MATH. SCAND. 51 (1982), 289–309

VITALI TYPE THEOREMS FOR A CLASS OF MEASURES IN $\mathbf{R}^N$

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1. Introduction and summary.

We consider the space $\mathbf{R}^N$ provided with the maximum norm $\|\cdot\|_\infty$. A closed ball with centre x and radius r is denoted by $B[x,r]$ and the corresponding open ball is denoted by $B(x,r)$. The special choice of norm is not essential for the results below, since any norm could do. It is only convenient for some of the proofs.

The Lebesgue measure is denoted by λ or λ_N. If $\mathbf{R}^p$, $p<N$, is imbedded in $\mathbf{R}^N$, the corresponding Lebesgue measure on $\mathbf{R}^p$ is denoted by λ_p. By μ we denote a measure on $\mathbf{R}^N$, which at least is defined on all Borel sets.

Let $\mathscr{B}$ denote a class of closed balls. In the notation from [5], [4] we say that *a set $A \subseteq \mathbf{R}^N$ can be packed with balls from $\mathscr{B}$ with respect to the measure μ*, if one can find a subclass (a *packing*) $\mathscr{B}^* \subseteq \mathscr{B}$ consisting of pairwise disjoint (closed) balls, such that

$$\mu^*(A \setminus \cup\{B \mid B \in \mathscr{B}^*\}) = 0,$$

where μ^* denotes the outer μ-measure. (In [1], $\mathscr{B}$ is said to be μ *adequate for* A.)

For given $\mathscr{B}$ we introduced in [4] the *local set* $A_{\mathrm{loc}}(\mathscr{B})$ by

$$A_{\mathrm{loc}}(\mathscr{B}) = \{x \in \mathbf{R}^N \mid \forall r \in \mathbf{R}_+ \exists B \in \mathscr{B} : B \subseteq B[x,r]\}.$$

(In [1], $\mathscr{B}$ is said *to cover a set A finely*, if $A \subseteq A_{\mathrm{loc}}(\mathscr{B})$.) We shall in the following only consider subsets of A_{loc}. Note that A_{loc} is independent of the measure under consideration.

For any subset $A \subseteq \mathbf{R}^N$, let $\tilde{A}$ denote the class of all closed balls B in $\mathbf{R}^N$, for which $A \cap B = \emptyset$, i. e.

$$\tilde{A} = \{B \text{ closed ball} \mid B \cap A = \emptyset\}.$$

This definition is especially used, when we want to characterize the μ-nullsets. If, however, in the applications, one is only interested in a differentiation theorem, the condition $B \cap A = \emptyset$ may be replaced by $\mu^*(A \cap B) = 0$.

Received May 11, 1981.

Let $\mathfrak{V}$ be a family of pairs $(A, \mathscr{B})$, where $A \subseteq \mathbf{R}^N$ and $\mathscr{B}$ is a class of closed balls in $\mathbf{R}^N$. The family $\mathfrak{V}$ is called a *Vitali system*, if the following two conditions are satisfied,

VS 1 $$\forall (A,\mathscr{B}) \in \mathfrak{V} \ \forall D \subseteq A : (D,\mathscr{B}) \in \mathfrak{V}\,,$$

VS 2 $$\forall (A,\mathscr{B}) \in \mathfrak{V} \ \forall F \text{ closed} : (A \setminus F, \mathscr{B} \cap \tilde{F}) \in \mathfrak{V}\,.$$

We note that if $\mathscr{B}$ is a class of closed balls, then $(A_{\mathrm{loc}}(\mathscr{B}), \mathscr{B})$, generates a Vitali system $\mathfrak{V}(\mathscr{B})$ by the axioms VS 1 and VS 2 above. Hence, if $\mathscr{B}$ and $A \subseteq A_{\mathrm{loc}}(\mathscr{B})$ are given, we can use all the theorems for general Vitali systems. This is sometimes done in the following without mentioning the corresponding Vitali system.

Let $\mathfrak{V}$ be a Vitali system. We say that *the packing theorem holds for* $\mathfrak{V}$ *with respect to* μ, if each pair $(A,\mathscr{B}) \in \mathfrak{V}$ satisfies the condition that there exists a μ-packing $\mathscr{B}^* \subseteq \mathscr{B}$ of A.

The fundamental (and classical) result for the packing theorem to hold for a Vitali system is the following lemma, which also was used in [5] and [4].

LEMMA 1. *Let* $\mathfrak{V}$ *be a Vitali system. The packing theorem holds for* $\mathfrak{V}$ *with respect to* μ, *if and only if there exists a constant* $c \in \mathbf{R}_+$, *such that one to each* $(A,\mathscr{B}) \in \mathfrak{V}$, *where* A *is bounded, can find a finite number of disjoint sets* $B_1,\ldots,B_n$ *from* $\mathscr{B}$, *such that*

$$\mu\left(\bigcup_{j=1}^{n} B_j\right) \geqq c \cdot \mu^*(A)\,.$$

We note the following simple and useful consequence of Lemma 1.

LEMMA 2. *Let* μ_1 *and* μ_2 *be two measures on* $\mathbf{R}^N$, *and let* $\mathfrak{V}$ *be a Vitali system, for which the packing theorem holds both with respect to* μ_1 *and with respect to* μ_2. *Then the packing theorem holds for* $\mathfrak{V}$ *with respect to* $\mu_1+\mu_2$.

PROOF. Let $(A,\mathscr{B}) \in \mathfrak{V}$, and let $U \supseteq A$ be an open set, such that

$$\mu_1(U) \leqq \tfrac{3}{2}\mu_1^*(A), \qquad \mu_2(U) \leqq \tfrac{3}{2}\mu_2^*(A)\,.$$

Then,

$$\mu^*(A) = (\mu_1+\mu_2)^*(A) \leqq (\mu_1+\mu_2)(U) = \mu_1(U)+\mu_2(U)\,.$$

Choose $B_1\ldots,B_n \in \mathscr{B}$ pairwise disjoint, such that

$$\mu_1\left(\bigcup_{j=1}^{n} B_j\right) \geqq \tfrac{1}{2}\mu_1^*(A)\,.$$

Then $(A \setminus \bigcup_{j=1}^n B_j, \mathscr{B} \cap (\bigcup_{j=1}^n B_j)^{\sim}) \in \mathfrak{V}$, so we can choose pairwise disjoint sets $B_{n+1}, \ldots, B_m \in \mathscr{B} \cap (\bigcup_{j=1}^n B_j)^{\sim}$, such that

$$\mu_2\left(\bigcup_{k=n+1}^{m} B_k\right) \geqq \tfrac{1}{2}\mu_2^*\left(A \setminus \bigcup_{j=1}^{n} B_j\right).$$

Then

$$\mu\left(\bigcup_{l=1}^{m} B_l\right) = \mu_1\left(\bigcup_{j=1}^{n} B_j\right) + \mu_2\left(\bigcup_{k=n+1}^{m} B_k\right) + \mu_1\left(\bigcup_{k=n+1}^{m} B_k\right) + \mu_2\left(\bigcup_{j=1}^{n} B_j\right)$$

$$\geqq \tfrac{1}{2}\mu_1^*(A) + \tfrac{1}{2}\mu_2^*\left(A \setminus \bigcup_{j=1}^{n} B_j\right) + \mu_1\left(\bigcup_{k=n+1}^{m} B_k\right) + \mu_2^*\left(A \cap \bigcup_{j=1}^{n} B_j\right)$$

$$\geqq \tfrac{1}{2}\mu_1^*(A) + \tfrac{1}{2}\mu_2^*(A) \geqq \tfrac{1}{3}\mu_1(U) + \tfrac{1}{3}\mu_2(U) \geqq \tfrac{1}{3}\mu^*(A),$$

and the result follows from Lemma 1.

