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CHARACTERIZATION OF MATRIX-ORDERED STANDARD FORMS OF W*-ALGEBRAS

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Abstract.

Let $\mathscr{M}$ be a von Neumann algebra operating in standard form on a Hilbert space $\mathscr{H}$ and let $\mathscr{H}^+$ be the corresponding selfdual cone. The algebra $\mathscr{M}_n = \mathscr{M} \otimes M_n$ operates on the Hilbert space $\mathscr{H}_n = \mathscr{H} \otimes M_n$ in such a way that $\mathscr{H}$ is matrix-ordered by the corresponding selfdual cones $\mathscr{H}_n^+$ $(n \in \mathbf{N})$. We call this situation a matrix-ordered standard form of $\mathscr{M}$.

On the other hand, to any matrix-ordered Hilbert space with selfdual cones we associate a von Neumann algebra which respects the matrix order. We call it the matrix multiplier algebra. It is a non-commutative analogue of the ideal center of an ordered vector space. If a von Neumann algebra operates in standard form on a matrix-ordered Hilbert space $\mathscr{H}$ with selfdual cones $\mathscr{H}_n^+$ then it is the matrix multiplier algebra of this space.

There is a one-to-one correspondence between the projections of the matrix multiplier algebra and the projectable faces of the cones. We obtain a characterization of matrix-ordered standard forms of von Neumann algebras in terms of the facial structure of the cones $\mathscr{H}_n^+$: The matrix multiplier algebra is in standard form iff every completed face of $\mathscr{H}_n^+$ $(n \in \mathbf{N})$ is projectable.

Introduction.

The striking analogy of order structure in C*- and W*-algebras and related spaces such as their duals, preduals or the selfadjoint cones of standard representations with the lattice structure of the corresponding commutative objects: $C(K)$, $L^\infty(\mu)$, and $L^2(\mu)$ has always been a rich source for inspiration in the theory of operator algebras. The theory of Banach lattices unifies the order properties of the classical function spaces. A general theory of non-commutative order is growing [10] fast, but a simple general definition comparable to "Banach lattice" has yet to be found. An important step in the development of Banach lattices and a main tool in the theory is the

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characterization of the classical function spaces in terms of AM-spaces, AL-spaces and Hilbert lattices. The corresponding program is carried out for selfdual cones of standard forms of W*-algebras by Connes [9] and for C*-algebras by Alfsen, Shultz and Hanche-Olsen [1]. Another method was developed by Werner [17].

An operator algebra is determined by its underlying ordered vector space uniquely up to a Jordan isomorphism. An additional condition often called an orientation, is necessary to determine the full algebraic structure. Without orientation one obtains characterizations of JBW-algebras (Alfsen, Shultz and Størmer [2] and Bellissard and Iochum [4], [5]).

At present we know of three different notions of orientation: Connes [9] introduced an orientation of a selfadjoint homogeneous cone in a Hilbert space. Alfsen, Shultz and Hanche-Olsen [1] defined an orientation of the state space of an order-unit space with the three-ball property. Choi and Effros [7] introduced matrix-ordered spaces as the appropriate objects to which completely positive maps apply. They showed that an injective matrix-ordered order-unit space is an injective C*-algebra. In [20] the injective W*-algebras are characterized by a matricial analogue of the Riesz separation property. Werner [17] characterizes C*- and W*-algebras in terms of the facial structure of the underlying matrix-ordered spaces. In [19] the physical meaning of matrix-order and the facial structure are discussed.

Matrix-order seems a very natural structure to describe order and orientation of operator algebras and related spaces. In this paper we characterize standard forms of von Neumann algebras (Araki [3], Connes [9], Haagerup [11]) by the facial properties of the selfdual cones of the associated matrix-ordered Hilbert space. Let us explain our procedure which is inspired by the commutative case: Let μ be a localizable measure. $L^2(\mu)$ is a Hilbert lattice. Its ideal center (Wils [18]) is the commutative von Neumann algebra of multiplication operators. There is a unique correspondence between the projections in the ideal center and the split-faces of the selfdual cone. (Penney [13], Bös [6]). We define the matrix multiplier algebra as a non-commutative analogue of the ideal center and study the faces which correspond to the projections of the matrix multiplier algebra. We are led to these definitions by the physical considerations in [19] and the corresponding results of Werner for matrix-ordered order-unit spaces. In [16] the Lie algebra of derivations of a 2-ordered Hilbert space with selfdual cones was studied.

1. Matrix-ordered standard forms.

Let $M_{m,n}$ be the space of complex $m \times n$ matrices and M_n be the $n \times n$ matrices, $m, n \in \mathbb{N}$. We will write st: $\alpha \rightarrow \alpha^*$ for the natural involution on $M_{m,n}$

and Tr: $M_n \to \mathbf{C}$ for the trace on M_n. If V is a complex vector space then $V_{m,n} = V \otimes M_{m,n}$ and $V_n = V \otimes M_n$ will denote the vector spaces of $m \times n$ and $n \times n$ matrices with entries in V. If J is an involution on V, then we have a natural involution

$$J_{m,n} = J \otimes \mathrm{st}: \ \mathrm{V}_{m,n} \to \mathrm{V}_{n,m}$$

defined by $[v_{kj}] \to [Jv_{j,k}]$ and we write J_n for $J_{n,n}$. We define

$$V_h = \{v \in V \mid v = Jv\}$$

to be the real vector space of selfadjoint (hermitian) elements.

A complex vector space V with an involution J is called *matrix-ordered*, if each $(V_n)_h$ is partially ordered by a cone V_n^+ and the following transformation law yields: if α is any $n \times m$ matrix of complex numbers, $m, n \in \mathbf{N}$, then

$$\alpha V_m^+ \alpha^* \subset V_n^+ . \tag{1.1}$$

Let V and W be matrix-ordered vector spaces and Φ: $V \to W$ be a linear map. If

$$\Phi_n := \Phi \otimes \mathrm{id}_n : \ V_n \to W_n$$

is defined by $\Phi_n[v_{ij}] \to [\Phi v_{ij}]$ and maps V_n^+ into W_n^+ for all $n \in \mathbf{N}$, then Φ is called *completely positive*.

The definition of matrix-order and basic results are due to M. D. Choi and E. G. Effros [7].

If $\mathscr{H}$ is a complex Hilbert space with a selfdual cone $\mathscr{H}^+$ (i.e. $\xi \in \mathscr{H}^+$ if and only if $\langle \xi, \mathscr{H}^+ \rangle \geqq 0$), then $\mathscr{H}_h := \mathscr{H}^+ - \mathscr{H}^+$ is a real Hilbert space, and we have that $\mathscr{H} = \mathscr{H}_h \oplus i\mathscr{H}_h$. $\mathscr{H}^+$ induces an involution $J = J_{\mathscr{H}^+}$ on $\mathscr{H}$, namely the map $J: \xi + i\eta \to \xi - i\eta$ for $\xi, \eta \in \mathscr{H}_h$. This can be found in [9], Proposition 4.1.

Let $\mathscr{M}$ be a W*-algebra that operates on a Hilbert space $\mathscr{H}$, $1_{\mathscr{H}} \in \mathscr{M}$, let $\mathscr{H}^+$ be a selfdual cone in $\mathscr{H}$ and $J = J_{\mathscr{H}^+}$ the induced involution. We call $(\mathscr{M}, \mathscr{H}, \mathscr{H}^+)$ a *standard form* of $\mathscr{M}$ if the following conditions are satisfied:

(1.2)
(i) $J\mathscr{M}J = \mathscr{M}'$,
(ii) $JzJ = z^*$ for every z in the center of $\mathscr{M}$,
(iii) $xJxJ\mathscr{H}^+ \subset \mathscr{H}^+$ for every $x \in \mathscr{M}$.

Every W*-algebra $\mathscr{M}$ has a standard representation $(\mathscr{M}, \mathscr{H}, \mathscr{H}^+)$ which is unique up to a spatial isomorphism. H. Araki [3], A. Connes [9] proved this for σ-finite W*-algebras and U. Haagerup [11, Theorem 1.6 and Theorem 2.3] presented the final form of the standard representation.

