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A RESULT ON TWO ONE-PARAMETER GROUPS OF AUTOMORPHISMS

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Abstract.

Let $\{\alpha_t \mid t \in \mathsf{R}\}$ and $\{\beta_t \mid t \in \mathsf{R}\}$ be strongly continuous one-parameter groups of *-automorphisms of a von Neumann algebra M, and suppose that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in \mathsf{R}$. If M is a factor we show that either $\alpha_t = \beta_t$ for all $t \in \mathsf{R}$ or $\alpha_t = \beta_{-t}$ for all $t \in \mathsf{R}$. In general we show that there exists a central projection p in M which is invariant under α and β and such that $\alpha_t(x) = \beta_t(x)$ for all $t \in \mathsf{R}$, when $x \in Mp$, and $\alpha_t(x) = \beta_{-t}(x)$ for all $t \in \mathsf{R}$, when $x \in M(1-p)$.

1. Introduction.

Let M be a von Neumann algebra and $\{\alpha_t \mid t \in \mathsf{R}\}$ a strongly continuous one-parameter group of *-automorphisms of M. The associated maps $\{\alpha_t + \alpha_{-t} \mid t \in \mathsf{R}\}$ occur naturally in various situations. Our original motivation for considering these maps was the new proof of the Tomika–Takesaki theorem as formulated in [8]. In that proof an important rôle was played by the integrals

$$\psi = \int \frac{2}{e^{\pi t} + e^{-\pi t}} \sigma_t \, dt ,$$

where $\{\sigma_t \mid t \in \mathsf{R}\}$ was the modular automorphism group. And the question arose how much σ was determined by ψ. In fact recently Haagerup and Skau in their paper on the geometric aspects of the Tomita–Takesaki theory ([2]) exactly met the same problem. A relatively simple L^1-functional calculus shows that ψ determines $\sigma_t + \sigma_{-t}$ for all $t \in \mathsf{R}$. Moreover in this paper we prove a result showing that to a great extent σ itself is determined by all those maps.

Our theorem is formulated as follows. Let $\{\alpha_t \mid t \in \mathsf{R}\}$ and $\{\beta_t \mid t \in \mathsf{R}\}$ be two strongly continuous one-parameter groups of *-automorphisms of a von Neumann algebra M such that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in \mathsf{R}$, then there exists a central projection p in M such that $\alpha_t(p) = p$ and $\beta_t(p) = p$ for all $t \in \mathsf{R}$ and such that

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$$\alpha_t(x) = \beta_t(x) \quad \text{for all } t \text{ when } x \in Mp \text{ and}$$

$$\alpha_t(x) = \beta_{-t}(x) \quad \text{for all } t \text{ when } x \in M(1-p)\,.$$

The theorem was first proved by the first author in the case where α_t and β_t were commuting automorphisms for all t [7]. This original proof can be slightly simplified and then becomes much simpler than the proof of the general case which we present here. Also Haagerup obtained a proof of the general case independently, but in fact Haagerup and Skau only use the result in the commuting case. The full proof we give here in this paper can be considered as a very nice application of Arveson's theory of spectral subspaces.

Our result may also be of interest in the theory of Jordan algebras. On the one hand recently Haagerup discovered an analogue of the Tomita–Takesaki theory for Jordan algebras, and while it is not possible to find operators like J and Δ, it turns out that the mappings $\{\sigma_t+\sigma_{-t} \mid t \in \mathrm{R}\}$ have generalization in this theory [3]. On the other hand the decomposition we obtain from M is similar to the one obtained by Kadison for Jordan isomorphisms [4].

Finally we remark that our result can also be formulated in terms of squares of derivations and that also here it might be of interest.

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2. The main result.

Let M be a von Neumann algebra and let $\{\alpha_t \mid t \in \mathrm{R}\}$ and $\{\beta_t \mid t \in \mathrm{R}\}$ be strongly continuous one-parameter groups of *-automorphisms of M. Throughout this section we will assume that

$$\alpha_t+\alpha_{-t} = \beta_t+\beta_{-t}$$

for all $t \in \mathrm{R}$.

Recall that in this context α is called strongly continuous if for all $x \in M$ the map $t \to \alpha_t(x)$ is continuous from R to M, when M is considered with one of the weaker topologies. In fact we will all the time use the σ-weak operator topology which is the weak topology $\sigma(M, M_*)$ induced by the unique predual M_* of M. The main tool will be Arveson's theory of spectral subspaces [1]. We will use the formulation of [9].