Let $\mathscr{B}$ be a given class of closed balls. If $x \in A_{\text{loc}}$ we need some information about the mass $\mu(B)$ which always can be chosen from $\mathscr{B}$ in the neighbourhood of x. This is given by the socalled *ϱ-function*, $\varrho_\mu(x,r)$, (also depending on $\mathscr{B}$), which is defined by

$$\varrho_\mu(x,r) = \sup\{\mu(B) \mid B \in \mathscr{B},\ B \subseteqq B[x,r]\}, \qquad x \in A_{\text{loc}}, \quad r \in \mathbf{R}_+,$$

and by the *neighbouring ϱ-function*, $\varrho_{\mu,c}(x,r)$, with parameter $c \in \mathbf{R}_+$, which is given by

$$\varrho_{\mu,c}(x,r) = \sup\{\min(\mu(B_1), \mu(B_2)) \mid B_1, B_2 \in \mathscr{B}; \quad B_1, B_2 \subseteqq B[x,r];$$
$$\|\operatorname{cen} B_1 - \operatorname{cen} B_2\| \geqq c \cdot r\},$$

where cen B denotes the centre of the ball B.

We shall in the following often compare $\varrho_\mu(x,r)$ and $\varrho_{\mu,c}(x,r)$ with the corresponding functions $\varrho_\lambda(x,r)$ and $\varrho_{\lambda,c}(x,r)$ for the Lebesgue measure, since we already know (cf. [4]) Vitali type theorems for the ϱ-functions with respect to λ.

It follows intrinsically from Lemma 1 that one is more interested in the quotient of $\varrho_\mu(x,r)$ and $\mu(B[x,r])$. This causes no problem if $\mu = \lambda$, but for general μ we may have $\mu(B[x,r]) = 0$ for some $r \in \mathbf{R}_+$ and hence also $\varrho_\mu(x,r) = 0$. We shall therefore define the *relative ϱ-function $\varrho_\mu^*(x,r)$ with respect to μ* by

$$\varrho_\mu^*(x,r) = \begin{cases} \varrho_\mu(x,r)/\mu(B[x,r]) & \text{if } \mu(B[x,r]) > 0 \\ 1 & \text{if } \mu(B[x,r]) = 0. \end{cases}$$

It is easily shown by Lemma 1 that if there exists a constant $c \in \mathbf{R}_+$, such that $\varrho_\mu^*(x,r) \geqq c$ for all $x \in A$ and all $r \in]0,1]$ (say), then A can be μ-packed with balls from $\mathscr{B}$. This is in fact the classical result.

In [5] it was proved that if $\varrho_\lambda^*(x,r) \geqq \varphi(r)$ for all $x \in A$ and all $r \in]0,1]$, where the non-negative function φ satisfies the condition

$$\int_0^1 \varphi(r)\frac{dr}{r} = +\infty, \tag{1}$$

then A can be λ-packed with balls from $\mathscr{B}$. (The classical result is included in this result with $\varphi(r) = c$.)

In [4] it was shown by a counterexample that

$$\int_0^1 \varrho_\lambda^*(x,r)\frac{dr}{r} = +\infty \qquad \text{for all } x \in A \tag{2}$$

does *not* imply a packing theorem, but if $A = A_1 \cup A_2$ and

$$\int_0^1 \varrho_\lambda^*(x,r)/|\log \varrho_\lambda(x,r)| \cdot \frac{dr}{r} = +\infty \qquad \text{for all } x \in A_1, \tag{3}$$

and for some $c \in \mathbf{R}_+$

$$\int_0^1 \varrho_{\lambda,c}^*(x,r)\frac{dr}{r} = +\infty \qquad \text{for all } x \in A_2, \tag{4}$$

then one again obtains a packing theorem. It was also proved that (1) follows from (4).

It is easy to see that (3) and (4) can be relaxed to hold for almost every $x \in A_1$ and almost every $x \in A_2$, since $A_1 \setminus N_1$ and $A_2 \setminus N_2$ can be packed, where N_1 and N_2 are suitable nullsets.

Definition 3. Let μ be a measure on $\mathbf{R}^N$, and let $\mathscr{B}$ be a class of closed balls. If $A \subseteq A_{\mathrm{loc}}(\mathscr{B})$ and if there exists a (not necessarily disjoint) decomposition $A = A_1 \cup A_2$ of A, such that for all $\varepsilon \in]0,1[$

$$\int_0^\varepsilon \varrho_\mu^*(x,r)/|\log \varrho_\mu(x,r)| \cdot \frac{dr}{r} = +\infty \qquad \text{for } (\mu)\text{-almost every } x \in A_1 \tag{5}$$

and for some $c \in \mathbf{R}_+$

$$\int_0^1 \varrho_{\mu,c}^*(x,r)\frac{dr}{r} = +\infty \qquad \text{for } (\mu)\text{-almost every } x \in A_2, \tag{6}$$

we say that A satisfies $P(\mathscr{B},\mu)$.

Then the main result i [4] can be formulated in the following way.

THEOREM 4. *Let $\mathscr{B}$ be a class of closed balls, and let $A \subseteqq R^N$. If A satisfies $P(\mathscr{B}, \lambda)$, then there exists a λ-packing $\mathscr{B}^* \subseteqq \mathscr{B}$ of A.*

EXAMPLE 1. If the conditions (5) and (6) are further relaxed only to hold in a dense set, one cannot conclude that we have a packing. In fact, let $\mu = \lambda$, let $Q = \{q_n \mid n \in N\}$ and let $\varepsilon \in R_+$ be any given positive constant. Let $U = \bigcup_{n=1}^{+\infty} B[q_n, \varepsilon \cdot 2^{-n-1}]$. Then $\lambda(U) \leqq \varepsilon$, so U does not cover R or any subset A, for which $\lambda^*(A) > \varepsilon$. Let

$$\mathscr{B} = \{I \text{ closed interval} \mid I \subseteqq U\} .$$

Especially, $B[q_n, r] \in \mathscr{B}$ for all $r \in]0, \varepsilon \cdot 2^{-n-1}]$, so $\varrho_\lambda^*(q_n, r) = 1$ for $r \in]0, \varepsilon \cdot 2^{-n-1}]$. Hence, we conclude that

$$\int_0^1 \varrho_\lambda^*(q_n, r)/|\log \varrho_\lambda(q_n, r)| \cdot \frac{dr}{r} \geqq \int_0^{\varepsilon \cdot 2^{-n-1}} 1/|\log(2r)| \cdot \frac{dr}{r} = +\infty .$$

so (3) is satisfied in a dense set. By an analogous argument one proves that (4) also is satisfied for all $c \in]0,2[$ and $x = q_n \in Q$.

In section 2 we introduce the strongly continuous measures μ with respect to λ and prove that if μ is strongly continuous and A satisfies $P(\mathscr{B}, \mu)$, then there exists a μ-packing from $\mathscr{B}$ of A. In section 3 this result is generalized to the socalled well-behaved measures, and in section 4 one further generalizes to the weakly well-behaved measures on R^N. In section 5 one considers weakly well-behaved atomic measures, for which one has fairly good results, and finally in section 6 some applications and the limits of the theory are mentioned.

2. Strongly continuous measures μ with respect to λ.

We first prove the following simple result, which we unfortunately forgot to include in [4].

THEOREM 5. *Let μ_1 and μ_2 be two measures on R^N, and suppose that μ_2 is absolutely continuous with respect to μ_1. Let $\mathfrak{V}$ be a Vitali system. If the packing theorem holds for $\mathfrak{V}$ with respect to μ_1, then the packing theorem also holds for $\mathfrak{V}$ with respect to μ_2.*

PROOF. Let $(A, \mathscr{B}) \in \mathfrak{V}$. By assumption there exists a μ_1-packing $\{B_j \mid j \in J\} \subseteqq \mathscr{B}$ of A, i. e.

$$\mu_1(A \setminus \bigcup\{B_j \mid j \in J\}) = 0 .$$

As μ_2 is absolutely continuous with respect to μ_1, each μ_1-nullset is also a μ_2-nullset, so

$$\mu_2(A \setminus \cup\{B_j \mid j \in J\}) = 0 .$$

Let μ be absolutely continuous with respect to the Lebesgue measure λ on $\mathbf{R}^N$. Let $f(x)$ be the Radon-Nikodym derivative of μ with respect to λ. By Lebesgue's differentiation theorem we get

$$f(x) = \lim_{n \to +\infty} \mu[B(x, 2^{-n}])/\lambda(B[x, 2^{-n}]) \quad \text{for a.e. } x \in \mathbf{R}^N . \tag{7}$$

DEFINITION 6. The measure μ is said to be strongly continuous with respect to λ, if μ is absolutely continuous with respect to λ, and the Radon-Nikodym derivative $f(x)$ is for almost every $x \in \mathbf{R}^N$ locally of class L^∞, i.e. for a.e. $x \in \mathbf{R}^N$ there exists an $r_x > 0$, such that $f \cdot \chi_{B[x, r_x]} \in L^\infty$.