Let φ be a faithful normal semifinite weight on $\mathscr{M}$ and

$$\mathscr{N}_\varphi := \{x \in \mathscr{M} \mid \varphi(x^*x) < \infty\}$$

the induced pre-Hilbert space. Let $\mathscr{H}_\varphi$ be the completion of $\mathscr{N}_\varphi$ and η_φ: $\mathscr{N} \to \mathscr{H}_\varphi$, π_φ: $\mathscr{M} \to B(\mathscr{H}_\varphi)$ the natural embeddings. Then $\mathscr{A} = \eta_\varphi(\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*)$ is a left Hilbert algebra. If J_φ is the involution defined by the Tomita-Takesaki theory and we put

$$(1.3) \qquad \mathscr{H}_\varphi^+ = \{\pi_\varphi(x)J_\varphi\eta_\varphi(x) \mid x \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*\}^- ,$$

then $(\pi_\varphi(\mathscr{M}), \mathscr{H}_\varphi, \mathscr{H}_\varphi^+)$ is a standard representation of $\mathscr{M}$ and $J_\varphi = J_{\mathscr{H}_\varphi^+}$ (cp. U. Haarerup [11, Theorem 1.6] and F. Combes [8] and F. Perdrizet [14]).

Also φ induces faithful normal semifinite weights φ_n on the W*-algebras $\mathscr{M}_n = \mathscr{M} \otimes M_n$ in a very natural way:

$$(1.4) \qquad \varphi_n = \varphi \otimes \mathrm{Tr} : \; x \to \sum \varphi(x_{kk}) \quad \text{for all } x = [x_{jk}] \in \mathscr{M}_n^+ .$$

To each φ_n exist similar objects as to φ which will be denoted by $\mathscr{N}_n$, $\mathscr{H}_n$, π_n, J_n, $\mathscr{H}_n^+$. It is easy to see that

$$(1.5) \qquad \mathscr{N}_n = \mathscr{N}_\varphi \otimes M_n \quad \text{and} \quad \mathscr{H}_n = \mathscr{H}_\varphi \otimes M_n .$$

If we define an inner product $\langle \alpha, \beta \rangle := \mathrm{Tr}\,(\beta^* \alpha)$ for $\alpha, \beta \in M_n$, then $\mathscr{H}_n$ is the tensor product of the Hilbert spaces $\mathscr{H}_\varphi$ and M_n. With similar methods as in [9, Lemma 2.3] we can show that

$$(1.6) \qquad J_n = J_\varphi \otimes \mathrm{st} .$$

1.1 LEMMA. *Suppose $\mathscr{M}$ is a* W*-*algebra and φ is a faithful normal semifinite weight on $\mathscr{M}$ and $(\pi_n(\mathscr{M}), \mathscr{H}_n, \mathscr{H}_n^+)$ the standard representation of $\mathscr{M}_n$ corresponding to the weight φ_n for every $n \in \mathbf{N}$. In this situation we can obtain the cones $\mathscr{H}_n^+$ from $\mathscr{H}_1^+ = \mathscr{H}_\varphi^+$ by the formula*

$$(1.7) \qquad \mathscr{H}_n^+ = \overline{\mathrm{co}}\,\{aJ_{n,1}aJ_1(\mathscr{H}_1^+) \mid a \in \pi_\varphi(\mathscr{M}) \otimes M_{n,1}\}$$

where $\overline{\mathrm{co}}$ denotes the closed convex hull. In addition

$$(1.8) \qquad aJ_{n,m}aJ_m(\mathscr{H}_m^+) \subset \mathscr{H}_n^+$$

holds for every operator valued matrix $a \in \pi_\varphi(\mathscr{M}) \otimes M_{n,m}$.

PROOF. Let $x \in (\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*) \otimes M_{m,k}$ $(m, k \in \mathbf{N})$ and define matrices $x_\varkappa \in (\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*) \otimes M_m$ for $\varkappa = 1, \ldots, k$ whose first column coincides with the $\varkappa$th column of x and whose remaining columns are zero. Let

$$\eta_{m,k} := \eta_\varphi \otimes \mathrm{id}_{m,k} : \; \mathscr{N}_\varphi \otimes M_{m,k} \to \mathscr{H}_\varphi \otimes M_{m,k}$$

and

$$\pi_{m,k} := \pi_\varphi \otimes \mathrm{id}_{m,k} : \; \mathscr{M} \otimes M_{m,k} \to B(\mathscr{H}_\varphi) \otimes M_{m,k}$$

be the naturally extended embeddings. We have

(1.9) $$\pi_{m,k}(x)J_{m,k}\eta_{m,k}(x) = \left[\sum_{\varkappa=1}^{k} \pi_\varphi(x_{\lambda\varkappa})J_\varphi\eta_\varphi(x_{\mu\varkappa})\right]_{\lambda,\mu=1\dots m}$$
$$= \sum_{\varkappa=1}^{k} \pi_m(x_\varkappa)J_m\eta_m(x_\varkappa) \in \mathscr{H}_m^+ .$$

If $m=k$ and $\tilde{x}_\varkappa \in (\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*)\otimes M_{m,1}$ is the $\varkappa$th column of x, then (1.9) shows that

$$\pi_m(x)J_m\eta_m(x) = \sum_{\varkappa=1}^{m} \pi_{m,1}(\tilde{x}_\varkappa)J_{m,1}\eta_{m,1}(\tilde{x}_\varkappa) .$$

Hence $\mathscr{H}_m^+$ is the closed convex hull of elements of the form

(1.10) $$\xi = \pi_{m,1}(y)J_{m,1}\eta_{m,1}(y), \quad y = [y_\mu]_{\mu=1\dots m} \in (\mathscr{N}_\varphi \cap \mathscr{N}_\varphi^*)\otimes M_{m,1} .$$

To show (1.7) we have to rewrite ξ in an appropriate way. Let

$$z = \left(\sum_{\mu=1}^{m} y_\mu^* y_\mu\right)^{\frac{1}{2}} \in \mathscr{N}_\varphi \cap \mathscr{N}_\varphi^* .$$

Then there exists $a=[a_\mu]_{\mu=1\dots m} \in \mathscr{M}\otimes M_{m,1}$ such that $y_\mu = a_\mu z$ (cp. [12, Proposition 1.45]). Therefore we can write

$$\xi = \pi_{m,1}(a)\pi_1(z)J_{m,1}\pi_{m,1}(a)\eta_1(z)$$
$$= \pi_{m,1}(a)J_{m,1}\pi_{m,1}(a)J_1(\pi_1(z)J_1\eta_1(z)) .$$

This proves (1.7).

Let $a \in \mathscr{M}\otimes M_{n,m}$ and ξ as in (1.10). Then we have by (1.9)

$$\pi_{n,m}(a)J_{n,m}\pi_{n,m}(a)J_m\xi = \pi_{n,1}(ay)J_{n,1}\eta_{n,1}(ay) \in \mathscr{H}_n^+ .$$

This completes the proof.

(1.6) and (1.8) show that $\mathscr{H}_\varphi$ is matrix-ordered in the sense of Choi and Effros [7]. Moreover (1.8) means that $\mathscr{H}_\varphi$ is a matrix ordered $\mathscr{M}$-module in the sence of Werner [17].

1.2 Definition. Suppose $\mathscr{H}$ is a complex Hilbert space and $\mathscr{H}_n^+ \subset \mathscr{H}_n$ ($n \in \mathbb{N}$) is a family of selfdual cones. Hereby $\mathscr{H}_n$ is meant as the Hilbert space tensor product of the Hilbert spaces $\mathscr{H}_\varphi$ and M_n (see below (1.5)). In addition suppose that the transformation law (1.1) yields, i.e.

(1.11) $$\alpha\mathscr{H}_m^+\alpha^* \subset \mathscr{H}_n^+$$

for every $n\times m$ matrix α with complex coefficients. Then we call $\mathscr{H}$ a *matrix ordered Hilbert space with selfdual cones.*

1.3. LEMMA. *Let $\mathscr{H}$ be a matrix ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ ($n \in \mathbb{N}$). If $J_n = J_{\mathscr{H}_n^+}$ is the induced involution on $\mathscr{H}_n$, then we have that $J_n = J_1 \otimes \mathrm{st}$. This implies that $\mathscr{H}$ is matrix-ordered in the sense of Choi and Effros* [7].