2.1. Definition. Denote by $M(a,b)$ the spectral subspace for α associated to the open interval (a,b) in R. Recall that $M(a,b)$ is the σ-weak operator closed subspace of M generated by the vectors

$$\pi_\alpha(f)x = \int f(t)\alpha_t(x)\,dt\,,$$

where $x \in M$ and $f \in L^1(\mathbf{R})$ and has a Fourier transform $\hat{f}$ with compact support in (a,b). Similarly by $N(a,b)$ we denote the spectral subspace for β of the interval (a,b). Here we will use π_β for the map defined by

$$\pi_\beta(f) = \int f(t)\beta_t\,dt$$

when $f \in L^1(\mathbf{R})$.

We will frequently use the following result on spectral subspaces.

2.2. LEMMA. *If $g, h \in L^1(\mathbf{R})$ and $\hat{g}=1$ on $[a,b]$ and $\hat{h}=0$ on $[a,b]$, then $\pi_\alpha(g)y = y$ and $\pi_\alpha(h)y=0$ for all $y \in M(a,b)$.*

PROOF. By continuity it is sufficient to consider vectors y of the form $\pi_\alpha(f)x$ with $x \in M$ and $f \in L^1(\mathbf{R})$ with $\hat{f}$ having compact support in (a,b). But

$$\pi_\alpha(g)y = \pi_\alpha(g)\pi_\alpha(f)x = \pi_\alpha(g*f)x \quad \text{and} \quad (g*f)\hat{} = \hat{g}\hat{f} = \hat{f}$$

so that

$$\pi_\alpha(g*f) = \pi_\alpha(f) \quad \text{and} \quad \pi_\alpha(g)y = \pi_\alpha(f)x = y\,.$$

Similarly because $\hat{h}\hat{f}=0$, we get $\pi_\alpha(h)y=0$.

We will now see what the implications of the relation $\alpha_t+\alpha_{-t}=\beta_t+\beta_{-t}$ for all $t \in \mathbf{R}$ are on the spectral subspaces. Apart from Lemma 2.2 we will also use often that $\pi_\alpha(h)=\pi_\beta(h)$, when $h(t)=h(-t)$ for all $t \in \mathbf{R}$. The next two lemma's are more or less immediate applications of this result.

2.3. LEMMA. *For all $a>0$ we have*

$$M(-a,a) = N(-a,a)$$

PROOF. Let $f \in L^1(\mathbf{R})$ such that $\hat{f}$ has compact support in $(-a,a)$. Choose a function $h \in L^1_r(\mathbf{R})$ such that $\hat{h}=1$ on the support of $\hat{f}$ and satisfies $\hat{h}(t)=\hat{h}(-t)$ for all t, but still has its support in $(-a,a)$. Then $h*f=f$, and for any $x \in M$ we have $\pi_\alpha(f)x=\pi_\alpha(h)\pi_\alpha(f)x$. Because also $h(t)=h(-t)$ for all t also $\pi_\alpha(h)=\pi_\beta(h)$, so that

$$\pi_\alpha(f)x = \pi_\beta(h)\pi_\alpha(f)x\,.$$

And since $\hat{h}$ has its support in $(-a, a)$ by definition, we get $\pi_\alpha(f)x \in N(-a, a)$. Since $N(-a, a)$ is σ-weak operator closed, also

$$M(-a, a) \subseteqq N(-a, a)\,,$$

and by symmetry we get equality.

2.4. Lemma. *If* $0 < a < b$, *then*

$$M(a, b) + M(-b, -a) = N(a, b) + N(-b, -a)$$

Proof. The proof is very similar to the previous one. Again let $f \in L^1(\mathbf{R})$, and suppose that the support of $\hat{f}$ lies in (a, b). Choose a function $h \in L^1(\mathbf{R})$ such that again $\hat{h} = 1$ on the support of $\hat{f}$, and $\hat{h}(t) = \hat{h}(-t)$ for all t and such that $\hat{h}$ has support in $(-b, -a) \cup (a, b)$. Then again for any $x \in M$ we have

$$\pi_\alpha(f)x = \pi_\alpha(h)\pi_\alpha(f)x = \pi_\beta(h)\pi_\alpha(f)x \in N(-b, -a) + N(a, b)\,.$$