EXAMPLE 2. Let $\varphi(t) = \chi_{]0,1[}(t)/\sqrt{t}$, and let

$$f(t) = \varphi(t) + \sum_{n=1}^{+\infty} \varphi(2^n t - 1) \qquad \text{and} \qquad g(t) = \sum_{n=1}^{+\infty} \varphi(2^n t - 2^n q_n) ,$$

where $\mathbf{Q} \cap [0,1] = \{q_n \mid n \in \mathbf{N}\}$. Then $f, g \in L^1(\lambda)$, and $f, g \geqq 0$. The measure μ_1 defined by $\mu_1(B) = \int_B f(x)\,dx$ is strongly continuous with respect to λ, while the measure μ_2 defined by $\mu_2(B) = \int_B g(x)\,dx$ is absolutely continuous but not strongly continuous with respect to λ.

We now have the following generalization of Theorem 4.

THEOREM 7. *Let μ be strongly continuous with respect to λ. Let $\mathscr{B}$ be a class of closed balls. Suppose that $A \subseteqq \mathbf{R}^N$ satisfies $P(\mathscr{B}, \mu)$. Then there exists a μ-packing of A.*

PROOF. Only $A \subseteqq \{x \mid f(x) > 0\}$ need to be considered. We shall only prove the theorem under the assumption.

(8)
$$\int_0^1 \varrho_\mu^*(x,r)/|\log \varrho_\mu(x,r)| \cdot \frac{dr}{r} = +\infty \quad \text{for } (\mu)-\text{a.e. } x \in A_1 \subseteqq \{x \mid f(x) > 0\} ,$$

as the proof in the case

$$\int_0^1 \varrho_{\mu,c}^*(x,r) \frac{dr}{r} = +\infty \qquad \text{for } (\mu)-\text{a.e. } x \in A_2 \subseteqq \{x \mid f(x) > 0\}$$

follows a similar pattern.

By assumption, (7), (8) and $f\cdot\chi_{B[x,r_x]}\in L^\infty$ are simultaneously fulfilled for (λ)-almost every $x\in A_1$. Let x be such a point. Let $a=\|f\cdot\chi_{B[x,r_x]}\|_\infty$. From (7) follows that $a>0$.

Choose $n_0\in\mathbf{N}_0$, such that $2^{-n_0}\leqq r_x$ and

$$\mu(B[x,2^{-n}])/\lambda(B[x,2^{-n}]) > \tfrac{1}{2}f(x) \qquad \text{and} \quad a\cdot\varrho_\lambda(x,2^{-n}) < 1 \quad \text{for all } n\geqq n_0\,.$$

For $n\geqq n_0$ we have

$$0 < \varrho_\mu(x,2^{-n}) \leqq a\cdot\varrho_\lambda(x,2^{-n}) < 1$$

and

$$\begin{aligned}\varrho_\mu^*(x,2^{-n}) &= \varrho_\mu(x,2^{-n})/\mu(B[x,2^{-n}]) \leqq a\cdot\varrho_\lambda(x,2^{-n})/\{\tfrac{1}{2}f(x)\cdot\lambda(B[x,2^{-n}])\}\\ &= \{2a/f(x)\}\cdot\varrho_\lambda^*(x,2^{-n})\,,\end{aligned}$$

so

$$\varrho_\mu^*(x,2^{-n})/|\log\varrho_\mu(x,2^{-n})| \leqq \{2a/f(x)\}\cdot\varrho_\lambda^*(x,2^{-n})/|\log[a\varrho_\lambda(x,2^{-n})]|\,.$$

Using the same technique as in [4] we conclude that (8) is satisfied if and only if

$$\sum_{n=1}^{+\infty}\varrho_\mu^*(x,2^{-n})/|\log\varrho_\mu(x,2^{-n})| = +\infty\,.$$

By the estimate above we conclude that also

$$\sum_{n=1}^{+\infty}\varrho_\lambda^*(x,2^{-n})/|\log[a\varrho_\lambda(x,2^{-n})]| = +\infty\,,$$

and since $a\in\mathbf{R}_+$ is a constant,

$$\sum_{n=1}^{+\infty}\varrho_\lambda^*(x,2^{-n})/|\log\varrho_\lambda(x,2^{-n})| = +\infty\,,$$

which is equivalent to

$$\int_0^1\varrho_\lambda^*(x,r)/|\log\varrho_\lambda(x,r)|\cdot\frac{dr}{r} = +\infty\,. \tag{9}$$

Since (9) is satisfied for (λ)-almost every $x\in A_1$, it follows from Theorem 4 that we have a λ-packing $\mathscr{B}^*$ of A_1, and since μ is absolutely continuous with respect to λ, we get that $\mathscr{B}^*$ is also a μ-packing. Note that if μ is strongly continuous with respect to λ and $A\subseteqq\{x \mid f(x)>0\}$ satisfies $P(\mathscr{B},\mu)$, then A also satisfies $P(\mathscr{B},\lambda)$.

The proof of Theorem 7 suggest the following

CONJECTURE. If μ is absolutely continuous with respect to λ, and $A \subseteqq \{x \mid f(x) > 0\}$ satisfies $P(\mathscr{B}, \mu)$, then A also satisfies $P(\mathscr{B}, \lambda)$.

3. Well-behaved measures on R^N.

In section 2 we considered the measure μ from a measure theoretical point of view. We shall now turn to the geometrical aspect in order to handle the case, where μ is not absolutely continuous with respect to λ. We shall first consider a very simple case.

LEMMA 8. *Let λ_N be the Lebesgue measure in R^N and λ_q the Lebesgue measure in R^q, $q < N$, where R^q is imbedded in R^N. We put $\mu = \lambda_N + \lambda_q$. Let $\mathscr{B}$ be a class of closed balls, and suppose that $A \subseteqq \mathsf{R}^N$ satisfies $P(\mathscr{B}, \mu)$. Then there exists a μ-packing $\mathscr{B}^* \subseteqq \mathscr{B}$ of A.*

PROOF. We may suppose that $N=2$ and $q=1$, and that supp λ_1 is the x-axis in R^2. First we note that if $y_0 \neq 0$, then

$$\varrho_\mu((x_0, y_0); r) = \varrho_{\lambda_2}((x_0, y_0); r) \qquad \text{for } r \in]0, |y_0|[$$

and

$$\varrho_\mu^*((x_0, y_0); r) = \varrho_{\lambda_2}^*((x_0, y_0); r) \qquad \text{for } r \in]0, |y_0|[\, .$$

Hence, there is nothing to prove if A does not intersect the x-axis.

If we can prove that $A \cap \operatorname{supp} \lambda_1$ also satisfies $P(\mathscr{B}, \lambda_1)$, we conclude, since $\{B \cap \operatorname{supp} \lambda_1 \mid B \in \mathscr{B}\}$ is a system of closed intervals on supp λ_1, that the packing theorem holds for $(A, \mathscr{B})$ both with respect to λ_2 and with respect to λ_1, and hence according to Lemma 2, also with respect to $\mu = \lambda_1 + \lambda_2$.