PROOF. Let $x = [x_{kj}] \in \mathscr{H}_n^+$. If one chooses suitable $\alpha \in M_{1,n}$ in (1.11) then one obtains that x_{kk}, $(x_{kk}+x_{jj}) \pm (x_{kj}+x_{jk})$, $(x_{kk}+x_{jj}) \pm i(x_{kj}-x_{jk})$ are elements of $\mathscr{H}_1^+$. This implies that $J_1 x_{kk} = x_{kk}$ and $J_1 x_{kj} = x_{jk}$ ($j, k = 1, \ldots, n$). Hence J_n and $J_1 \otimes \mathrm{st}$ coincide on $\mathscr{H}_n^+$.

1.4 DEFINITION. Let $\mathscr{M}$ be a W*-algebra operating on a Hilbert space $\mathscr{H}$, $1_{\mathscr{H}} \in \mathscr{M}$ and let $\mathscr{H}_n^+$ ($n \in \mathbb{N}$) be a family of selfdual cones in $\mathscr{H}_n$. If $(\mathscr{M}, \mathscr{H}, \mathscr{H}_1^+)$ is a standard representation of $\mathscr{M}$ and if for every $a \in \mathscr{M} \otimes M_{n,m}$ ($m, n \in \mathbb{N}$)

$$aJ_{n,m}aJ_m(\mathscr{H}_m^+) \subset \mathscr{H}_n^+ \tag{1.12}$$

holds, then we call $(\mathscr{M}, \mathscr{H}, \mathscr{H}_n^+)$ a *matrix-ordered standard form.*

Suppose $(\mathscr{M}, \mathscr{H}, \mathscr{H}_n^+)$ is a matrix-ordered standard form. By Lemma 1.1 formula (1.7) we obtain that $(\mathscr{M}_n, \mathscr{H}_n, \mathscr{H}_n^+)$ is a standard form for every $n \in \mathbb{N}$. We will show that $\mathscr{M}$ can be reconstructed from the family $\mathscr{H}_n^+$ ($n \in \mathbb{N}$) of selfdual cones. In addition it is our objective to characterize those matrix-ordered Hilbert spaces with selfdual cones $\mathscr{H}_n^+$ which belong to a matrix ordered standard form $(\mathscr{M}, \mathscr{H}, \mathscr{H}_n^+)$.

2. The matrix multiplier algebra.

In [17], K.-H. Werner has constructed the matrix multiplier algebra of a matrix-ordered complete order unit or base normed space. He could show that it is a C*-algebra or a W*-algebra, respectively. We will define the matrix multiplier algebra of a matrix-ordered Hilbert space with selfdual cones and we will prove that it is a W*-algebra. It is easy to see that some parts of our construction carry over to general matrix-ordered spaces with proper archimedian cones. But one needs some additional structure to define an involutove algebra.

Let $\mathscr{H}$ be a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ ($n \in \mathbb{N}$). Let

$$J := J_1 = J_{\mathscr{H}_1^+}, \quad J_n = J_{\mathscr{H}_n^+} = J \otimes \mathrm{st}$$

(Lemma 1.3) and

$$J_{m,n} := J \otimes \mathrm{st} : \mathscr{H} \otimes M_{m,n} \to \mathscr{H} \otimes M_{n,m}$$

be the induced involutions. In order to write (1.12) similarly to (1.11) we have to define a multiplication from the right hand side for operator valued matrices. We let

$$\xi x^J := JxJ\xi \tag{2.1}$$

for $\xi \in \mathcal{H}$ and every linear (not necessarily bounded) map $x \in L(\mathcal{H})$. If $\xi = [\xi_{\mu\nu}] \in \mathcal{H} \otimes M_{m,n}$, $x = [x_{\varkappa\mu}] \in L(\mathcal{H}) \otimes M_{k,m}$, and $y = [y_{\lambda\nu}] \in L(\mathcal{H}) \otimes M_{l,n}$ $(k, l, m, n \in \mathbf{N})$, then we define

$$x\xi := \left[\sum_{\mu=1}^{m} x_{\varkappa\mu}\xi_{\mu\nu}\right] \in \mathcal{H} \otimes M_{k,n} \tag{2.2}$$

and

$$\xi y^J := J_{l,m} y J_{m,n} \xi = \left[\sum_{\nu=1}^{n} \xi_{\mu\nu} y_{\lambda\nu}^J\right] \in \mathcal{H} \otimes M_{m,l}\,. \tag{2.3}$$

Finally we define a sesquilinear Jordan product for x and y by the formula

$$\{x\xi y^J\} := \tfrac{1}{2}((x\xi)y^J + x(\xi y^J))\,. \tag{2.4}$$

If $x_1, \ldots, x_n \in L(\mathcal{H})$ $(n \in \mathbf{N})$, then we let

$$\operatorname{diag}(x_1, \ldots, x_n) = \begin{bmatrix} x_1 & 0 & \ldots & 0 \\ & x_2 & & \\ & & \ddots & \\ 0 & \ldots & 0 & x_n \end{bmatrix} \in L(\mathcal{H}) \otimes M_n\,. \tag{2.5}$$

We write 1 for the identity map on $\mathcal{H}$.

2.1 Definition. Let $\mathcal{H}$ be a matrix-ordered Hilbert space with selfdual cones $\mathcal{H}_n^+$ $(n \in \mathbf{N})$ and induced involution J. The matrix multiplier algebra $\mathcal{M}$ of $\mathcal{H}$ is the set of all linear maps $x \in L(\mathcal{H})$ that satisfy (see (2.4))

$$\{\operatorname{diag}(x, 1, \ldots, 1)\xi \operatorname{diag}(x, 1, \ldots, 1)^J\} \in \mathcal{H}_n^+ \tag{2.6}$$

for every $\xi \in \mathcal{H}_n^+$ and all $n \in \mathbf{N}$.

The next theorem shows that this set carries a rich structure. $\mathcal{M}$ is in fact a W*-algebra and $\mathcal{H}$ is a two-sided $\mathcal{M}$-module.

2.2 Theorem. *Suppose $\mathcal{H}$ is a matrix-ordered Hilbert space with selfdual cones $\mathcal{H}_n^+$ and J is the induced involution. If $\mathcal{M}$ is the matrix multiplier algebra of $\mathcal{H}$, then we have*

(i) $J\mathcal{M}J \subset \mathcal{M}'$, i.e. $(x\xi)y^J = x(\xi y^J)$ *for* $x, y \in \mathcal{M}$, $\xi \in \mathcal{H}$.
(ii) $\mathcal{M}$ *is a* W*-*algebra.*
(iii) $a\mathcal{H}_m^+ a^J \subset \mathcal{H}_n^+$ *for every* $a \in \mathcal{M} \otimes M_{n,m}$ $(m, n \in \mathbf{N})$.

PROOF. (i) If one chooses suitable scalar matrices $\alpha \in M_n$, one sees by (1.11) that $\mathcal{H}_n^+$ is invariant under simultaneous permutations of rows and columns. Hence (2.6) implies for $x \in \mathcal{M}$

(2.7) $\{\operatorname{diag}(1,\ldots,1,x,1\ldots,1)\mathcal{H}_n^+ \operatorname{diag}(1,\ldots,1,x,1\ldots,1)^J\} \subset \mathcal{H}_n^+$.

Let $\xi \in \mathcal{H}_1^+$ and $x, y \in \mathcal{M}$. We define

$$d_1 = \operatorname{diag}(x,1,1), \quad d_2 = \operatorname{diag}(1,1,y), \quad d_3 = \operatorname{diag}(1,x,1) \in L(\mathcal{H}) \otimes M_3,$$

and

$$\alpha = [1,-1,1], \quad \beta = [\lambda^{-1}, \lambda^{-1}, \lambda\varepsilon] \in \mathrm{M}_{1,3}$$

for $\lambda > 0$ and $\varepsilon = \pm 1, \pm i$. It follows that the iterated Jordan product

$$\beta\{d_3\{d_2\{d_1(\alpha^*\xi\alpha)d_1^J\}d_2^J\}d_3^J\}\beta^*$$
$$= \lambda^2\{y\xi y^J\} + \varepsilon(y(\xi x^J) - (y\xi)x^J) + \bar{\varepsilon}((x\xi)y^J - x(\xi y^J))$$

is an element of $\mathcal{H}_1^+$. Since $\mathcal{H}_1^+$ is closed proper cone this holds for $\lambda \to 0$ and we obtain $(x\xi)y^J - x(\xi y^J) = 0$. This proves (i). Left and right multiplication with elements of $\mathcal{M}$ commute and we write $x\xi y^J$ for $\{x\xi y^J\}$ in this case.

