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ON AN INEQUALITY OF FRIEDRICHS

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0. Introduction.

Let $\Omega \subset \mathbf{R}^3$ be a smooth bounded domain. An inequality of Friedrichs states that

$$\|u\|_{1,\Omega} \leqq c(\|\operatorname{curl} u\|_{0,\Omega} + \|\operatorname{div} \varepsilon u\|_{0,\Omega} + \|u\|_{0,\Omega}) \tag{0.1}$$

for all vector fields $u \in L^2(\Omega)^3$ satisfying $\operatorname{curl} u \in L^2(\Omega)^3$, $\operatorname{div} \varepsilon u \in L^2(\Omega)$ and $n \wedge u|_\Gamma = 0$, $\Gamma = \partial\Omega$, [2]. Here n denotes the unit outward normal to Γ and ε is a symmetric positive definite matrix. The condition $n \wedge u|_\Gamma = 0$ is understood in an appropriate weak sense. The above result in [2] is a special case of the corresponding inequalities for differential forms in Riemannian manifolds with smooth boundaries.

Later Leis (see [11], [12]) derived this inequality by elementary methods and the proof covers besides all smooth domains also some domains having a piecewise smooth boundary and a special geometric configuration, compare pp. 17–18 in [12]. More precisely, it was required that the solution w of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(\varepsilon \nabla w) + w = f \in \mathscr{D}(\Omega), \\ w|_\Gamma = 0 \end{cases} \tag{0.2}$$

has the strong regularity $w \in C^3(\bar{\Omega})$.

Moreover, it was shown by Saranen in [14] that inequality (0.1) holds for domains having a smooth boundary except for a conical point, where the corresponding inner cone is convex, when $\varepsilon = 1$.

Originally, the inequality of Friedrichs was proved in order to show that the imbedding of the space of the fields u satisfying $\operatorname{curl} u \in L^2(\Omega)^3$, $\operatorname{div} \varepsilon u \in L^2(\Omega)$, $n \wedge u|_\Gamma = 0$ in the space $L^2(\Omega)^3$ is compact. In this direction the inequality has loosed its importance since the compactness is valid for a large class of domains, where the inequality is not true, cf. Remark 3.6.

However, the question of the validity of the Friedrichs inequality for nonsmooth domains is interesting by its own right. It for example provides a regularity result for the solutions of the system

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$$
\begin{cases} \operatorname{curl} u = j \in L^2(\Omega)^3\,, \\ \operatorname{div} \varepsilon u = \varrho \in L^2(\Omega)\,, \\ n \wedge u|_\Gamma = 0 \end{cases} \tag{0.3}
$$

which appears e.g. in the magnetohydrodynamics. The same is true also for other problems — such as Maxwell's problem — concerning vector fields u with the electric boundary condition $n \wedge u|_\Gamma = 0$. It is of central importance when the finite element approximation of solutions for problems of this type is considered, cf. Saranen [15], Neittaanmäki and Saranen [13].

In this article we shall show that the Friedrichs inequality (0.1) is valid for all bounded convex domains.

The well-studied regularity property $v \in H^2(\Omega)$ with the estimate

$$
\|v\|_{2,\Omega} \leqq c\|f\|_{0,\Omega} \tag{0.4}
$$

for the solution $v \in H_0^1(\Omega)$ of the Dirichlet problem

$$
\begin{cases} \operatorname{div}(\varepsilon \nabla v) = f\,, \\ v|_\Gamma = 0 \end{cases} \tag{0.5}
$$

is a necessary condition for the validity of the Friedrichs inequality. Our proof is largely based on the fact that the above $H^2(\Omega)$-regularity is valid for all bounded convex domains.

Essentially due to the simple connectivity of the convex domains our final result takes the refined form

$$
\|u\|_{1,\Omega} \leqq c(\|\operatorname{curl} u\|_{0,\Omega} + \|\operatorname{div} \varepsilon u\|_{0,\Omega}) \tag{0.6}
$$

since (0.3) admits at most one solution in this case.

We achieve inequality (0.6) under the Lipschitz continuity of the matrix ε whereas the strong regularity $\varepsilon_{ij} \in C^2(\bar{\Omega})$ was needed in [11], [12].

The result is valid also in the two-dimensional setting. For completeness this case is also included into the consideration.