Now, let $(x, 0) \in A$. Then

$$\mu(B[(x,0); r]) = 2r + 4r^2 \qquad \text{and} \qquad \lambda_1(B[(x,0); r]) = 2r \, ,$$

and from the definition of the ϱ-function we get

$$\text{(10)} \quad \varrho_\mu((x,0); r) \leqq \varrho_{\lambda_1}((x,0); r) + \lambda_2(B[(x,0); r]) = \varrho_{\lambda_1}((x,0); r) + 4r^2 \, ,$$

and similarly

$$\varrho_{\mu,c}((x,0); r) \leqq \varrho_{\lambda_1,c}((x,0); r) + 4r^2 \, ,$$

so

$$\text{(11)} \qquad \begin{aligned} \varrho_\mu^*((x,0);\, r) &= \varrho_\mu((x,0);\, r)/(2r + 4r^2) \\ &\leqq \varrho_{\lambda_1}((x,0); r)/\{2r(1+2r)\} \\ &\qquad + \lambda_2(B[(x,0); r])/\{2r(1+2r)\} \\ &\leqq \varrho_{\lambda_1}^*((x,0); r) + 2r/(1+2r) \, , \end{aligned}$$

and similarly

$$\varrho^*_{\mu,c}((x,0);r) \leqq \varrho^*_{\lambda_1,c}((x,0);r)+2r/(1+2r) ,$$

from which we derive that

$$\varrho^*_{\lambda_1,c}((x,0);r)/r \geqq \varrho^*_{\mu,c}((x,0);r)/r-2/(1+2r) .$$

Since $\int_0^{1} 2/(1+2r)dr = \log 3$, it follows from the assumption that

$$\int_0^1 \varrho^*_{\lambda_1,c}((x,0);r)\frac{dr}{r} = +\infty \qquad \text{for} \quad (\mu)\text{-almost every } (x,0) \in A_2 .$$

In the same way one proves using (10) and (11) that

$$\int_0^1 \varrho^*_{\lambda_1}((x,0);r)/|\log \varrho_{\lambda_1}((x,0);r)| \cdot \frac{dr}{r} = +\infty$$

for (μ)-almost every $(x,0) \in A_1$.

In fact, assume that (8) holds, and assume that $\varrho_\mu(r)<\frac{1}{2}$ for $r \in]0,1]$. Then also

$$\int_0^{1/4} \varrho^*_\mu(r)/|\log \varrho_\mu(r)| \cdot \frac{dr}{r} = +\infty ,$$

where we have used the shorter notation $\varrho_\mu(r)$ for $\varrho((x,0);r)$ etc. Let

$$B_1 = \{r \in [0,\tfrac{1}{4}] \mid \varrho_\mu(r) \geqq 8r^2\}, \qquad B_2=[0,\tfrac{1}{4}]\setminus B_1 .$$

Since $u/|\log u|$ is increasing for $u \in]0,1[$, we get

$$\begin{aligned}\int_{B_2} \frac{\varrho^*_\mu(r)}{|\log \varrho_\mu(r)|}\frac{dr}{r} &= \int_{B_2} \frac{\varrho_\mu(r)}{|\log \varrho_\mu(r)|}\frac{1}{\mu(B[(x,0);\, r])}\frac{dr}{r}\\ &\leqq \int_{B_2} \frac{8r^2}{|\log (8r^2)|}\frac{1}{2r+4r^2}\frac{1}{r}\,dr\\ &\leqq 2\int_{B_2}\frac{1}{|\log r|-\frac{1}{2}\log 8}\,dr < +\infty ,\end{aligned}$$

so

$$(*) \qquad \int_{B_1} \frac{\varrho^*_\mu(r)}{|\log \varrho_\mu(r)|}\frac{dr}{r} = +\infty .$$

For $r \in B_1$ we have from (10),

$$\tfrac{1}{2} > \varrho_{\lambda_1}(r) \geqq \varrho_\mu(r)-4r^2 \geqq \tfrac{1}{2}\varrho_\mu(r) ,$$

hence,

$$|\log \varrho_{\lambda_1}(r)| \leqq |\log(\tfrac{1}{2}\varrho_\mu(r))|, \qquad r \in B_1 .$$

From (11) we get

$$\frac{1}{r}\varrho^*_{\lambda_1}(r) \geqq \frac{1}{r}\varrho^*_\mu(r) - \frac{2}{1+2r},$$

hence for $r \in B_1$,

$$\frac{\varrho^*_{\lambda_1}(r)}{|\log \varrho_{\lambda_1}(r)|}\frac{1}{r} \geqq \frac{\varrho^*_\mu(r)}{|\log(\frac{1}{2}\varrho_\mu(r))|}\frac{1}{r} - \frac{2}{1+2r}\frac{1}{|\log(\frac{1}{2}\varrho_\mu(r))|}.$$

Since $\int_{B_1} 2\{(1+2r)|\log(\frac{1}{2}\varrho_\mu(r))|\}^{-1}\,dr < +\infty$, it follows from (*) that

$$\int_0^1 \frac{\varrho^*_{\lambda_1}(r)}{|\log \varrho_{\lambda_1}(r)|}\frac{dr}{r} \geqq C\int_{B_1} \frac{\varrho^*_\mu(r)}{|\log \varrho_\mu(r)|}\frac{dr}{r} = +\infty,$$

and the proof is complete.

COROLLARY 9. *Let μ_N be a strongly continuous measure with respect to λ_N, and μ_q, $q < N$, a strongly continuous measure with respect to λ_q, and let $\mu = \mu_N + \mu_q$. If $\mathscr{B}$ is a class of closed balls, and $A \subseteq \mathsf{R}^N$ satisfies $P(\mathscr{B}, \mu)$, then there exists a μ-packing $\mathscr{B}^* \subseteq \mathscr{B}$ of A.*

PROOF. If $x \notin \operatorname{supp} \mu_q$, there exists an $r_0 \in \mathsf{R}_+$, such that

$$\mu|B[x, r_0] = \mu_N|B[x, r_0],$$

so when $x \notin \operatorname{supp} \mu_q$, the result follows from Theorem 7.

Since, by assumption, $\mu_N(\operatorname{supp} \mu_q) = 0$, we shall only prove that $A \cap \operatorname{supp} \mu_q$ satisfies $P(\mathscr{B}, \mu_q)$, when A satisfies $P(\mathscr{B}, \mu)$.

Let f_q be the Radon-Nikodym derivative of μ_q with respect to λ_q, and let f_N be the Radon-Nikodym derivative of μ_N with respect to λ_N. When $f_N(x) = 0$, there is nothing to prove, and the set where either $f_q(x) = +\infty$ or $f_N(x) = +\infty$ is a nullset by assumption. Hence, we need only consider the case, where $f_q(x) \in \mathsf{R}_+$ and $f_N(x) \in \mathsf{R}_+$. Furthermore, for (μ)-almost every

$$x \in \{x \in \operatorname{supp} \mu_q \mid f_q(x) \in \mathsf{R}_+,\ f_N(x) \in \mathsf{R}_+\}$$

there exists an $r_0 \in \mathsf{R}_+$, such that

$$\frac{1}{2}\frac{\mu_q(B[x,r])}{\lambda_q(B[x,r])} \leqq f_q(x) \leqq 2\frac{\mu_q(B[x,r])}{\lambda_q(B[x,r])} \qquad \text{for all } r \in]0, r_0]$$

and

$$\frac{1}{2}\frac{\mu_N(B[x,r])}{\lambda_N(B[x,r])} \leqq f_N(x) \leqq 2\frac{\mu_N(B[x,r])}{\lambda_N(B[x,r])} \qquad \text{for all } r \in]0,r_0] ,$$

so for $r \in]0,r_0]$ we get

$$\begin{aligned}\varrho_\mu(x,r) &\leqq \varrho_{\mu_q}(x,r)+\mu_N(B[x,r]) \leqq \varrho_{\mu_q}(x,r)+2f_N(x)\cdot\lambda_N(B[x,r]) \\ &= \varrho_{\mu_q}(x,r)+2f_N(x)\cdot 2^N\cdot r^N ,\end{aligned}$$

and

$$\begin{aligned}\mu(B[x,r]) &= \mu_q(B[x,r])+\mu_N(B[x,r]) \\ &\geqq \tfrac{1}{2}f_q(x)\cdot\lambda_q(B[x,r])+\tfrac{1}{2}f_N(x)\cdot\lambda_N(B[x,r]) \\ &= \tfrac{1}{2}f_q(x)\cdot 2^q r^q+\tfrac{1}{2}f_N(x)\cdot 2^N r^N ,\end{aligned}$$

so

$$\begin{aligned}\varrho_\mu^*(x,r) &\leqq \varrho_{\mu_q}^*(x,r)+2f_N(x)\cdot 2^N r^N/\{\tfrac{1}{2}f_q(x)\cdot 2^q r^q+\tfrac{1}{2}f_N(x)\cdot 2^N r^N\} \\ &\leqq \varrho_{\mu_q}^*(x,r)+2^{N-q+2}\cdot\{f_N(x)/f_q(x)\}\cdot r^{N-q} .\end{aligned}$$