(ii) Let again $x, y \in \mathcal{M}$. Part (i) shows that $xy \in \mathcal{M}$. Also $\lambda x \in \mathcal{M}$ for every $\lambda \in \mathbf{C}$. We define

$$d_1 = \operatorname{diag}(x,1,\ldots,1), \quad d_2 = \operatorname{diag}(1,y,1,\ldots,1) \in \mathcal{M}_{n+1}$$

and

$$\alpha = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1_{n-1} \end{bmatrix} \in M_{n,n+1},$$

where 1_n denotes the unit matrix in M_n. Then we have for $\xi \in \mathcal{H}_n^+$

$$\operatorname{diag}(x+y,1,\ldots,1)\xi \operatorname{diag}(x+y,1,\ldots,1)^J = \alpha d_2 d_1 \alpha^* \xi \alpha d_1^J d_2^J \alpha^* \in \mathcal{H}_n^+.$$

Hence $\mathcal{M}$ is closed under addition.

The selfdual cone $\mathcal{H}_2^+$ is generating and weakly normal in $(\mathcal{H}_2)_h$. It follows that every positive linear map from $(\mathcal{H}_2)_h$ into itself is bounded (cp. [15, Chapter V, Section 5.5 and 5.6]). Hence

$$\operatorname{diag}(x,1)J_2 \operatorname{diag}(x,1)J_2 \in B(\mathcal{H}_2) \quad \text{for every } x \in \mathcal{M}$$

and therefore $x \in B(\mathcal{H})$.

If $x \in \mathcal{M} \subset B(\mathcal{H})$, then $x^* \in \mathcal{M}$ since the cones $\mathcal{H}_n^+$ are selfdual. Kaplansky's density theorem implies that $\mathcal{M}$ is weakly closed.

(iii) This part of the proof is due to K. H. Werner [17, Proposition 4.1]. Our assertion is clear in the case $n = m$ and $a = \operatorname{diag}(x_1, \ldots, x_n) \in \mathcal{M}_n$ ($n \in \mathbb{N}$). The general case follows from this in the following way. Let $a \in \mathcal{M} \otimes M_{n,m}$ and $\xi \in \mathcal{H}_n^+$.

We define a block diagonal matrix

$$\tau := \begin{bmatrix} 1\ldots1 & 0\ldots0 & \ldots & \ldots & \ldots & 0\ldots0 \\ 0\ldots0 & 1\ldots1 & 0\ldots0 & \ldots & \ldots & 0\ldots0 \\ \ldots & 0\ldots0 & \ldots & & & \ldots \\ \ldots & \ldots & & \ldots & & \ldots \\ \ldots & \ldots & & & \ldots & 0\ldots0 \\ 0\ldots0 & 0\ldots0 & \ldots & \ldots & 0\ldots0 & 1\ldots1 \end{bmatrix} \in M_{m,mn}$$

a matrix

$$\gamma := [1_n, \ldots, 1_n] \in M_{n,mn}$$

and

$$d := \operatorname{diag}(a_{11}, a_{21}, \ldots, a_{n1}, a_{12}, a_{22}, \ldots, a_{n2}, \ldots, a_{1m}, \ldots, a_{nm})$$

in $\mathcal{M}_{mn}$. Since $a = \gamma d \tau^*$ we obtain

$$a\xi a^J = \gamma d\tau^* \xi \tau d^J \gamma^* \in \mathcal{H}_n^+ . \tag{2.8}$$

2.3 COROLLARY. *$\mathcal{H}_k = \mathcal{H} \otimes M_k$ ($k \in \mathbb{N}$) is a matrix-ordered Hilbert space with selfdual cones $\mathcal{H}_{kn}^+$ ($n \in \mathbb{N}$). The matrix multiplier algebra of $\mathcal{H}_k$ is $\mathcal{M}_k = \mathcal{M} \otimes M_k$ operating by left multiplication.*

PROOF. We identify $B(\mathcal{H}) \otimes M_k \subset B(\mathcal{H}_k)$ by (2.2) and $\mathcal{H}_k \otimes M_n = \mathcal{H} \otimes M_{kn}$ ($k, n \in \mathbb{N}$). It is clear that $\mathcal{H}_k$ is a matrix ordered Hilbert space with selfdual cones $\mathcal{H}_{kn}^+$. Theorem 2.2 (iii) implies that $\mathcal{M} \otimes M_k$ is contained in the matrix multiplier algebra $\mathcal{N}_k$ of $\mathcal{H}_k$.

M_k is the tensor product of the Hilbert spaces $M_{k,1}$ and $M_{1,k}$ (which are both isometric to $\mathbb{C}^k$ with the standard scalar product). The complex $k \times k$ matrices operate on $M_{k,1}$ by left multiplication and on $M_{1,k}$ by right multiplication. Hence $B(M_{k,1}) = M_k$ and $B(M_{1,k}) = \tilde{M}_k$, where $\tilde{M}_k$ is the opposite algebra to M_k. We obtain

$$B(\mathcal{H}_k) = B(\mathcal{H} \otimes M_{k,1} \otimes M_{1,k}) = B(\mathcal{H}) \otimes M_k \otimes \tilde{M}_k . \tag{2.9}$$

Let β be a $k \times k$ matrix, $\tilde{\beta} \in \tilde{M}_k$ right multiplication with β. It follows from (1.11) that $1_{\mathcal{H}} \otimes M_k \otimes \tilde{1}_k \subset \mathcal{N}_k$. We obtain

$$1_{\mathscr{H}}\otimes 1_k\otimes\tilde{\beta} = J_k(1_{\mathscr{H}}\otimes\beta^*\otimes\tilde{1}_k)J_k \in J_k(1_{\mathscr{H}}\otimes M_k\otimes\tilde{1}_k)J_k \subset J_k\mathscr{N}_kJ_k \subset \mathscr{N}'_k$$

by theorem 2.2 (i). Hence

$$\mathscr{N}_k \subset (1_{\mathscr{H}}\otimes 1_k\otimes\tilde{M}_k)' = B(\mathscr{H})\otimes M_k\otimes\tilde{1}_k .$$

Since $1_{\mathscr{H}}\otimes M_k$ is contained in $\mathscr{N}_k$ we obtain $\mathscr{N}_k=\mathscr{N}_1\otimes M_k$, where $\mathscr{N}_1$ is a subspace of $B(\mathscr{H})$. It is easy to see (again by (1.11)) that $\mathscr{N}_1\subset\mathscr{M}$. This gives $\mathscr{N}_k\subset\mathscr{M}\otimes M_k\otimes\tilde{1}_k=\mathscr{M}\otimes M_k$.

In the sequel we will write $\mathscr{M}_k$ for the matrix multiplier algebra of $\mathscr{H}_k$.

2.4 THEOREM. *If* $(\mathscr{N}, \mathscr{H}, \mathscr{H}_n^+ \ (n \in \mathsf{N}))$ *is a matrix-ordered standard form of the* W*-*algebra* $\mathscr{N}$, *then* $\mathscr{N}$ *is the matrix multiplier algebra of the matrix-ordered Hilbert space* $\mathscr{H}$ *with selfdual cones* $\mathscr{H}_n^+$.

PROOF. Let $\mathscr{M}$ be the matrix multiplier algebra of $\mathscr{H}$. Definition 1.4 implies that $\mathscr{N}\subset\mathscr{M}$. By (1.2) and Theorem 2.2 (i) we obtain $\mathscr{N}'=J\mathscr{N}J\subset J\mathscr{M}J\subset\mathscr{M}'$. The von Neumann commutation theorem implies that $\mathscr{M}\subset\mathscr{N}$.

3. Projectable faces.

In this section we want to examine the relationship between the projections of the matrix multiplier algebra $\mathscr{M}$ of a matrix-ordered Hilbert space $\mathscr{H}$ with selfdual cones $\mathscr{H}_n^+$ $(n \in \mathsf{N})$ and the faces of the cone $\mathscr{H}^+=\mathscr{H}_1^+$.