1. Notations.

Let $\Omega \subset \mathbf{R}^p$, $p=3$ or 2 be an open set. Further, let $\varepsilon = (\varepsilon_{ij}(x))$, $\varepsilon_{ij}(x) \in \mathbf{R}$ be a continuous positive definite bounded matrix valued function in Ω. More precisely, there exist two constants $m(\varepsilon)$, $M(\varepsilon) > 0$ such that

$$
m(\varepsilon)|\xi|^2 \leqq \langle \xi \,|\, \varepsilon(x)\xi \rangle \leqq M(\varepsilon)|\xi|^2 \tag{1.1}
$$

for all $(x, \xi) \in \Omega \times \mathbf{R}^p$. Here $\langle . | . \rangle$ denotes the Euclidean inner product in $\mathbf{R}^p$ with the norm $|\cdot|$. For the uniqueness, the constants in (1.1) are fixed by

$$m(\varepsilon) = \inf\{\langle\xi\,|\,\varepsilon(x)\xi\rangle \;\big|\; |\xi|=1, \quad x \in \Omega\}\ ,$$
$$M(\varepsilon) = \sup\{\langle\xi\,|\,\varepsilon(x)\xi\rangle \;\big|\; |\xi|=1, \quad x \in \Omega\}\ .$$

Later on, we shall assume that ε is symmetric as well as Lipschitz continuous.

For the space of the square integrable functions,—fields and the Sobolev spaces with the corresponding inner products and norms we use the familiar notations $L^2(\Omega)$, $L^2(\Omega)^p$, $H^k(\Omega)$, $H_0^1(\Omega)$ etc., [1]. In particular, we abbreviate $(\cdot|\cdot)_k=(\cdot|\cdot)_{k,\Omega}$ and $\|\cdot\|_k=\|\cdot\|_{k,\Omega}$.

Besides the above spaces we introduce

$$H(\operatorname{div}\varepsilon) = \{u \in L^2(\Omega)^p \;\big|\; \operatorname{div}\varepsilon u \in L^2(\Omega)\} \tag{1.2}$$

and for $p=3$

$$H(\operatorname{curl}) = \{u \in L^2(\Omega)^3 \;\big|\; \operatorname{curl} u \in L^2(\Omega)^3\}\ , \tag{1.3}$$

$$H_0(\operatorname{curl}) = \{u \in H(\operatorname{curl}) \;\big|\; (\operatorname{curl} u\,|\,\varphi)_0=(u\,|\operatorname{curl}\varphi)_0,\ \varphi \in H(\operatorname{curl})\}\ . \tag{1.4}$$

The operations $\operatorname{curl} u$ and $\operatorname{div}\varepsilon u$ above are defined in the usual weak sense. In the two-dimensional case the weak curl generalizes the notation $\operatorname{curl} u=\partial_1 u_2 -\partial_2 u_1$ for the differentiable fields. It is given as follows. For $\varphi \in H^1(\Omega)$ we set $\operatorname{curl}\varphi=(\partial_2\varphi, -\partial_1\varphi)$ and introduce

$$H(\operatorname{curl}) = \{u \in L^2(\Omega)^2 \;\big|\; \exists\, g \in L^2(\Omega)\colon (u\,|\operatorname{curl}\varphi)_0=(g\,|\,\varphi)_0,\ \varphi \in \mathscr{D}(\Omega)\} \tag{1.5}$$

with $\operatorname{curl} u := g$. Here $\mathscr{D}(\Omega)$ denotes the space of all infinitely differentiable functions with a compact support in Ω. Analogous to (1.4) we abbreviate for $p=2$

$$H_0(\operatorname{curl}) = \{u \in H(\operatorname{curl}) \;\big|\; (\operatorname{curl} u\,|\,\varphi)_0 = (u\,|\operatorname{curl}\varphi)_0,\ \varphi \in H^1(\Omega)\}\ . \tag{1.6}$$

The statement $u \in H_0(\operatorname{curl})$ describes for both dimensions the boundary condition $n \wedge u|_\Gamma=0$, where "$\wedge$" denotes the exterior product such that $n \wedge u =n_1u_2-n_2u_1$ for $p=2$.

The space $H_0(\operatorname{curl})$ has also another characterization. Define namely in $H(\operatorname{curl})$ the norm $\|\cdot\|_c$ by

$$\|u\|_c^2 = \|\operatorname{curl} u\|_0^2+\|u\|_0^2\ . \tag{1.7}$$

Then it holds that $H_0(\operatorname{curl})=\overline{\mathscr{D}(\Omega)^p}^{\|\cdot\|_c}$, Weber [16, Lemma 3.3].