Since x is fixed and $q<N$, we conclude that

$$\int_0^{r_0} 2^{N-q+2}\{f_N(x)/f_q(x)\}r^{N-q}\cdot\frac{dr}{r} < +\infty ,$$

so using the same method as in the proof of Lemma 8 we conclude that

$$A\cap\{x \in \operatorname{supp}\mu_q \mid f_q(x)\in\mathbf{R}_+,\ f_N(x)\in\mathbf{R}_+\}$$

satisfies $P(\mathscr{B},\mu_q)$, and the proof is complete.

Let V_j, $j<N$, be a j-dimensional (orientable) C^1-manifold imbedded in $\mathbf{R}^N$. The manifold V_j need not necessarily be connected. Then by the C^1-structure V_j inherits a measure λ_j', which is uniquely determined by the Lebesgue measure in $\mathbf{R}^j$ (cf. e. g. [3]).

When we have a j-dimensional C^1-manifold V_j as above we can by the local charts define the absolutely continuous and the strongly continuous measures μ_j with respect to λ_j'.

If to every $x \in V_j$ one can find $r_x \in \mathbf{R}_+$, such that $V_j\cap B(x,r)$ is connected for all $r \in]0,r_x[$, it follows by the C^1-structure, which assures the existence of the tangent space, that if A satisfies $P(\mathscr{B},\lambda_j')$, then one has a λ_j'-packing $\mathscr{B}^*\subseteqq\mathscr{B}$ of A. In fact, condition $P(\mathscr{B},\lambda_j')$ is local, and if $x \in A\cap V_j$, let $(U_\varkappa,\varkappa)$, $U_\varkappa\subseteqq\mathbf{R}^j$, be the corresponding chart. Then $\varkappa$ is extended to a C^1-map $\tilde\varkappa$ in a neighbourhood of x in $\mathbf{R}^N$, such that $\tilde\varkappa$ is bijective on $B(x,r_x')$. If

$$\mathscr{F} = \{\tilde\varkappa(B) \mid B\in\mathscr{B},\ B\subseteqq B(x,r_x')\} ,$$

then $\mathscr{F}$ is regular in the sense of [2], hence $\tilde{\varkappa}(A \cap B(x, r'_x))$ satisfies $P(\mathscr{F}, \lambda_j)$, so $\tilde{\varkappa}(A \cap B(x, r'_x))$ can be λ_j-packed with sets from $\mathscr{F}$. Using $\tilde{\varkappa}^{-1}$, we get a λ'_j-packing of $A \cap B(x, r'_x)$.

This result, together with Theorem 7, Lemma 8 and Corollary 9, leads to the following definition.

DEFINITION 10. A measure μ on $\mathbf{R}^N$ is called well-behaved, if there exist $N+1$ measures $\mu_0, \mu_1, \ldots, \mu_N$, such that

$$\mu = \mu_N + \mu_{N-1} + \ldots + \mu_1 + \mu_0 ,$$

where supp $\mu_j \subseteqq V_j$, $j = 0, 1, \ldots, N$, and each V_j is a j-dimensional C^1-manifold satisfying

(12) $\qquad \forall x \in V_j \exists r_x \in \mathbf{R}_+ \forall r \in]0, r_x[\; : \; V_j \cap B(x, r)$ is connected ,

and where μ_j is strongly continuous with respect to the measure λ'_j induced by λ_j.

REMARK. In the definition above supp μ_j is considered as a closed set relative to the manifold V_j, which need not be closed in $\mathbf{R}_N$.

Note that the V_j are not necessarily disjoint. This idea of decomposition of μ according to dimension was also used in [6] in a simpler case, though in a different context.

Note also that if the conjecture in section 2 is true, we may allow that μ_j is only absolutely continuous with respect to λ'_j. In that case the concept of well-behavior would only depend on the geometry.

The reason for introduction of well-behaved measures is that we have a unique decomposition $\mu = \mu_a + \mu_s$ of every measure μ on $\mathbf{R}^N$, where μ_a is absolutely continuous with respect to λ_N and μ_s is singular. If supp μ_s is contained in $\bigcup_{j=0}^{N-1} V_j$, where V_j is a j-dimensional C^1-manifold of the considered kind, we may again split $\mu_s | V_{N-1}$ into $\mu_a^{N-1} + \mu_s^{N-1}$, where μ_a^{N-1} is absolutely continuous with respect to λ'_{N-1}, and after a finite number of steps and under suitable assumptions we would get $\mu = \mu_N + \mu_{N-1} + \ldots + \mu_1 + \mu_0$ as in Definition 10.

THEOREM 11. *Let μ be a well-behaved measure on $\mathbf{R}^N$, and let $\mathscr{B}$ be a class of closed balls in $\mathbf{R}^N$. If A satisfies $P(\mathscr{B}, \mu)$, then there exists a μ-packing $\mathscr{B}^* \subseteqq \mathscr{B}$ of A.*

PROOF. It is enough to assume that A is bounded, and $\mu^*(A) < +\infty$. As A satisfies $P(\mathscr{B}, \mu)$ we must have $A \subseteqq A_{\text{loc}}(\mathscr{B})$, so $(A, \mathscr{B})$ defines a Vitali system $\mathfrak{V}$.

Let

$$\mu = \mu_N + \mu_{N-1} + \ldots + \mu_1 + \mu_0 ,$$

where supp $\mu_j \subseteqq V_j$ and $\mu_k(V_j) = 0$ for all $k > j$. First we choose disjoint balls $B_1, \ldots, B_{n_0} \in \mathscr{B}$, such that

$$\mu_0\left(\bigcup_{j=1}^{n_0} B_j\right) \geqq \tfrac{1}{2}\mu_0(A \cap V_0)$$

(cf. Corollary 9). Since

$$\left(A \setminus \bigcup_{j=1}^{n_0} B_j,\quad \mathscr{B} \cap \left(\bigcup_{j=1}^{n_0} B_j\right)^{\sim}\right) \in \mathfrak{B} ,$$

we can choose disjoint balls

$$B_{n_0+1}, \ldots, B_{n_1} \in \mathscr{B} \cap \left(\bigcup_{j=1}^{n_0} B_j\right)^{\sim} ,$$

such that

$$\mu_1\left(\bigcup_{j=n_0+1}^{n_1} B_j\right) \geqq \tfrac{1}{2}\mu_1^*\left([A \cap V_1] \setminus \bigcup_{j=1}^{n_0} B_j\right) ,$$

and hence (cf. the proof of Lemma 2)

$$(\mu_0 + \mu_1)\left(\bigcup_{j=1}^{n_1} B_j\right) \geqq \tfrac{1}{3}(\mu_0 + \mu_1)^*(A) .$$

After a finite number of steps we have chosen n disjoint closed balls $B_1 \ldots, B_n$ from $\mathscr{B}$, such that

$$\mu\left(\bigcup_{j=1}^{n} B_j\right) \geqq \tfrac{1}{3}\mu^*(A) ,$$

and the theorem follows from Lemma 1.