3.1 DEFINITION. Suppose $\mathscr{H}$ is a complex Hilbert space and $\mathscr{H}^+\subset\mathscr{H}$ is a selfdual cone. Let $F\subset\mathscr{H}^+$ be a face of $\mathscr{H}^+$ and P_F be the selfadjoint projection on the closed subspace of $\mathscr{H}$ generated by F. We call F a *projectable face* if

$$P_F\mathscr{H}^+ = F .$$

If $S\subset\mathscr{H}^+$ is a subset then the set

$$S^\perp := \{\xi \in \mathscr{H}^+ \mid \langle\xi, S\rangle=0\}$$

is called the orthocomplementary face to S. A face $F\subset\mathscr{H}^+$ is called *completed* if $F=F^{\perp\perp}$

3.2 LEMMA. *Let* $\mathscr{H}$ *be a matrix-ordered Hilbert space with selfdual cones* $\mathscr{H}_n^+$ $(n \in \mathsf{N})$. *Suppose* $\xi=[\xi_{\mu\nu}] \in \mathscr{H}_n^+$ *and* $\xi_{kk}=0$ *for a fixed* $k \in \{1,\ldots,n\}$. *Then* $\xi_{\nu k}=\xi_{k\nu}=0$ *for* $\nu=1,\ldots,n$.

PROOF. Take $\lambda>0$ and $\varepsilon = 1, i$ and define

$$\alpha = (0,\ldots,0,\lambda^{-1},0,\ldots,0,\varepsilon\lambda,0,\ldots 0) \in \mathrm{M}_{1,n}\,,$$

where $\alpha_{1,k}=\lambda^{-1}$ and $\alpha_{1,\nu}=\varepsilon\lambda$. Then we obtain

$$\alpha\xi\alpha^* = \lambda^2\xi_{\nu\nu}+(\varepsilon\xi_{\nu k}+\bar{\varepsilon}\xi_{k\nu}) \in \mathscr{H}_1^+\,.$$

Since $\mathscr{H}_1^+$ is a proper closed cone this holds for $\lambda \to 0$ and we obtain $\xi_{\nu k}=\xi_{k\nu}=0$.

3.3. Corollary. *Let $\mathscr{H}$ be a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ $(n \in \mathrm{N})$, $J=J_1$ the induced involution, and $\mathscr{M}$ its matrix multiplier algebra. If $x \in \mathscr{M}$, $\xi \in \mathscr{H}_1^+$ and $xJxJ\xi=0$, then $x\xi=JxJ\xi=0$.*

Proof. By (1.11) we have

$$\Xi = \begin{bmatrix} \xi & \xi \\ \xi & \xi \end{bmatrix} \in \mathscr{H}_2^+\,.$$

Hence

$$\operatorname{diag}(x,1)\Xi\operatorname{diag}(x,1)^J = \begin{bmatrix} 0 & x\xi \\ JxJ\xi & \xi \end{bmatrix} \in \mathscr{H}_2^+$$

by Theorem 2.2. Lemma 3.2 implies that $x\xi=JxJ\xi=0$.

3.4. Proposition. *Let $\mathscr{H}$ be a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ $(n \in \mathrm{N})$, $\mathscr{H}^+=\mathscr{H}_1^+$, $J=J_1$ the induced involution, and $\mathscr{M}$ its matrix multiplier algebra. Suppose $p \in \mathscr{M}$ is a selfadjoint projection. Then*

(i) *$F:=pJpJ\mathscr{H}^+$ is a completed projectable face and $P_F=pJpJ$.*
(ii) *$F^\perp=(1-p)J(1-p)J\mathscr{H}^+$ and $P_{F^\perp}=(1-p)J(1-p)J$.*

Proof. (i) $pJpJ$ is a selfadjoint projection, since p and JpJ commute. $pJpJ\mathscr{H}^+\subset\mathscr{H}^+$ by Theorem 2.2. Let $\xi \in \mathscr{H}^+$ with $\xi=pJpJ\xi$, $\eta \in \mathscr{H}^+$ such that $\xi-\eta \in \mathscr{H}^+$, and

$$\zeta = \begin{bmatrix} \xi & \eta \\ \eta & \eta \end{bmatrix} = \begin{bmatrix} \xi-\eta & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \eta & \eta \\ \eta & \eta \end{bmatrix} \in \mathscr{H}_2^+\,.$$

Then $\operatorname{diag}(1-p,1)\zeta\operatorname{diag}(1-p,1)^J \in \mathscr{H}_2^+$. Since $(1-p)\xi(1-p)^J=0$, Corollary 3.3 implies now that $(1-p)\eta=J(1-p)J\eta=0$. Hence $\eta=pJpJ\eta$ and F is a face.

(ii) It is clear that $(1-p)J(1-p)J\mathscr{H}^+\subset F^\perp$. If $\eta \in F^\perp$, then η is perpendicular to the linear span of F, and therefore $pJpJ\eta=0$. By Corollary 3.3, $p\eta=JpJ\eta=0$. Hence $\eta=(1-p)J(1-p)J\eta$.

3.5. LEMMA. *Let $\mathscr{H}$ be a complex Hilbert space and $\mathscr{H}^+$ be a selfdual cone in $\mathscr{H}$. If F, G are projectable faces and $P_G F \subset F$, then $F \cap G$ is a projectable face and $P_{G\cap F} = P_G P_F = P_F P_G$.*

PROOF. $P_G F \subset F$ is equivalent to $P_G P_F \mathscr{H}^+ \subset P_F \mathscr{H}^+$, and therefore

$$P_G P_F = P_F P_G P_F = (P_G P_F)^2 \quad \text{and} \quad P_G P_F = P_F P_G P_F = (P_G P_F)^* .$$

Hence $P_G P_F = P_F P_G$ is a selfadjoint projection. It follows that

$$P_G P_F \mathscr{H}^+ \subset P_F \mathscr{H}^+ \cap P_G \mathscr{H}^+ = F \cap G \subset P_G P_F \mathscr{H}^+ .$$

If $\xi \in \mathscr{H}_m^+$ and $\eta \in \mathscr{H}_n^+$ $(m, n \in \mathrm{N})$ we define

$$\xi \oplus \eta := \begin{bmatrix} \xi & 0 \\ 0 & \eta \end{bmatrix} \in \mathscr{H}_{m+n}^+ .$$

Analoguously $S \oplus T := \{\xi \oplus \eta \mid \xi \in S, \eta \in T\}$ for $S \subset \mathscr{H}_m$ and $T \subset \mathscr{H}_n$. Let S be a subset of $\mathscr{H}^+$. We define the face in $\mathscr{H}^+$ generated by S to be the set

$$F_S = \{\xi \in \mathscr{H}^+ \mid \text{exists } \lambda > 0, \eta \in S \text{ such that } \lambda\eta - \xi \in \mathscr{H}^+\} .$$

3.6. LEMMA. *Suppose that $\mathscr{H}$ is a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n$ $(n \in \mathrm{N})$ and $S \subset \mathscr{H}_m^+$, $T \subset \mathscr{H}_n^+$. Then*

(i) $(S \oplus T)^{\perp\perp} = F_{S^{\perp\perp} \oplus T^{\perp\perp}}$,
(ii) $(S \oplus T)^{\perp} = F_{S^{\perp} \oplus T^{\perp}}$.

PROOF. Let $\xi_{11}, \eta_{11} \in \mathscr{H}_m$, $\xi_{12}, \eta_{12} \in \mathscr{H}_{m,n}$, $\xi_{21}, \eta_{21} \in \mathscr{H}_{n,m}$, $\xi_{22}, \eta_{22} \in \mathscr{H}_n$. Suppose that

$$(3.1) \qquad \xi = \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} \in (S \oplus T)^{\perp\perp}, \qquad \eta = \begin{bmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{bmatrix} \in (S \oplus T)^{\perp} .$$

It follows that $\eta_{11} \in S^{\perp}$, $\eta_{22} \in T^{\perp}$, and $\xi_{11} \in S^{\perp\perp}$, $\xi_{22} \in T^{\perp\perp}$, since $S^{\perp} \oplus T^{\perp} \subset (S \oplus T)^{\perp}$. Now

$$2\begin{bmatrix} \xi_{11} & 0 \\ 0 & \xi_{22} \end{bmatrix} - \xi = \begin{bmatrix} \xi_{11} & -\xi_{12} \\ -\xi_{21} & \xi_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \xi \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in \mathscr{H}_{n+m}^+ .$$

Hence

$$\xi \in F_{S^{\perp\perp} \oplus T^{\perp\perp}} \quad \text{and} \quad (S \oplus T)^{\perp\perp} \subset F_{S^{\perp\perp} \oplus T^{\perp\perp}} .$$

Similarly

$$\eta \in F_{S^{\perp} \oplus T^{\perp}} \quad \text{and} \quad (S \oplus T)^{\perp} \subset F_{S^{\perp} \oplus T^{\perp}} .$$

The opposite inclusions are easily verified and left to the reader.

