**}, A^*)$-continuous isomorphism τ of A^- onto $A^{**}z$. Let $\varrho = \delta^{**} \circ \tau$. By

(3.1), ϱ is a derivation, it is evidently compact, and since $\delta^{**}(z)=0$, which follows upon evaluating (3.1) at $a=b=z$, ϱ extends δ. It is hence the extension that we seek.

Let R denote the von Neumann subalgebra of $B(H)$ generated by the type II_1 component M of A^- and the identity operator. Let a be a fixed self-adjoint element of M, and let Z be a maximal abelian self-adjoint subalgebra of R containing a. Z is type I, and so by [3], there is an $S \in B(H)$ for which $\delta|_Z = \mathrm{ad}_S|_Z$. Now

$$Z = Z_1 \oplus \{\lambda(1-p) : \lambda \in \mathbf{C}\},$$

where p is the minimal central support of M in A^- and Z_1 is maximal abelian in M. Since $\delta|_Z$ is compact and Z_1 has no nonzero minimal projections, we conclude by Theorem 2.12 that $S=U\oplus V$, where $U \in B((1-p)(H))$ and V commutes with Z_1 in $B(p(H))$. Thus $\delta(a)=0$, and since a is an arbitrary self-adjoint element in M, δ vanishes on M. We hence conclude by [3] and [5] that there exists a $T \in B(H)$ with $\delta = \mathrm{ad}_T|_{A^-}$. Since δ is compact, T asymptotically commutes with A^-.

3.2. COROLLARY. *Let A be a C^*-subalgebra of $B(H)$, and let p denote the purely discrete support of A^-. Then A admits a nonzero compact derivation into $B(H)$ if and only if $pA^- \neq \{\lambda p : \lambda \in \mathbf{C}\}$.*

If we combine the characterization of the asymptotic commutant of A^- given by Theorem 2.12 with Theorem 3.1, we obtain the following corollary, which answers Question 1 of [14] affirmatively for the case $X=B(H)$.

3.3. COROLLARY. *Every compact derivation of a C^*-subalgebra A of $B(H)$ into $B(H)$ is the norm limit of finite-rank derivations of A into $B(H)$.*

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