4. Weakly well-behaved measures on R^N.

In Theorem 11 we proved that for well-behaved measures μ on R^N we have the same type of Vitali theorems as the theorems proved in [4] for Lebesgue measure. If the μ_j are only absolutely continuous with respect to λ_j', we may put $\tilde{\mu} = \lambda_N' + \lambda_{N-1}' + \ldots + \lambda_1' + \lambda_0'$, because if A satisfies $P(\mathscr{B}, \tilde{\mu})$ we get a $\tilde{\mu}$-packing of A, which of course also is a μ-packing.

We shall now turn to extensions of the geometrical structure of supp μ. Let

$$\mu = \mu_N + \mu_{N-1} + \ldots + \mu_1 + \mu_0 ,$$

where supp $\mu_j \subseteq V_j$, and each V_j is a j-dimensional C^1-manifold imbedded in R^N, and suppose that each μ_j is strongly continuous with respect to λ'_j. Suppose that V_j does not satisfy (12), i. e.

$$\text{(13)} \qquad \exists x \in V_j \forall r_0 \in \mathsf{R}_+ \exists r \in]0, r_0] : V_j \cap B(x,r) \text{ is disconnected} .$$

Let W_j be the subset, for which (13) holds for all $x \in W_j$. In order to avoid too many complications we shall assume that [cf. (12)]

$$\forall x \in W_j \exists r_x \in \mathsf{R}_+ \forall r \in]0, r_x[: W_j \cap B(x,r) \text{ is connected.}$$

Let $\tilde{\mu}_j = \mu_j | W_j$. Then we have the following lemma.

LEMMA 12. *Let* $\mu = \mu_j$, supp $\mu_j \subseteq V_j$, *where* V_j *is given as above, including a subset* W_j *characterized as above. Suppose that*

$$\text{(14)} \qquad \{1 - \tilde{\mu}_j(B[x,r])/\mu_j(B[x,r])\}/r \in L^1(]0,1])$$

for (μ_j)*-almost every* $x \in W_j$. *If* A *satisfies* $P(\mathscr{B}, \mu_j)$, *then* A *can be* μ_j*-packed with balls from* $\mathscr{B}$.

PROOF. Since $\mu_j - \tilde{\mu}_j$ is a well-behaved measure, it is enough to prove that A also satisfies $P(\mathscr{B}, \tilde{\mu}_j)$. Let $x \in W_j$ and let $r \in]0,1]$. Then for geometrical reasons,

$$\varrho_{\mu_j}(x,r) \leqq \varrho_{\tilde{\mu}_j}(x,r) + \{\mu_j(B[x,r]) - \tilde{\mu}_j(B[x,r])\} ,$$

so

$$\varrho^*_{\mu_j}(x,r) \leqq \varrho^*_{\tilde{\mu}_j}(x,r) + 1 - \tilde{\mu}_j(B[x,r])/\mu_j(B[x,r]) ,$$

hence

$$\varrho^*_{\tilde{\mu}_j}(x,r)/r \geqq \varrho^*_{\mu_j}(x,r)/r - \{1 - \tilde{\mu}_j(B[x,r])/\mu_j(B[x,r])\}/r ,$$

and a similar estimate, when ϱ^* is replaced by the relative neighbouring function. From (14) follows that $A \cap W_j$ and hence also A satisfies $P(\mathscr{B}, \tilde{\mu}_j)$. The packing result follows from that W_j is a connected C^1-manifold.

Let V_j be a j-dimensional C^1-manifold and let $W_{j,1}$ be the subset of V_j for which (13) holds for all $x \in W_{j,1}$. Let $W_{j,2}$ be the subset of $W_{j,1}$, for which

$$\forall x \in W_{j,2} \forall r_0 \in \mathsf{R}_+ \exists r \in]0, r_0]: W_{j,1} \cap B(x,r) \text{ is disconnected} .$$

By induction, define $W_{j,n}$ by

$$\forall x \in W_{j,n} \forall r_0 \in \mathsf{R}_+ \exists r \in]0, r_0] : W_{j,n-1} \cap B(x,r) \text{ is disconnected.}$$

The $W_{j,k}$ are generalizations of the set of accumulation points, the set of the accumulation points of the accumulation points etc. in e. g. the 1-dimensional case.

DEFINITION 13. The manifold V_j is called weakly well-behaved, if there exists an $n \in \mathbf{N}$, such that $W_{j,n}=\emptyset$. If $W_{j,1}=\emptyset$, we also say that V_j is well-behaved.

The definition means that we can achieve any *finite* order of complexity of each V_j.

DEFINITION 14. A measure μ on $\mathbf{R}^N$ is called weakly well-behaved, if there exists a decomposition

$$\mu = \mu_N+\mu_{N-1}+\ldots+\mu_1+\mu_0 ,$$

such that $\operatorname{supp} \mu_j \subseteqq V_j$, where each V_j is a weakly well-behaved j-dimensional C^1-manifold, μ_j is strongly continuous with respect to λ_j', and if furthermore $W_{j,k}$, $k=1,2,\ldots,n$, are non-empty, while $W_{j,n+1}=\emptyset$, we require that $\tilde{\mu}_{j,k} =\mu_j| W_{j,k}$ satisfies

$$\{1-\tilde{\mu}_{j,k}(B[x,r])/\mu(B[x,r])\}/r \in L^1(]0,1])$$

for (μ)-almost every $x \in W_{j,k}$ for all $k=1,2,\ldots,n$.

We shall later (cf. Example 4) construct a measure μ on $\mathbf{R}$, which satisfies all the conditions for a weakly well-behaved measure except that $V_0=\operatorname{supp} \mu$ is not a well-behaved 0-dimensional manifold, and we shall construct a system $\mathscr{B}$ of closed intervals and a set A satisfying $P(\mathscr{B},\mu)$ such that A has no μ-packing from $\mathscr{B}$.

THEOREM 15. *Let μ be a weakly well-behaved measure on $\mathbf{R}^N$, and let $\mathscr{B}$ be a class of closed balls. If $A\subseteqq\mathbf{R}^N$ satisfies $P(\mathscr{B},\mu)$, then there exists a μ-packing $\mathscr{B}^*\subseteqq\mathscr{B}$ of A.*

PROOF. First we split μ into the sum of measures

$$\mu = \mu_N+\mu_{N-1}+\ldots+\mu_1+\mu_0$$

as described in Definition 14. Then each μ_j is again split into a sum $\mu_j=\tilde{\mu}_{j,1} + (\mu_j-\tilde{\mu}_{j,1})$. The measure $\tilde{\mu}_{j,1}$ is again split into a sum of $\tilde{\mu}_{j,1}-\tilde{\mu}_{j,2}$ and $\tilde{\mu}_{j,2}$. However, as $W_{j,n_j+1}=\emptyset$ for some $n_j \in \mathbf{N}$ by assumption, we end up with $\tilde{\mu}_{j,n_j}$ after a finite number of steps. Hence,

$$\mu = \{\tilde{\mu}_{0,n_0} + (\tilde{\mu}_{0,n_0-1} - \tilde{\mu}_{0,n_0}) + \ldots + (\mu_0 - \tilde{\mu}_{0,1})$$
$$+ \ldots$$
$$+ \{\tilde{\mu}_{N,n^N} + (\tilde{\mu}_{N,n^N-1} - \tilde{\mu}_{N,n^N}) + \ldots + (\mu_N - \tilde{\mu}_{N,1})\} .$$

Using the same technique as in the proof of Lemma 12, we conclude that we can find a finite number of disjoint balls $B_1, \ldots, B_m \in \mathscr{B}$, such that

$$\tilde{\mu}_{0,n_0}\left(\bigcup_{j=1}^{m} B_j\right) \geqq \tfrac{1}{2}\tilde{\mu}^*_{0,n_0}(A) .$$