From now on o_n will denote the zero vector and O_n the zero subspace in $\mathscr{H}_n$.

3.7. LEMMA. *Suppose that $\mathscr{H}$ is a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ ($n \in \mathbb{N}$) and $S \subset \mathscr{H}_m^+$. Then*

(i) $(S \oplus O_n)^{\perp\perp} = S^{\perp\perp} \oplus O_n$
(ii) $(S \oplus \mathscr{H}_n^+)^{\perp} = S^{\perp} \oplus O_n$.

PROOF. Let $\xi \in (S \oplus O_n)^{\perp\perp}$ as in (3.1). Then $\xi_{11} \in S^{\perp\perp}$ and $\xi_{22} \in O_n^{\perp\perp} = O_n$. Lemma 3.2 implies now that $\xi_{12} = \xi_{21} = o$. Hence $\xi \in S^{\perp\perp} \oplus O_n$. The opposite way is straightforward.

Let $\eta \in (S \oplus \mathscr{H}_n^+)$ as in (3.1). Then $\eta_{11} \in S^{\perp}$ and $\eta_{22} \in (\mathscr{H}_n^+)^{\perp} = O_n$. Now the remainder of the proof is clear.

The methods used in the proof of the following theorem were developed by K. H. Werner [17]. The physical aspects of faces with the projection property considered in the following theorem were investigated in [19]. The corresponding projections are called there interactive projections.

3.8. THEOREM. *Suppose $\mathscr{H}$ is a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ ($n \in \mathbb{N}$) and $F = F_1$ is a completed face in $\mathscr{H}^+ = \mathscr{H}_1^+$. Let*

$$F_n := (F \oplus \mathscr{H}_{n-1})^{\perp\perp} \qquad (n = 2, 3, \dots)\,.$$

If F_n is a projectable face for every $n \in \mathbb{N}$, then there exists a selfadjoint projection p in the matrix multiplier algebra $\mathscr{M}$ of $\mathscr{H}$ which satisfies

$$P_{F_n} = \operatorname{diag}(p, 1, \dots, 1) J_n \operatorname{diag}(p, 1, \dots, 1) J_n\,. \tag{3.2}$$

PROOF. Define $\varepsilon_{j,k} \in \mathrm{M}_n$ as the usual matrix units. We consider three selfadjoint projections $Q, Q', Q''\colon \mathscr{H}_n \to \mathscr{H}_n$ defined by

$$Q\xi = \varepsilon_{11}\xi\varepsilon_{11}, \quad Q'\xi = (1-\varepsilon_{11})\xi(1-\varepsilon_{11}) \quad \text{and} \quad Q''\xi = (1-\varepsilon_{nn})\xi(1-\varepsilon_{nn})$$

for $\xi \in \mathscr{H}_n$. (1.11) implies that all three projections map $\mathscr{H}_n^+$ into $\mathscr{H}_n^+$. We have $Q\mathscr{H}_n^+ = \mathscr{H}_1^+ \oplus O_{n-1}$, $Q'\mathscr{H}_n^+ = O_1 \oplus \mathscr{H}_{n-1}^+$, and $Q''\mathscr{H}_n^+ = \mathscr{H}_{n-1}^+ \oplus O_1$. Lemma 3.7.(i) shows that $Q\mathscr{H}_n^+$, $Q'\mathscr{H}_n^+$, and $Q''\mathscr{H}_n^+$ are faces which are projectable by their definition. Lemma 3.7 (ii) implies $Q(F_n^{\perp}) = Q(F^{\perp} \oplus O_{n-1}) = F_n^{\perp}$ and this implies $QF_n \subset F_n$. Hence we have by Lemma 3.5

$$QP_{F_n} = P_{F_n}Q\,, \tag{3.3}$$

and analoguously $Q'F_n \subset F_n$, respectively $Q''F_n \subset F_n$ implies

$$Q'P_{F_n} = P_{F_n}Q' \quad \text{and} \quad Q''P_{F_n} = P_{F_n}Q'' . \tag{3.4}$$

In addition Lemma 3.5 and Lemma 3.7(ii) imply that

$$QP_{F_n}\mathscr{H}_n^+ = F_n \cap (\mathscr{H}_1^+ \oplus O_{n-1}) = F_1 \oplus O_{n-1} = (P_{F_1}\mathscr{H}_1^+) \oplus O_{n-1}$$

$$Q''P_{F_n}\mathscr{H}_n^+ = F_n \cap (\mathscr{H}_{n-1} \oplus O_1) = F_{n-1} \oplus O_1 = (P_{F_{n-1}}\mathscr{H}_{n-1}^+) \oplus O_1 .$$

Hence we have for every $\xi \in \mathscr{H}_1$, $\eta \in \mathscr{H}_{n-1}$

$$P_{F_n}(\xi \oplus o_{n-1}) = P_{F_n}Q(\xi \oplus o_{n-1}) = (P_{F_1}\xi) \oplus o_{n-1} \tag{3.5}$$

$$P_{F_n}(\eta \oplus o_1) = P_{F_n}Q''(\eta \oplus o_1) = (P_{F_{n-1}}\eta) \oplus o_1 . \tag{3.6}$$

The identity

$$Q'P_{F_n}(\mathscr{H}_n^+) = F_n \cap (O_1 \oplus \mathscr{H}_{n-1}^+) = O_1 \oplus \mathscr{H}_{n-1}^+$$

implies

$$Q'P_{F_n} = Q' . \tag{3.7}$$

Let $v \in M_{n-1}$ be a unitary matrix and define $u = 1 \oplus v \in M_n$. The map $U\colon \mathscr{H}_n \to \mathscr{H}_n$ defined by $U\xi = u^*\xi u$ ($\xi \in \mathscr{H}_n$) is unitary and $U^*\xi = u\xi u^*$. (1.11) implies that $U\mathscr{H}_n^+ = \mathscr{H}_n^+$ and $U^*\mathscr{H}_n^+ = \mathscr{H}_n^+$. This and Lemma 3.7(ii) imply $U^*F_n^\perp = U^*(F^\perp \oplus O_{n-1}) = F_n^\perp$ and therefore $UF_n = F_n$. Analoguosly we obtain $U^*F_n = F_n$. Hence

$$UP_{F_n}U^* = P_{F_n} . \tag{3.8}$$

Since $P_{F_n}\mathscr{H}_n^+ = F_n \subset \mathscr{H}_n^+$ we have

$$J_nP_{F_n} = P_{F_n}J_n . \tag{3.9}$$

Let $\eta \in \mathscr{H} \otimes M_{1,n-1}$ and define $\xi \in \mathscr{H}_n$ by

$$\xi = \begin{bmatrix} o_1 & \eta \\ o & o_{n-1} \end{bmatrix} .$$

It is easily checked that

$$QU^*\xi = Q'U^*\xi = 0 . \tag{3.10}$$

Using (3.8), (3.3), (3.4), and (3.10) we obtain

$$\begin{aligned} P_{F_n}\xi &= UP_{F_n}U^*\xi \\ &= U\{(\varepsilon_{11} + (1-\varepsilon_{11}))(P_{F_n}U^*\xi)(\varepsilon_{11} + (1-\varepsilon_{11}))\} \\ &= UQP_{F_n}U^*\xi + UQ'P_{F_n}U^*\xi \\ &\quad + U\{\varepsilon_{11}(P_{F_n}U^*\xi)(1-\varepsilon_{11}) + (1-\varepsilon_{11})(P_{F_n}U^*\xi)\varepsilon_{11}\} \\ &= U\{\varepsilon_{11}(P_{F_n}U^*\xi)(1-\varepsilon_{11}) + (1-\varepsilon_{11})(P_{F_n}U^*\xi)\varepsilon_{11}\} . \end{aligned}$$

If we choose $v=1_{n-1}$ and $v=i1_{n-1}$ respectively, then the last identity implies

$$P_{F_n}\xi = \varepsilon_{11}(P_{F_n}\xi)(1-\varepsilon_{11}) \pm (1-\varepsilon_{11})(P_{F_n}\xi)\varepsilon_{11} .$$