If we can show that $N(-b, -a) + N(a, b)$ still is σ-weak operator closed, then it will follow that

$$M(a, b) \subseteqq N(-b, -a) + N(a, b)\,.$$

Similarly also $M(-b, -a) \subseteqq N(-b, -a) + N(a, b)$ so that

$$M(a, b) + M(-b, -a) \subseteqq N(-b, -a) + N(a, b)$$

and by symmetry we would get equality. To prove that $N(a, b) + N(-b, -a)$ is closed, we consider a function $g \in L^1(\mathbf{R})$ such that $\hat{g} = 1$ on $[a, b]$, but $\hat{g} = 0$ on $[-b, -a]$. If now $\{x_\gamma \mid \gamma \in \Gamma\}$ and $\{y_\gamma \mid \gamma \in \Gamma\}$ are nets in $N(a, b)$ and $N(-b, -a)$ such that $x_\gamma + y_\gamma$ converges, then $\pi_\beta(g)(x_\gamma + y_\gamma)$ will also converge, but by Lemma 2.2 we have $\pi_\beta(g)(x_\gamma + y_\gamma) = x_\gamma$ so that x_γ and y_γ converge. Therefore $N(a, b) + N(-b, -a)$ is closed because $N(a, b)$ and $N(-b, -a)$ are closed.

The two previous results are completely natural in view of our condition on α and β. The next one is rather unexpected but it is the main step towards the proof of our theorem. We will use here for the first tome that α_t and β_t are *-automorphisms by means of the following two facts about the spectral subspaces. If $x \in M(a, b)$ and $y \in M(c, d)$, then $x^* \in M(-b, -a)$ and $xy \in M(a + c, b + d)$. The first fact is easy to obtain and for the second one an argument as in [9, Lemma 4.3] must be used. Also these two results on spectral subspaces for groups of *-automorphisms will frequently be used in what follows.

2.5. PROPOSITION. *If* $0<a<b<2a$, *then*

$$M(a,b) = M(a,b) \cap N(a,b)+M(a,b) \cap N(-b,-a) .$$

PROOF. Let $x \in M(a,b)$. By the previous lemma we can write $x=x_1+x_2$, where $x_1 \in N(a,b)$ and $x_2 \in N(-b,-a)$. Again by this lemma we can write $x_1=y_1+y_2$, where $y_1 \in M(a,b)$ and $y_2 \in M(-b,-a)$. We will show that $y_2=0$ so that $x_1=y_1$ and $x_1 \in M(a,b)\cap N(a,b)$. A similar argument gives that $x_2 \in M(a,b)\cap N(-b,-a)$.

First consider the formula

$$x^*x = x_1^*x_1+x_2^*x_2+x_1^*x_2+x_2^*x_1 .$$

By the preceding remarks we know that $x_1^*x_1$, $x_2^*x_2 \in N(a-b,b-a)$ and that $x^*x \in M(a-b,b-a)$. By Lemma 2.3, however, those two spaces are the same so that $x^*x, x_1^*x_1, x_2^*x_2 \in N(a-b,b-a)$. On the other hand $x_1^*x_2 \in N(-2b, -2a)$, while $x_2^*x_1 \in N(2a,2b)$. Because of the condition $0<a<b<2a$, the three intervals $[-2b,-2a]$, $[a-b,b-a]$, and $[2a,2b]$ are mutually disjoint. Then we can find a function $h \in L^1(\mathbf{R})$ such that $\hat{h}=1$ on $[2a,2b]$ and vanishes on the two others. By lemma 2.2 then $\pi_\beta(h)$ will be 1 on $N(2a,2b)$ and 0 on the other spaces and if we apply $\pi_\beta(h)$ to the above formula we obtain $x_2^*x_1=0$. Similarly, or by taking adjoints we get also $x_1^*x_2=0$. By considering xx^* we would also obtain $x_1x_2^*=0$ and $x_2x_1^*=0$.

For the same reason we have $y_1^*y_2=y_2^*y_1=y_1y_2^*=y_2y_1^*=0$. All those relations together give us

$$\begin{aligned} xx_1^*x_1 &= (x_1+x_2)x_1^*x_1 \\ &= x_1x_1^*x_1 \\ &= y_1y_1^*y_1+y_2y_2^*y_2 . \end{aligned}$$

But since $y_1 \in M(a,b)$, we have $y_1y_1^*y_1 \in M(2a-b,2b-a)$, while $y_2 \in M(-b, -a)$ implies $y_2y_2^*y_2 \in M(a-2b,b-2a)$. On the other hand $x \in M(a,b)$ and $x_1^*x_1 \in N(a-b,b-a)=M(a-b,b-a)$ so that $xx_1^*x_1 \in M(2a-b,2b-a)$. So

$$xx_1^*x_1, y_1y_1^*y_1 \in M(2a-b,2b-a) \quad \text{and} \quad y_2y_2^*y_2 \in M(a-2b,b-2a) .$$

Again because of the condition $0<a<b<2a$ the intervals $[a-2b,b-2a]$ and $[2a-b,2b-a]$ are disjoint and as before it follows from the relation

$$xx_1^*x_1 = y_1y_1^*y_1+y_2y_2^*y_2$$

that $y_2y_2^*y_2=0$. Therefore $y_2=0$ and $x_1=y_1$ and the proof is complete.