Then use the same method on $\tilde{\mu}_{0,n_0-1} - \tilde{\mu}_{0,n_0}$ and the set $A \setminus \bigcup_{j=1}^m B_j$. Combined with the proof of Lemma 2 and possibly an iteration of the usual kind we get a finite number of disjoint balls $B_1, \ldots, B_p \in \mathscr{B}$ including $B_1, \ldots, B_m$, such that

$$\tilde{\mu}_{0,n_0-1}\left(\bigcup_{j=1}^{p} B_j\right) \geqq \tfrac{1}{2}\tilde{\mu}^*_{0,n_0-1}(A) .$$

[Note that $\tilde{\mu}_{0,n_0-1} = \tilde{\mu}_{0,n_0} + (\tilde{\mu}_{0,n_0-1} - \tilde{\mu}_{0,n_0})$.] In this way we continue, until we have got disjoint balls $B_1, \ldots, B_q \in \mathscr{B}$, such that

$$\mu_0\left(\bigcup_{j=1}^{q} B_j\right) \geqq \tfrac{1}{2}\mu^*_0(A) .$$

Then consider $\tilde{\mu}_{1,n_1}$ and the set $A \setminus \bigcup_{j=1}^q B_j$. By the same procedure we are able to continue through all the $\tilde{\mu}_{1,j-1} - \tilde{\mu}_{1,j}$, so we can turn to $\tilde{\mu}_{2,n_2}$, etc. After a finite number of steps we have chosen disjoint balls $B_1, \ldots, B_n \in \mathscr{B}$, such that

$$\mu\left(\bigcup_{j=1}^{n} B_j\right) \geqq \tfrac{1}{2}\mu^*(A) ,$$

and the theorem follows from Lemma 1.

5. Weakly well-behaved atomic measures.

In the case of μ_0 above we have a much better description, and this may help us to understand the higher dimensional cases. Let

$$\mu = \sum_{n=1}^{+\infty} a_n \delta_{(x_n)} , \tag{15}$$

where $a_n \geqq 0$ and $\delta_{(x_n)}$ is the Dirac measure in $x_n \in \mathbf{R}^N$, and $\mu(B) < +\infty$ for all bounded sets B. Then

$$V_0 = \{x_n \mid n \in \mathbf{N}\}$$

is a 0-dimensional manifold.

Let $\mathscr{B}$ be a class of closed balls in $\mathbf{R}^N$, and let A satisfy condition $P(\mathscr{B},\mu)$. Suppose that $A \cap V_0 \neq \emptyset$ and let $x_n \in A \cap V_0$. If x_n is isolated in V_0 in the topology of $\mathbf{R}^N$, there exists an $r_0 \in \mathbf{R}_+$, such that $B(x_n,r_0) \cap V_0 = \{x_n\}$, and hence

$$\mu(B(x_n,r_0)) = \mu(\{x_n\}) = a_n .$$

Suppose that $a_n > 0$. Since A satisfies $P(\mathscr{B},\mu)$, we must have $\varrho_\mu(x_n,r) > 0$ for $r \in]0,r_0[$, but this is only possible if there exists a $B \in \mathscr{B}$, such that $x_n \in B \subseteq B[x_n,r]$. Therefore, if x_n is isolated in V_0 (in the topology of $\mathbf{R}^N$) and $a_n > 0$, then $\varrho_\mu(x_n,r) > 0$ for all $r \in \mathbf{R}_+$ implies that there exists a $B \in \mathscr{B}$, such that $x_n \in B$.

THEOREM 16. *Suppose that the set of accumulation points of* $\{x_n \mid n \in \mathbf{N}\}$ *only consists of isolated points in* $\mathbf{R}^N$, *and let these be included in* $\{x_n \mid n \in \mathbf{N}\}$. *Let*

$$\mu = \sum_{n=1}^{+\infty} a_n \delta_{(x_n)} ,$$

where $a_n \geqq 0$ *and* $\mu(B) < +\infty$ *for* B *bounded. Suppose that for every accumulation point* x_p,

$$\sum_{0<\|x_n-x_p\|<r_p} a_n a_p |\log \|x_n - x_p\|| < +\infty \tag{16}$$

for some $r_p \in]0,1]$. *If* $\mathscr{B}$ *is a class of closed balls in* $\mathbf{R}^N$, *and* A *satisfies* $P(\mathscr{B},\mu)$, *then there exists a* μ*-packing* $\mathscr{B}^* \subseteq \mathscr{B}$ *of* A.

The proof may easily be extended to the case, where one considers the set of accumulation points of the set of accumulation points etc. a finite number of steps. This extension is left to the reader.

PROOF. If $a_p = 0$ there is nothing to prove, as for some $r_p > 0$ the set $\{x_n \mid 0 < \|x_n - x_p\| < r_p\}$ only consists of isolated points. We shall prove that when $a_p > 0$, there exists a $B \in \mathscr{B}$, such that $x_p \in B \subseteq B[x_p, r_p]$.

Without loss of generality we may assume that there exists only one accumulation point and that

$$\{x_n \mid n \in \mathbf{N}\} \subset B(x_p, r_p) = B(x_p, 1) .$$

Also, let $x_p = 0$ and $a_p = 1$.

Suppose that $0 \notin B$ for all $B \in \mathscr{B}$. After a change of index we may assume that $x_0 = 0$ and that $\{x_n\}$ has been arranged, such that $\{\|x_n\|\}$ is decreasing. We shall for a while assume that $\{\|x_n\|\}$ is strictly decreasing and later remove this restriction. Hence, $\|x_{n+1}\| < \|x_n\|$ for all $n \in \mathbf{N}$.

For $r \in [\|x_{n+1}\|, \|x_n\|$ we have

$$\mu(B[0,r]) = 1+\sum_{j=n+1}^{+\infty} a_j \geqq 1 \qquad \text{and} \qquad \varrho_\mu(0,r) \leqq \sum_{j=n+1}^{+\infty} a_j ,$$

since by assumption $x_0 = 0 \notin B$ for all $B \in \mathscr{B}$.

Choose a sequence $r_n \in \mathbf{R}_+$, such that

i) $\{B[x_n, r_n] \mid n \in \mathbf{N}\}$ is a system of disjoint balls;

ii) $\|x_{n+1}\| + r_{n+1} < \|x_n\| - r_n$ for all $n \in \mathbf{N}$;

iii) $\log(\{\|x_n\| + r_n\}/\{\|x_n\| - r_n\}) < 2^{-n}$ for all $n \in \mathbf{N}$.

Then we get the estimate

$$\int_0^{\|x_1\|+r_1} \varrho_\mu^*(0,r)\frac{dr}{r}$$

$$= \sum_{n=1}^{+\infty}\left\{\int_{\|x_{n+1}\|+r_{n+1}}^{\|x_n\|-r_n} \frac{\varrho_\mu(0,r)}{\mu(B[0,r])}\frac{dr}{r}\right\} + \sum_{n=1}^{+\infty}\left\{\int_{\|x_n\|-r_n}^{\|x_n\|+r_n} \frac{\varrho_\mu(0,r)}{\mu(B[0,r])}\frac{dr}{r}\right\}$$

$$\leqq \sum_{n=1}^{+\infty}\int_{\|x_{n+1}\|+r_{n+1}}^{\|x_n\|-r_n} \varrho_\mu(0,\|x_{n+1}\|)\frac{dr}{r} + \sum_{n=1}^{+\infty}\int_{\|x_n\|-r_n}^{\|x_n\|+r_n} \varrho_\mu(0,\|x_n\|)\frac{dr}{r}$$

$$\leqq \sum_{n=1}^{+\infty}\left(\sum_{j=n+1}^{+\infty} a_j\right)\log\left(\frac{\|x_n\|-r_n}{\|x_{n+1}\|+r_{n+1}}\right) + \sum_{n=1}^{+\infty}\left(\sum_{j=n}^{+\infty} a_j\right)\log\left(\frac{\|x_n\|+r_n}{\|x_n\|-r_n}\right)$$

$$\leqq \sum_{n=1}^{+\infty}\left(\sum_{j=n+1}^{+\infty} a_j\right)\log(\|x_n\|/\|x_{n+1}\|) + M\sum_{n=1}^{+\infty} 2^{-n}$$

$$= \sum_{n=1}^{+\infty}\left\{\sum_{j=n+1}^{+\infty} a_j\log\|x_n\| - a_{n+1}\log\|x_{n+1}\| - \sum_{j=n+2}^{+\infty} a_j\log\|x_{n+1}\|\right\} + M$$

$$= \sum_{j=2}^{+\infty} a_j\log\|x_1\| + \sum_{n=1}^{+\infty} a_{n+1}\,|\log\|x_{n+1}\|| + M$$

$$\leqq 0 + \sum_{n=1}^{+\infty} a_n|\log\|x_n\|| + M < +\infty ,$$

contradicting the assumption that A satifies $P(\mathscr{B},\mu)$ since this assumption implies that

$$\int_0^{\|x_1\|+r_1} \varrho_\mu^*(0,r)\frac{dr}{r} = +\infty .$$

Hence, there exists a $B \in \mathscr{B}$, such that $x_0 = 0 \in B$, and the proof is complete in this case.