We abbreviate $X(\varepsilon)=H_0(\operatorname{curl})\cap H(\operatorname{div}\varepsilon)$. The space $X(\varepsilon)$ is a Hilbert space with respect to the inner product

$$(u\,|\,v)_{c+\operatorname{div}\varepsilon} := (\operatorname{curl} u\,|\operatorname{curl} v)_0+(\operatorname{div}\varepsilon u\,|\operatorname{div}\varepsilon v)_0+(\varepsilon u\,|\,v)_0\ .$$

Let $\|\cdot\|_{c+\operatorname{div}\varepsilon}$ be the corresponding norm.

As earlier, the notation c refers to a generic constant independent on the functions or fields under discussion. The notations $X(\varepsilon, \Omega)$, $H_0(\mathrm{curl}, \Omega)$ etc. are used when necessary. Since the entries ε_{ij} are real valued we may assume that all the fields have real components and the functions are real valued. The partial derivatives are denoted by $\partial_i = \partial/\partial x_i$.

2. Preliminary lemmas.

The Friedrichs inequality which we are going to prove for a class of domains states that the space $X(\varepsilon)$ is continuously imbedded in $H^1(\Omega)^p$, that is $X(\varepsilon) \subset H^1(\Omega)^p$ with

$$\|u\|_1 \leqq c\|u\|_{c+\mathrm{div}\,\varepsilon}, \qquad u \in X(\varepsilon)\,. \tag{2.1}$$

We first point out that this imbedding property implies a well-studied regularity result for the second order problem (0.5). Define namely the corresponding closed operator $\mathrm{div}_0\,(\varepsilon\nabla)$ in $L^2(\Omega)$ with the domain

$$D(\varepsilon) = \{w \in H_0^1(\Omega) \mid \exists\, f \in L^2(\Omega)\colon (\varepsilon\nabla w \mid \nabla\varphi)_0 = -(f \mid \varphi)_0,\ \varphi \in H_0^1(\Omega)\}$$

and with $\mathrm{div}_0\,(\varepsilon\nabla w) = f$. Here the subscript in $\mathrm{div}_0\,(\varepsilon\nabla)$ refers to the Dirichlet boundary condition. Since ε is positive definite the inverse $(\mathrm{div}_0\,(\varepsilon\nabla))^{-1}\colon L^2(\Omega) \to D(\varepsilon)$ exists by the inequality of Poincaré and by the theorem of Lax and Milgram. If imbedding (2.1) is valid, then holds $D(\varepsilon) \subset H^2(\Omega)$ with

$$\|w\|_2 \leqq c\|f\|_0 \tag{2.2}$$

for $w = (\mathrm{div}_0\,(\varepsilon\nabla))^{-1} f$. This follows from the fact that $u \in H_0(\mathrm{curl}) \cap H(\mathrm{div}\,\varepsilon)$ and $\mathrm{curl}\, u = 0$, $\mathrm{div}\,\varepsilon u = f$, if $u = \nabla w$.

Conversely, the subsequent characterization of the space $X(\varepsilon)$ enables one to utilize the relation $D(\varepsilon) \subset H^2(\Omega)$ and inequality (2.2) (when they are true) in the proof of the imbedding result (2.1). Since the inverse $(-\mathrm{div}_0\,(\varepsilon\nabla) + 1)^{-1}\colon L^2(\Omega) \to D(\varepsilon)$ also exists (Ω bounded or not) we can define

$$V(\varepsilon) = \{\varphi = \psi + \nabla((-\mathrm{div}_0\,(\varepsilon\nabla) + 1)^{-1}(\mathrm{div}\,\varepsilon\psi) \mid \psi \in \mathscr{D}(\Omega)^p\}\,.$$

It holds

LEMMA 2.1. *Let $\Omega \subset \mathbf{R}^p$, $p = 3$ or 2 be an open set. Then the space $V(\varepsilon)$ is dense in $X(\varepsilon)$.*

PROOF. Take $\varphi = \psi + \nabla w$ with $-\mathrm{div}_0\,(\varepsilon\nabla w) + w = \mathrm{div}\,\varepsilon\psi$, $\psi \in \mathscr{D}(\Omega)^p$. As $\nabla H_0^1(\Omega) \subset H_0(\mathrm{curl})$, and as $\mathrm{div}\,\varepsilon\nabla w \in L^2(\Omega)$, we have $V(\varepsilon) \subset X(\varepsilon)$. Since $X(\varepsilon)$ is a Hilbert space, it suffices to establish that $V(\varepsilon)^\perp \cap X(\varepsilon) = \{0\}$, where $V(\varepsilon)^\perp$