Simple examples show that the previous result is not true anymore if we do not require the α_t and β_t to be automorphisms but just isometries (see example 2.13 below).

The next result show how close Proposition 2.5 brings us already to the final result.

2.6. PROPOSITION. *If* $0<a<b$ *and* $x \in M(a,b) \cap N(a,b)$, *then* $\alpha_t(x)=\beta_t(x)$ *for all* $t \in \mathbf{R}$, *while if* $x \in M(a,b) \cap N(-b,-a)$, *then* $\alpha_t(x)=\beta_{-t}(x)$ *for all* $t \in \mathbf{R}$.

PROOF. First let $x \in M(a,b) \cap N(a,b)$ and take any $f \in L^1(\mathbf{R})$. We will show that $\pi_\alpha(f)x=\pi_\beta(f)x$ so that

$$\int f(t)(\alpha_t(x)-\beta_t(x))\,dt = 0,$$

and since this will be true for all $f \in L^1(\mathbf{R})$, we get $\alpha_t(x)-\beta_t(x)=0$ for all $t \in \mathbf{R}$.

Choose $g \in L^1(\mathbf{R})$ such that $\hat{g}(t)=1$ when $a \leqq t \leqq b$ and $\hat{g}(t)=0$ if $t \leqq 0$. Let also $g_1(t)=g(-t)$ and $f_1(t)=f(-t)$ for all t, and define further $h=f*g+f_1*g_1$. Because $\hat{g}=1$ on $[a,b]$ and $x \in M(a,b)$, we have $\pi_\alpha(g)x=x$ by Lemma 2.2, and because $\hat{g}_1=0$ on $[a,b]$ we have $\pi_\alpha(g_1)x=0$. As also $x \in N(a,b)$ similarly we have $\pi_\beta(g)x=x$ and $\pi_\beta(g_1)x=0$. All together we obtain

$$\begin{aligned}\pi_\alpha(f)x &= (\pi_\alpha(f)\pi_\alpha(g)+\pi_\alpha(f_1)\pi_\alpha(g_1))x \\ &= \pi_\alpha(h)x\end{aligned}$$

and similarly $\pi_\beta(f)x=\pi_\beta(h)x$. But $\pi_\alpha(h)=\pi_\beta(h)$, because $h(t)=h(-t)$ for all t and this completes the proof. The case $x \in M(a,b) \cap N(-b,-a)$ is completely similar.

So from Propositions 2.5 and 2.6 we see that each subspace $M(a,b)$ with $0<a<b<2a$ can be decomposed into a part, where $\alpha_t=\beta_t$ for all t and another part where $\alpha_t=\beta_{-t}$ for all t. The condition $b<2a$ is not very serious. If simply $0<a<b$, we can cover $[a,b]$ by a finite union of open intervals

$$\{(p_i,q_i) \mid i=1,2,\dots n\}$$

all having the property that $0<p_i<q_i<2p_i$, and if $\{h_i \mid i=1,\dots n\}$ are L^1-functions such that $\sum_{i=1}^n \hat{h}_i=1$ on $[a,b]$ and $\hat{h}_i$ has support in (p_i,q_i) (see appendix A of [9]), then

$$M(a,b) = \pi_\alpha\left(\sum_{i=1}^{n} h_i\right)M(a,b) \subseteqq \sum_{i=1}^{n} \pi_\alpha(h_i)M(a,b) \subseteqq \sum_{i=1}^{n} M(p_i,q_i)$$

and since each $M(p_i,q_i)$ can be decomposed, also $M(a,b)$ can.

Of course also the spaces $M(a,b)$ with $a<b<0$ can be decomposed in the right way. Then how far are we from a decomposition of all of M? In general it is not true that M is the smallest σ-weak operator closed subspace containing all the spectral subspaces $M(a,b)$ with $a<b<0$ or $0<a<b$. We will certainly miss the fixed points. However we can show that α and β precisely have the same fixed points. (Clearly this would follow from our theorem).

2.7. LEMMA. *If $x \in M$ and $\alpha_t(x)=x$ for all $t \in \mathbf{R}$, then also $\beta_t(x)=x$ for all $t \in \mathbf{R}$.*

PROOF. Since in fact $\{x \in M \mid \alpha_t(x)=x \text{ for all } t\}$ is the spectral subspace for α associated to the closed set $\{0\}$, which is by definition $\bigcap_{a>0} M(-a,a)$, the result immediately follows from Lemma 2.3. We can also give the following more direct proof.