If there exist points $x_{n+1}, \ldots, x_{n+p}$, such that

$$\|x_n\| > \|x_{n+1}\| = \ldots = \|x_{n+p}\| > \|x_{n+p+1}\| ,$$

condition ii) above is replaced by

iv) choose $r_{n+1} = \ldots = r_{n+p}$, such that

$$\|x_n\| - r_n > \|x_{n+1}\| + r_{n+1} > \|x_{n+1}\| - r_{n+1} > \|x_{n+p+1}\| + r_{n+p+1} .$$

Some of the terms in the estimate above then drop out, and we still get our contradiction.

EXAMPLE 3. We shall by an example show that (16) is more or less necessary for the validity of a packing theorem of the considered type. In fact, we shall construct an atomic measure μ on R, for which (16) is not fulfilled, and we shall define a system of closed intervals $\mathscr{B}$ and a set A, such that $\mu(A) > 0$, A satifies $P(\mathscr{B}, \mu)$, and $A \cap B = \emptyset$ for every $B \in \mathscr{B}$, so A cannot be μ-packed with intervals from $\mathscr{B}$.

Let $x_0 = 0$ and $x_n = 2^{-n}$ and consider the measure

$$\mu = \delta_{(0)} + \sum_{n=1}^{+\infty} \frac{1}{n^2} \delta_{(2^{-n})} .$$

Then $M = 1 + \pi^2/6$. It follows immediately that

$$\sum_{n=1}^{+\infty} a_0 a_n |\log x_n| = \log 2 \cdot \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty ,$$

so (16) is not satisfied for x_0.

Let $I_n = [2^{-2n-4}, 2^{-n}]$. Then a small computation shows that

$$\mu(I_n) \geqq \frac{1}{2n} .$$

Let $\mathscr{B} = \{[\frac{1}{4}, \frac{1}{2}]\} \cup \{I_n \mid n \in \mathrm{N}\}$, and let $A = A_{\mathrm{loc}}(\mathscr{B}) = \{0\}$. Then $\mu(A) = 1$ and $A \cap I_n = A \cap [\frac{1}{4}, \frac{1}{2}] = \emptyset$, so A cannot be μ-packed.

Finally, we prove that $A = \{0\}$ satisfies $P(\mathscr{B}, \mu)$. We note that

$$\varrho_\mu(0, 2^{-n}) = \mu(I_n) \geqq \frac{1}{2n}$$

and

$$1 \leqq \mu([-2^{-n}, 2^{-n}]) \leqq 1 + M ,$$

so

$$\int_0^1 \varrho_\mu^*(0,r)/|\log \varrho_\mu(0,r)|\cdot\frac{dr}{r} = \sum_{n=1}^{+\infty}\int_{2^{-n-1}}^{2^{-n}} \frac{\varrho_\mu(0,r)}{\mu([-r,r])}\cdot\frac{1}{|\log\varrho_\mu(0,r)|}\frac{dr}{r}$$

$$\geqq \sum_{n=1}^{+\infty}\frac{1}{1+M}\cdot\frac{1}{2n}\cdot\frac{1}{\log(2n)}\int_{2^{-n-1}}^{2^{-n}}\frac{dr}{r} = \frac{\log 2}{1+M}\sum_{n=1}^{+\infty}\frac{1}{2n\log(2n)} = +\infty\,,$$

and A satisfies $P(\mathscr{B},\mu)$.

EXAMPLE 4. Let $x_0=0$ and $\{x_n \mid n\in \mathsf{N}\}=]0,1]\cap Q$. Let $\mathscr{B}$ be as in Example 3. Construct

$$\mu = \delta_{(0)} + \sum_{n=1}^{+\infty} a_n\delta_{(x_n)}\,,$$

such that $\mu(I_n)=\sum_{j=n}^{2n+4} 1/j^2$, and such that $a_n>0$ for each n. Then Example 3 shows that $A=\{0\}$ satisfies $P(\mathscr{B},\mu)$ and that $\mathscr{B}$ does not cover A. The set W_1 introduced in section 4 is in this case given by $W_1=[0,1]\cap Q=V_0$, hence $W_k=V_0$ for $k\in\mathsf{N}$. Especially, $\mu_0=\mu_{0,1}^*=\ldots=\mu_{0,k}^*=\ldots$, and (14) becomes trivial. This example shows, why we must have an extra condition on the geometrical structure in definition 13 and definition 14.

6. Applications and limits of the theory.

The applications of Vitali type theorems have already been mentioned in [5] and [4]. They are connected with the characterization of some class of μ-nullsets and the generalization of Lebesgue's differentiation theorem. The class of μ-nullsets is described by the class $\mathfrak{Z}$ of $\mathfrak{V}$-*meagre* sets defined by

$$Z\in\mathfrak{Z} \Leftrightarrow (Z,\tilde{Z})\in\mathfrak{V}\,,$$

where $\mathfrak{V}$ is a Vitali system for which the packing theorem holds. We may choose any class of closed balls $\mathscr{B}$ and look at $A\subseteq A_{\mathrm{loc}}(\mathscr{B})$. If A satisfies $P(\mathscr{B},\mu)$, then [4] and the results in this paper describe some class of μ-nullsets, since Z and the sets in $\tilde{Z}$ are disjoint and we have a packing result.

In the same way, when we have a packing result for a Vitali system $\mathfrak{V}$, we may use the general result in [4], namely Proposition 1, to get a differentiation theorem with respect to μ. The results of this paper have thus extended the class of measures for which it is possible to achieve such results.

By introducing regularity (cf. [2]) we may extend the results to classes $\mathscr{B}$ of closed sets B, where each B satisfies a suitable regularity condition with respect to the balls in R^N. As this follows a known pattern, we shall not repeat the construction here.

Though the class of well-behaved measures and the class of weakly well-behaved measures are fairly extensive, especially when compared with our starting point the Lebesgue measure in $\mathbf{R}^N$, they do not cover all possible measures on $\mathbf{R}^N$, since we had to introduce some geometrical limits. It is not known what happens, when for instance supp μ_1 is a curve without double points for which $\lambda_2(\operatorname{supp}\mu_1)>0$ (say, Osgood's curve), or if $\operatorname{supp}\mu_1$ is a Cantor set. Also, the case where $\mu=\sum_{q\in Q}a_q\delta_{(q)}$ and $\mu(B)<+\infty$ for B bounded is not included in the theory. In all three cases we see that there is "something wrong" with the geometry of $\operatorname{supp}\mu_1$. The assumption that μ_j should be strongly continuous with respect to the induced measure λ'_j on V_j seems to be a minor obstacle, which always can be avoided by considering $\tilde{\mu}=\lambda'_N+\lambda'_{N-1}+\ldots+\lambda'_1+\lambda'_0$ instead. However, we here again assume that the V_j are of class C^1 or at least C^1 except for a nullset. It would be interesting if one could allow the manifolds V_j to satisfy a still weaker condition, say some kind of Hölder continuity.

The author wishes to thank Ole Jørsboe for helpful discussions.

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