Hence

$$P_{F_n}\xi = \begin{bmatrix} o_1 & \zeta \\ o & o_{n-1} \end{bmatrix} \quad \text{for some } \zeta \in \mathscr{H}\otimes M_{1,n-1} . \tag{3.11}$$

This defines a bounded linear map $r\colon \eta \to \zeta$ from $\mathscr{H}\otimes M_{1,n-1} \cong \mathscr{H}^{n-1}$ into $\mathscr{H}^{n-1}$. (3.8) implies that for every unitary $v \in M_{n-1}$ $r(\eta v) = (r\eta)v$, and therefore $r(\eta\alpha) = (r\eta)\alpha$ for every $\alpha \in M_{n-1}$. Hence

$$r = p\otimes \mathrm{id}_{M_{1,n-1}} \quad \text{for some } p \in B(\mathscr{H})$$

(3.6) implies that p is independent from $n \in \mathbf{N}$ ($n \geqq 2$). It is easy to check that $p = p^2 = p^*$, since

$$P_{F_n} = P_{F_n}^2 = P_{F_n}^* .$$

Let $\xi_{11} \in \mathscr{H}_1$, $\xi_{12} \in \mathscr{H}_{1,n-1}$, $\xi_{21} \in \mathscr{H}_{n-1,1}$, $\xi_{22} \in \mathscr{H}_{n-1}$, and $\xi = [\xi_{jk}] \in \mathscr{H}_n$ ($j,k = 1,2$). Then we have by (3.5), (3.7), (3.11), and (3.9) that

$$P_{F_n}\xi = \begin{bmatrix} P_{F_1}\xi_{11} & (p\otimes \mathrm{id}_{M_{1,n-1}})\xi_{12} \\ (JpJ\otimes \mathrm{id}_{M_{n-1,1}})\xi_{21} & \xi_{22} \end{bmatrix} . \tag{3.12}$$

We want to show that $P_{F_1} = pJpJ = JpJp$. In order to prove this identity it is sufficient to consider (3.12) in the case $n=2$. Let $G = (\mathscr{H}_1^+ \oplus F_1)^{\perp\perp}$. Changing the coordinates in $\mathscr{H}_2$ we obtain from (3.12)

$$P_G \begin{bmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{bmatrix} = \begin{bmatrix} \xi_{11} & JpJ\xi_{12} \\ p\xi_{21} & P_{F_1}\xi_{22} \end{bmatrix} .$$

This implies $P_G F_2^\perp = P_G(F_1^\perp \oplus O_1) = F_2^\perp$ by Lemma 3.7(ii). Hence $P_G F_2 \subset F_2$. Lemma 3.5 implies that $P_{F_2 \cap G} = P_G P_{F_2} = P_{F_2} P_G$, and therefore $pJpJ = JpJp$.

We have by Lemma 3.7(ii) and Lemma 3.6(ii) that

$$F_2 \cap G = (F_1^\perp \oplus O_1)^\perp \cap (O_1 \oplus F_1^\perp)^\perp = (F_1^\perp \oplus F_1^\perp)^\perp = F_{F_1 \oplus F_1} .$$

Let

$$\Xi_1 \in F_{F_1\oplus F_1}, \quad \Xi_2 = \xi_1 \oplus \xi_2 \in F_1 \oplus F_1 ,$$

such that $\Xi_2 - \Xi_1 \in \mathscr{H}_2^+$, and

$$\Xi_3 = (\xi_1+\xi_2)\oplus(\xi_1+\xi_2) = (\xi_1+\xi_2)\otimes 1_2 .$$

Then $\Xi_3 - \Xi_1 \in \mathscr{H}_2^+$ and $\|\alpha\alpha^*\|\Xi_3 - \alpha\Xi_3\alpha^* \in \mathscr{H}_2^+$ for every $\alpha \in M_2$. The map $A\colon \Xi \to \alpha\Xi\alpha^*$ maps $\mathscr{H}_2^+$ into $\mathscr{H}_2^+$. Hence $\|\alpha\alpha^*\|\Xi_3 - A\Xi_1 \in \mathscr{H}_2^+$, and therefore $A\Xi_1 \in F_{F_1 \oplus F_1}$. This implies that $P_{F_2 \cap G}$ and, A commute for every $\alpha \in M_2$. Hence it is of the form $r \otimes \mathrm{id}_{M_2}$ for some $r \in B(\mathscr{H})$. Finally we obtain

$$P_G P_{F_2} = P_{F_2 \cap G} = P_{F_1} \otimes \mathrm{id}_{M_2} \quad \text{and} \quad P_{F_1} = pJpJ\ .$$

3.9 LEMMA. *Suppose $\mathscr{H}$ is a matrix ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ ($n \in \mathbb{N}$), and $\mathscr{M}$ its matrix multiplier algebra. If $p \in B(\mathscr{H})$ is a selfadjoint projection, then the following three statements are equivalent:*

(i) *p and $1-p$ are completely positive,*
(ii) *p and JpJ are in $\mathscr{M}$,*
(iii) *$p = pJpJ$ is in the center of $\mathscr{M}$.*

PROOF. (i) $\Rightarrow$ (iii) $p\mathscr{H}_1^+ \subset \mathscr{H}_1^+$ implies $p = JpJ = p^2 = pJpJ$. Let

$$\xi = [\xi_{jk}]_{j,k=1,\ldots,n} \in \mathscr{H}_n^+\ ,$$

$$\xi' := [\xi_{j,k}]_{j,k=2,\ldots,n} \in \mathscr{H}_{n-1}^+ \quad \text{and} \quad \eta := o_1 \oplus \xi' \in \mathscr{H}_n^+\ .$$

Now the Jordan product

$$\{\mathrm{diag}\,(p,1,\ldots,1)\xi\,\mathrm{diag}\,(p,1,\ldots,1)^J\} = (p \otimes \mathrm{id}_n)\xi + ((1-p)\otimes \mathrm{id}_n)\eta \in \mathscr{H}_n^+\ .$$

Hence $p \in \mathscr{M}$ and $p = JpJ \in \mathscr{M} \cap J\mathscr{M}J \subset \mathscr{M} \cap \mathscr{M}'$.

(iii) $\Rightarrow$ (ii) is clear.

(ii) $\Rightarrow$ (i) p and JpJ commute. Hence

$$r := (1-p)J(1-p)J = JrJ \in \mathscr{M} \cap J\mathscr{M}J$$

is a selfadjoint projection. Theorem 2.2(iii) implies that

$$(r \otimes \mathrm{id}_n)\mathscr{H}_n^+ = \mathrm{diag}\,(r,\ldots,r)\mathscr{H}_n^+\,\mathrm{diag}\,(r,\ldots,r)^J \subset \mathscr{H}_n^+\ .$$

Hence r and (exactly by the same argument) $1-r$ are completely positive. We have that $p(1-r) = p$ and $(1-r)JpJ = JpJ$. Hence

$$((1-r)-p)J((1-r)-p)J = r - r = 0\ .$$

Now Corollary 3.3 implies $1-r = p$.

4. Characterization of matrix-ordered standard forms.

Let $\mathscr{H}$ be a Hilbert space with a selfdual cone $\mathscr{H}^+ \subset \mathscr{H}$ and $J = J_{\mathscr{H}^+}$ the induced involution. A. Connes [9, Lemma 5.3] gave the following characterization of one-parameter groups of order automorphisms of $(\mathscr{H}, \mathscr{H}^+)$ with a bounded infinitesimal generator $\delta \in B(\mathscr{H})$:

The statements

(4.1) (a) $\exp(t\delta)\mathscr{H}^+ \subset \mathscr{H}^+$ for every $t \in \mathbf{R}$,
(b) $J\delta = \delta J$ and $\xi, \eta \in \mathscr{H}^+$, $\xi \perp \eta$ implies $\delta\xi \perp \eta$

are equivalent. Observe [6] that this equivalence is true even if $\mathscr{H}^+$ is not homogeneous in the sense of [9, Definition 5.1]. A simple proof is given by Evans and Hanche-Olsen in [21, Theorem 1].

The idea to use (4.1) in connection with matrix-ordered cones was developed in [16].

4.1 LEMMA. *Let $\mathscr{H}$ be a matrix-ordered Hilbert space with selfdual cones $\mathscr{H}_n^+$ ($n \in \mathbf{N}$), J the induced involution, and $\mathscr{M}$ its matrix multiplier algebra. Suppose that all completed faces of $\mathscr{H}_n^+$ are projectable for every $n \in \mathbf{N}$. If $x \in \mathscr{M}'$, then we have*

(i) $\exp(t(x + JxJ))\colon \mathscr{H} \to \mathscr{H}$ is completely positive for every $t \in \mathbf{R}$.
(ii) $JxJ \in \mathscr{M}$.