Assume $\beta_t(x)+\beta_{-t}(x)=2x$ for all $t \in \mathbf{R}$. Then

$$(\beta_t-1)^2x = \beta_{2t}(x)-2\beta_t(x)+x = 0 .$$

Put $y=\beta_t(x)-x$ so that $\beta_t(y)=y$ and $\beta_t(x)=x+y$. Then also

$$\beta_{2t}(x) = \beta_t(x)+y = x+2y ,$$

and by induction $\beta_{nt}(x)=x+ny$ for all $n=1,2,3 \ldots$. But

$$n\|y\| = \|\beta_{nt}(x)-x\| \leqq 2\|x\|$$

for all n so that $y=0$ and $\beta_t(x)=x$.

Still in general it is not true that the whole space is generated by the fixed points, i.e. the spectral subspace of $\{0\}$, and the spaces $M(a,b)$ with $a<b<0$ or $0<a<b$. In our special situation however this can be proved as we do essentially in the following proposition.

2.8. PROPOSITION. *The fixed points and the spaces $M(a,b)$ with $a<b<0$ or $0<a<b$ span a σ-weak operator dense subspace of M.*

PROOF. Suppose that φ is an element in the predual M_* of M such that $\varphi(x)=0$, when x is a fixed point under α, and when $x \in M(a,b)$ with $a<b<0$ or with $0<a<b$. We will show that $\varphi=0$.

Because $\mathbf{R}\setminus\{0\}$ is the union of intervals (a,b) in $\mathbf{R}\setminus\{0\}$, the spectral subspace of $\mathbf{R}\setminus\{0\}$ is the σ-weak operator closure of the linear span of the spaces $M(a,b)$ with $(a,b)\subseteqq\mathbf{R}\setminus\{0\}$. Therefore also $\varphi(x)=0$ if x is an element of the spectral subspace of $\mathbf{R}\setminus\{0\}$. But then φ is a fixed point for the dual action, i.e. φ is invariant under α, see e.g. [9, page 223].

Now let $\varphi=u|\varphi|$ be the polar decomposition of φ (see [5, p. 32]). Then by the invariance of φ and the uniqueness of the polar decomposition, also $|\varphi|$ and u will be invariant. But $|\varphi|(uu^*)=\varphi(u^*)=0$ because u^* is a fixed point and φ is also zero on fixed points. Finally because uu^* is the support of $|\varphi|$, it also follows that $|\varphi|=0$ and hence $\varphi=0$.

(We remark that this result is very similar to the one obtained in [6]).

So we have now obtained a decomposition of a dense subspace of the von Neumann algebra. This is not yet what we want. On the one hand it is not clear that elements in the closure still can be decomposed. On the other hand we want a central decomposition. Fortunately we have the following proposition which is the next main step towards the final result.

2.9. PROPOSITION. *If $x\in M(a,b)\cap N(a,b)$ and $y\in M(c,d)\cap N(-d,-c)$ for some open intervals (a,b) and (c,d) not containing* 0, *then $xy=0$.*

PROOF. We must consider four different cases:

i)	$0<a<b$	and	$0<c<d$
ii)	$0<a<b$	and	$c<d<0$
iii)	$a<b<0$	and	$0<c<d$
iv)	$a<b<0$	and	$c<d<0$

We will deal with cases iii) and iv) by also proving that $yx=0$ in cases i) and ii) which is sufficient by taking adjoints.

On the other hand case ii) can be obtained from case i) if we interchange α and β. So we may restrict to the case $0<a<b$ and $0<c<d$ but we have to prove that $xy=0$ and $yx=0$.

So let $x\in M(a,b)\cap N(a,b)$ and $y\in M(c,d)\cap N(-d,-c)$. Then xy and yx belong to $M(a+c,b+d)$ and to $N(a-d,b-c)$. Also here we must consider different possibilities. Either $a-d<b-c\leqq 0$, or $a-d<0<b-c$ but $b-c\leqq d-a$, or $a-d<0<b-c$ and $d-a<b-c$, or $0\leqq a-d<b-c$. In the first two cases we have $(a-d,b-c)\subseteqq(a-d,d-a)$ and in the other two we have $(a-d,b-c)\subseteqq(c-b,b-c)$. So we have either

$$N(a-d,b-c)\subseteqq N(a-d,d-a)=M(a-d,d-a)$$

or

$$N(a-d, b-c) \subseteq N(c-b, b-c) = M(c-b, b-c) .$$

If now $d-c<2a$ so that $d-a<a+c$, we get

$$M(a-d, d-a) \cap M(a+c, b+d) = \{0\}$$

and this would give $xy=yx=0$ in the first case. If on the other hand $b-a<2c$ so that $b-c<a+c$, we get

$$M(c-b, b-c) \cap M(a+c, b+d) = \{0\} ,$$

and this would imply $xy=yx=0$ in the second case. So we obtain the result if $d-c<2a$ and $b-a<2c$.

We will finally show that the general case can be obtained from this special case. So assume $0<a<b$ and $0<c<d$. Cover the closed interval $[a,b]$ by a finite number of open intervals (p,q) such that $\frac{1}{2}a<p$ and $q-p<c$. Do the same for $[c,d]$ with intervals (r,s) such that $\frac{1}{2}c<r$ and $s-r<a$. Then we will have $0<p<q$ and $0<r<s$ but also $s-r<2p$ and $q-p<2r$. By the following lemma the space $M(a,b) \cap N(a,b)$ will be spanned by the spaces $M(p,q) \cap N(p,q)$, while the space $M(c,d) \cap N(-d,-c)$ will be spanned by the spaces $M(r,s) \cap N(-s,-r)$.

2.10. LEMMA. *If* $0<a<b$ *and* $[a,b] \subseteq \bigcup_{i=1}^{n} (p_i, q_i)$, *where also* $0<p_i<q_i$, *then*

$$M(a,b) \cap N(a,b) \subseteq \sum_{i=1}^{n} M(p_i, q_i) \cap N(p_i, q_i)$$

and

$$M(a,b) \cap N(-b,-a) \subseteq \sum_{i=1}^{n} M(p_i, q_i) \cap N(-q_i, -p_i) .$$

PROOF. By replacing β_t by β_{-t} for all t again it is sufficient to prove only the first statement. Choose functions $\{f_i \mid i=1,n\}$ in $L^1(\mathbf{R})$ such that $\sum_{i=1}^{n} \hat{f}_i=1$ on $[a,b]$ and $\hat{f}_i$ has support in (p_i, q_i) for all $i=1,2,\ldots n$. Because $\sum \hat{f}_i=1$ on $[a,b]$ we have

$$M(a,b) \cap N(a,b) \subseteq \sum_{i=1}^{n} \pi_\alpha(f_i)(M(a,b) \cap N(a,b)) .$$

But because $\alpha_t=\beta_t$ for all t on $M(a,b) \cap N(a,b)$ we will also have $\pi_\alpha(f_i)=\pi_\beta(f_i)$ on $M(a,b) \cap N(a,b)$ for all i. So

$$\pi_\alpha(f_i)(M(a,b) \cap N(a,b)) \subseteq M(p_i, q_i)$$

but also

$$\pi_\alpha(f_i)(M(a,b) \cap N(a,b)) = \pi_\beta(f_i)(M(a,b) \cap N(a,b)) \subseteqq N(p_i, q_i) .$$

This completes the proof.

We are now able to prove the main result.

2.11. THEOREM. *If* $\{\alpha_t \mid t \in \mathrm{R}\}$ *and* $\{\beta_t \mid t \in \mathrm{R}\}$ *are strongly continuous one-parameter groups of *-automorphisms of* M *such that* $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ *for all* $t \in \mathrm{R}$, *then there exists a projection* p *in the centre of* M *such that* $\alpha_t(p) = p$ *and* $\beta_t(p) = p$ *for all* t *and* $\alpha_t(x) = \beta_t(x)$ *for all* t *if* $x \in Mp$ *and* $\alpha_t(x) = \beta_{-t}(x)$ *for all* t *if* $x \in M(1-p)$.

PROOF. Denote by K_0 the *-subalgebra of M generated by the spaces $M(a,b) \cap N(a,b)$ with (a,b) not containing 0, and by L_0 the *-subalgebra generated by the spaces $M(a,b) \cap N(-b, -a)$ with (a,b) not containing 0. Also denote

$$M_0 = \{x \in M \mid \alpha_t(x) = \beta_t(x) = x \text{ for all } t \in \mathrm{R}\} .$$

We claim that $M_0 K_0 \subseteqq K_0$ as well as $M_0 L_0 \subseteqq L_0$. To see this, let $x \in M_0$, $y \in M$, and $f \in L^1(\mathrm{R})$ such that $\hat{f}$ has support in an interval (a,b). Then

$$x(\pi_\alpha(f)y) = x \int f(t)\alpha_t(y)\,dt = \int f(t)\alpha_t(xy)\,dt$$

so that $x(\pi_\alpha(f)y) \in M(a,b)$. So

$$xM(a,b) \subseteqq M(a,b) \quad \text{if } \alpha_t(x) = x \text{ for all } t \in \mathrm{R} .$$

Similarly

$$xN(a,b) \subseteqq N(a,b) \quad \text{if } \beta_t(x) = x \text{ for all } t \in \mathrm{R} .$$

And the claim follows easily from this.