PROOF. (i) Let $\delta = x + JxJ$. Then $J\delta = \delta J$. Let $\xi, \eta \in \mathscr{H}_1^+$, $\xi \perp \eta$ and $F_1 = (F_{\{\xi\}})^{\perp\perp}$ the completed face generated by ξ. Then $\xi \in F_1$ and $\eta \in F_1^\perp$. We write P_{F_1} for the selfadjoint projection on the closed subspace of $\mathscr{H}_1$ generated by F_1. By hypothesis $F_n = (F_1 \oplus \mathscr{H}_{n-1}^+)^{\perp\perp}$ is a projectable face in $\mathscr{H}_n^+$ for every $n \in \mathbf{N}$. Theorem 3.8 shows that there exists a selfadjoint projection $p \in \mathscr{M}$, which satisfies $P_{F_1} = pJpJ$ and $P_{F_1^\perp} = (1-p)J(1-p)J$. With the aid of Corollary 3.3 we conclude from $P_{F_1^\perp}(\xi) = 0$ and $P_{F_1}(\eta) = 0$ that $p\xi = JpJ\xi = \xi$ and $p\eta = JpJ\eta = 0$. Hence

$$\langle \delta\xi, \eta \rangle = \langle xp\xi, \eta \rangle + \langle JxJJpJ\xi, \eta \rangle = \langle x\xi, p\eta \rangle + \langle JxJ\xi, JpJ\eta \rangle = 0 .$$

(4.1) implies now that $\exp(t\delta)\mathscr{H}_1^+ \subset \mathscr{H}_1^+$ for every $t \in \mathbf{R}$.

By Corollary 2.3 and formula 2.9, the commutant $\mathscr{M}_k'$ of the matrix multiplier algebra $\mathscr{M}_k$ of $(\mathscr{H}_k, \mathscr{H}_{kn}^+, n \in \mathbf{N})$ is $\mathscr{M}' \otimes \tilde{\mathrm{M}}_k$. $\tilde{\mathrm{M}}_k$ is the opposite algebra to M_k and formally $\mathscr{M}' \otimes \tilde{\mathrm{M}}_k$ operates on $\mathscr{H}_k$ by matrix multiplication from the right hand side. Hence $x \otimes \tilde{1}_k \in \mathscr{M}_k'$ and we obtain as above that

$$\exp(t(\delta \otimes \mathrm{id}_{M_k}) = (\exp(t\delta)) \otimes \mathrm{id}_{M_k}$$

maps $\mathscr{H}_k^+$ into $\mathscr{H}_k^+$. This proves (i).

(ii) Let $x \in \mathscr{M}'$, $x_k = x \otimes \tilde{1}_k = x \otimes \mathrm{id}_{M_k} \in \mathscr{M}_k'$ and $X_k = x_k \otimes \tilde{\varepsilon}_{12} \in \mathscr{M}_{2k}'$, where $\varepsilon_{\mu\nu}$, $\mu, \nu = 1, 2$, are the usual matrix units in M_2. We define $\delta = X_k + J_{2k}X_kJ_{2k} \in B(\mathscr{H}_{2k})$. It is easy to check that $X_k^2 = 0$ and $X_kJ_{2k}X_kJ_{2k}X_k = 0$. Part (i) shows that $\exp\delta$ maps $\mathscr{H}_{2k}^+$ into $\mathscr{H}_{2k}^+$.

If $\xi \in \mathscr{H}_k^+$, then we define $\Xi = o_k \oplus \xi \in \mathscr{H}_{2k}^+$. We obtain

$$(\exp\delta)\Xi = \begin{bmatrix} \frac{1}{2}(x_k J_k x_k J_k + J_k x_k J_k x_k)\xi & J_k x_k J_k \xi \\ x_k \xi & \xi \end{bmatrix} \in \mathscr{H}_{2k}^+ .$$

Using the notations (2.4) and (2.5) we have

$$\{\operatorname{diag}(JxJ,\ldots,JxJ,1,\ldots,1)\Xi \operatorname{diag}(JxJ,\ldots,JxJ,1,\ldots,1)^J\} = (\exp\delta)\Xi \in \mathscr{H}_{2k}^+ .$$

Let $\alpha = (\varepsilon'_{11}, 1_k - \varepsilon'_{11}) \in M_{k,2k}$, where $\varepsilon'_{\mu\nu}$ denotes the matrix units in M_k. Then (1.11) implies that

$$\{\operatorname{diag}(JxJ,1,\ldots,1)\xi \operatorname{diag}(JxJ,1,\ldots,1)^J\} = \alpha((\exp\delta)\Xi)\alpha^* \in \mathscr{H}_k^+ .$$

Hence $JxJ \in \mathscr{M}$.

4.2 LEMMA. *Let $\mathscr{H}$ be a Hilbert space and $\mathscr{H}^+$ a selfdual cone in $\mathscr{H}$. Suppose that for every $\xi \in \mathscr{H}^+$, the completed face $(F_{\{\xi\}})^{\perp\perp}$ generated by ξ is projectable. Then all completed faces of $\mathscr{H}^+$ are projectable.*

PROOF. Let F be a completed face in $\mathscr{H}^+$. Then

$$F = \bigcup \{(F_{\{\xi\}})^{\perp\perp} : \xi \in F\} .$$

We write P_ξ for the selfadjoint projection on the closed subspace of $\mathscr{H}$ generated by $(F_{\{\xi\}})^{\perp\perp}$. If $\xi, \eta \in \mathscr{H}^+$, $\xi - \eta \in \mathscr{H}^+$, then $F_{\{\eta\}} \subset F_{\{\xi\}}$, $(F_{\{\eta\}})^{\perp\perp} \subset (F_{\{\xi\}})^{\perp\perp}$, and $P_\eta \leqq P_\xi$. Hence $\lim_{\xi\in F} P_\xi$ exists and equals P_F the selfadjoint projection on the closed subspace generated by F. In addition $P_F\mathscr{H}^+ = F$.

If the selfdual cones $\mathscr{H}_n^+$ $(n \in \mathbb{N})$ are all facial homogeneous (Connes [9, Definition 5.1]), then every completed face $F = F^{\perp\perp}$ in $\mathscr{H}_n^+$ $(n \in \mathbb{N})$ is projectable. The latter condition is formally much weaker and has a simpler geometric meaning. In the following we have no assumptions on general non-completed faces and obtain as a corollary that the cones are homogeneous (cp. Connes [9, Theorem 4.6]).

4.3 THEOREM. *Suppose that $\mathscr{H}$ is a matrix-ordered Hilbert space with selfudal cones $\mathscr{H}_n^+$ $(n \in \mathbb{N})$, $J = J_{\mathscr{H}_1^+}$ the induced involution, and $\mathscr{M}$ its matrix multiplier algebra. Then the following statements are equivalent:*

(i) *$(\mathscr{M}, \mathscr{H}, \mathscr{H}_n^+, n \in \mathbb{N})$ is a matrix ordered standard form.*
(ii) *The completed face $(F_{\{\xi\}})^{\perp\perp}$ generated by ξ in $\mathscr{H}_n^+$ is projectable for every $\xi \in \mathscr{H}_n^+$, $n \in \mathbb{N}$.*

PROOF. (i) $\Rightarrow$ (ii) $(\mathscr{M}_n, \mathscr{H}_n, \mathscr{H}_n^+)$ is a standard form for every $n \in \mathbf{N}$. In the case that $\mathscr{M}_n$ has a cyclic separating vector $\xi_0 \in \mathscr{H}_n^+$, property (ii) was proved by A. Connes [9]. For the general case see U. Haagerup [11, Chapter 2].

(ii) $\Rightarrow$ (i). Lemma 4.2 and Lemma 4.1 imply $J\mathscr{M}J = \mathscr{M}'$. Let p be in the center of $\mathscr{M}$, i.e. $p \in \mathscr{M} \cap J\mathscr{M}J$. Then JpJ is in the center of $\mathscr{M}$ and Lemma 3.9 implies $p = JpJ$. The spectral theorem shows that $JxJ = x^*$ for every x in the center of $\mathscr{M}$.

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