By Proposition 2.9, we also know that $L_0 K_0 = \{0\}$ and so $(K_0 + M_0 + L_0)K_0 \subseteqq K_0$. Then by Proposition 2.5, Lemma 2.7, and Proposition 2.8 we know that $K_0 + M_0 + L_0$ is σ-weak operator dense. Therefore the σ-weak operator closure K of K_0 is a left ideal, and because it is self-adjoint, it is also a two-sided ideal. Similarly the closure L of L_0 is a two-sided ideal. Then there exist central projections p and q such that $K = Mp$ and $L = Mq$.

Because the spectral subspaces are invariant under the actions, also K_0 and L_0 will be invariant, and so $\alpha_t(p) = \beta_t(p) = p$ and $\alpha_t(q) = \beta_t(q) = q$ for all $t \in \mathrm{R}$. Since $\alpha_t = \beta_t$ on $M(a,b) \cap N(a,b)$ and $\alpha_t = \beta_{-t}$ on $M(a,b) \cap N(-b, -a)$ for all t if

$0 \notin (a,b)$ we will also have that $\alpha_t = \beta_t$ on K and $\alpha_t = \beta_{-t}$ on L for all t. Because $K_0 L_0 = \{0\}$ also $KL = \{0\}$ and therefore $pq = 0$.

So we arrive to a central decomposition $M = Mp + M(1-p-q) + Mq$ of M. We know that p and q are invariant and that $\alpha_t = \beta_t$ on Mp and $\alpha_t = \beta_{-t}$ on Mq for all t. It remains to consider $M(1-p-q)$ and because $K_0 + M_0 + L_0$ is dense and M_0 is closed, we get $M(1-p-q) \subseteq M_0$. Moreover on M_0 we have $\alpha_t = \beta_t = 1$, and so $\alpha_t = \beta_{-t}$ for all t and this completes the proof.

It is clear that the decomposition is not unique in general. If e.g. M_0 contains a non-zero projection e of the centre of M, then $M_0 e$ is a part on which $\alpha_t = \beta_t$ and $\alpha_t = \beta_{-t}$ for all t.

The theorem is particularly nice in the case of a factor.

2.12. COROLLARY. *If* $\{\alpha_t \mid t \in \mathbf{R}\}$ *and* $\{\beta_t \mid t \in \mathbf{R}\}$ *are two strongly continuous one-parameter groups of *-automorphisms of a factor* M, *then either* $\alpha_t = \beta_t$ *for all* $t \in \mathbf{R}$ *or* $\alpha_t = \beta_{-t}$ *for all* $t \in \mathbf{R}$.

We conclude this main section by giving a simple example to show that the result is not true for general one-parameter groups of isometries.

2.13. EXAMPLE. Choose two projections p and q on a Hilbert space $\mathscr{H}$ such that any two of the subspaces $p\mathscr{H}$, $(1-p)\mathscr{H}$, $q\mathscr{H}$ and $(1-q)\mathscr{H}$ have $\{0\}$ as their intersection. Take $\lambda > 0$ and define

$$u_t = e^{it\lambda} p + e^{-it\lambda}(1-p)$$

and

$$v_t = e^{it\lambda} q + e^{-it\lambda}(1-q) \qquad \text{for all } t \in \mathbf{R}\,.$$

This gives two strongly continuous one-parameter groups of unitaries such that $u_t + u_{-t} = v_t + v_{-t}$ for all $t \in \mathbf{R}$. If now $\xi \in \mathscr{H}$ such that $u_t \xi = v_t \xi$ for all t, then by differentiating we obtain $p\xi - (1-p)\xi = q\xi - (1-q)\xi$ and so $p\xi = q\xi$. But $p\mathscr{H} \cap q\mathscr{H} = \{0\}$ so that $p\xi = 0$ and $q\xi = 0$. Hence $\xi = (1-p)\xi = (1-q)\xi$, but as also $(1-p)\mathscr{H} \cap (1-q)\mathscr{H} = \{0\}$, we get $\xi = 0$. Similarly if $u_t \xi = v_{-t}\xi$ for all t we obtain

$$p\xi - (1-p)\xi = -q\xi + (1-q)\xi$$

so that $p\xi = (1-q)\xi$ and because $p\mathscr{H} \cap (1-q)\mathscr{H} = \{0\}$ and $(1-p)\mathscr{H} \cap q\mathscr{H} = \{0\}$, also here we get $\xi = 0$.

If instead of u_t and v_t we consider left multiplication on $\mathscr{B}(\mathscr{H})$ by u_t and v_t, respectively, we get a counter example, where the underlying space is a von Neumann algebra.

3. Formulation in terms of derivations.

We now formulate Theorem 2.11 in terms of derivations.

3.1. THEOREM. *Let δ_1 and δ_2 be derivations of a von Neumann algebra which are generators of a strongly continuous one-parameter group of *-automorphisms. If $\delta_1^2 = \delta_2^2$, then δ_1 and δ_2 have the same domaon and there exists a central projection p in their common domain such that*

i) $\delta_1(p) = \delta_2(p) = 0$
ii) $\delta_1(xp) = \delta_2(xp)$
iii) $\delta_1(x(1-p)) = -\delta_2(x(1-p))$

for all x in their common domain.

PROOF. Suppose that δ_1 is the generator of $\{\alpha_t \mid t \in \mathbf{R}\}$ and δ_2 the generator of $\{\beta_t \mid t \in \mathbf{R}\}$. Since formally $\alpha_t = e^{it\delta_1}$ and $\beta_t = e^{it\delta_2}$ we see that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ should hold for all $t \in \mathbf{R}$. We can prove it if we use e.g. that, if $\lambda \in \mathbf{C}$ and $\operatorname{Im} \lambda > 0$ we have

$$(\delta_1 - \lambda)^{-1} x = i \int_0^\infty e^{it\lambda} \alpha_{-t}(x)\, dt .$$

If we replace α_t by α_{-t} for all t we also get

$$(-\delta_1 - \lambda)^{-1} x = i \int_0^\infty e^{it\lambda} \alpha_t(x)\, dt .$$

and so

$$\begin{aligned} 2\lambda(\delta_1^2 - \lambda^2)^{-1} &= (\delta_1 - \lambda)^{-1} - (\delta_1 + \lambda)^{-1} \\ &= i \int_0^\infty e^{it\lambda} (\alpha_t + \alpha_{-t})(x)\, dt . \end{aligned}$$

Because the same formula holds for β and because $\delta_1^2 = \delta_2^2$ we get

$$\int_0^\infty e^{it\lambda} (\alpha_t + \alpha_{-t} - \beta_t - \beta_{-t})(x)\, dt = 0$$

for all $\lambda \in \mathbf{C}$ with $\operatorname{Im} \lambda > 0$. In particular, for any $\varphi \in M_*$ and any $a > 0$ and $s \in \mathbf{R}$ we get

$$\int_0^{\infty} e^{its} e^{-at} f(t)\, dt = 0 ,$$

where $f(t) = \varphi((\alpha_t + \alpha_{-t} - \beta_t - \beta_{-t})(x))$. As f is continuous, this implies that $f=0$. And this is true for all $x \in M$ and $\varphi \in M_*$ so that $\alpha_t + \alpha_{-t} = \beta_t + \beta_{-t}$ for all $t \in \mathbf{R}$.

Then apply Theorem 2.11. Because p is invariant under α and β we have that $p \in \mathscr{D}(\delta_1)$ and $p \in \mathscr{D}(\delta_2)$ and $\delta_1(p) = \delta_2(p) = 0$. Now let $x \in \mathscr{D}(\delta_2)$, then also xp and $x(1-p)$ are in $\mathscr{D}(\delta_2)$. For any $t \neq 0$ we have

$$\begin{aligned} \frac{1}{it}(\alpha_t(x) - x) &= \frac{1}{it}(\alpha_t(xp) - xp) + \frac{1}{it}(\alpha_t(x(1-p)) - x(1-p)) \\ &= \frac{1}{it}(\beta_t(xp) - xp) + \frac{1}{it}\beta_{-t}(x(1-p)) - x(1-p)) . \end{aligned}$$

So by taking the limit for $t \to 0$, we see that also $x \in \mathscr{D}(\delta_1)$ and that

$$\delta_1(x) = \delta_2(xp) - \delta_2(x(1-p)) .$$

Then by symmetry we get $\mathscr{D}(\delta_1) = \mathscr{D}(\delta_2)$, and if we first replace x by xp and then x by $x(1-p)$ we get the desired result.

Of course also here, in the case of a factor we get either $\delta_1 = \delta_2$ or $\delta_1 = -\delta_2$. And even for bounded derivations, this result is far from obvious, although in that case, because the derivations are inner and using Lemma 2.7, it is easy to obtain that they must commute.

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