denotes the orthogonal complement of $V(\varepsilon)$ with respect to the inner product $(\cdot|\cdot)_{c+\operatorname{div}\varepsilon}$. Let $u \in V(\varepsilon)^{\perp} \cap X(\varepsilon)$ be given. Then we obtain

$$
\begin{aligned}
(2.3) \qquad 0 &= (u \mid \psi + \nabla w)_{c+\operatorname{div}\varepsilon} \\
&= (\operatorname{curl} u \mid \operatorname{curl} \psi)_0 + (\operatorname{div} \varepsilon u \mid \operatorname{div} \varepsilon\psi)_0 \\
&\quad + (\operatorname{div} \varepsilon u \mid \operatorname{div} \varepsilon \nabla w)_0 + (\varepsilon u \mid \psi)_0 + (\varepsilon u \mid \nabla w)_0 \\
&= (\operatorname{curl} u \mid \operatorname{curl} \psi)_0 + (\varepsilon u \mid \psi)_0
\end{aligned}
$$

for all $\psi \in \mathscr{D}(\Omega)^p$. By $H_0(\operatorname{curl}) = \overline{\mathscr{D}(\Omega)^p}^{\|\cdot\|_c}$ relation (2.3) implies $u = 0$.

As a consequence of Lemma 2.1 we note that when the regularity $D(\varepsilon) \subset H^2(\Omega)$ is valid, then holds $V(\varepsilon) \subset H^1(\Omega)^p$. Accordingly, for the imbedding $X(\varepsilon) \subset H^1(\Omega)^p$ with (2.1) it is enough to prove that the estimate

$$
(2.4) \qquad \|u\|_1 \leqq c\|u\|_{c+\operatorname{div}\varepsilon}
$$

holds for every $u \in H_0(\operatorname{curl}) \cap H^1(\Omega)^p$ or for all $u \in V(\varepsilon)$.

Let $|\cdot|_{2,\Omega}$ be the usual semi-norm such that

$$
|w|_{2,\Omega}^2 = \sum_{i,j=1}^{p} \|\partial_i \partial_j w\|_{0,\Omega}^2 .
$$

By the above remarks the following result of Kadlec ([9, Teorema 10 and Teorema 14]) is relevant.

THEOREM 2.2. *Let $\Omega \subset \mathbf{R}^p$, $p \in \mathbf{N}$ be a bounded convex domain. If ε is a Lipschitz continuous positive definite matrix in Ω, then holds $D(\varepsilon) \subset H^2(\Omega)$ with*

$$
(2.5) \qquad \|w\|_{2,\Omega} \leqq c\|f\|_{0,\Omega}
$$

for all $f \in L^2(\Omega)$, $w = (\operatorname{div}_0(\varepsilon\nabla))^{-1} f$. Particularly, if ε is constant, we have

$$
(2.6) \qquad |w|_{2,\Omega} \leqq m(\varepsilon)^{-1}\|f\|_{0,\Omega} .
$$

3. Friedrichs inequality.

In this section we shall prove the imbedding $X(\varepsilon) \subset H^1(\Omega)^p$ with (2.1) for all bounded convex domains if the matrix ε is symmetric positive definite and Lipschitz continuous.

We fix some conventions for the vector fields. If $u = (u_i) \in H^1(\Omega)^p$, then $\nabla u := (\nabla u_i) \in (L^2(\Omega)^p)^p$. Especially holds

$$\|u\|_{1,\Omega}^2 = \|\nabla u\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2$$

for $u \in H^1(\Omega)^p$. Furthermore, if $w \in H^2(\Omega)$, then

$$|w|_{2,\Omega}^2 = \|\nabla^2 w\|_{0,\Omega}^2 ,$$

where $\nabla^2 w := \nabla(\nabla w)$. For the field $u \in H^2(\Omega)^p$, Δu is the vectorial Laplacian which satisfies

$$\Delta u = -\operatorname{curl}(\operatorname{curl} u) + \nabla \operatorname{div} u .$$

Let us begin with the case $\varepsilon = 1$.

THEOREM 3.1. *Let $\Omega \subset \mathbf{R}^p$, $p=3$ or 2 be a bounded convex domain. Then holds $X(1) \subset H^1(\Omega)^p$, where the imbedding is continuous such that*

$$\|\nabla u\|_0^2 \leqq \|\operatorname{curl} u\|_0^2 + \|\operatorname{div} u\|_0^2 \tag{3.1}$$

for all $u \in X(1)$.

PROOF. According to Lemma 2.1 and Theorem 2.2 it suffices to consider the fields $u \in V(1)$. We take $u = \psi + \nabla w$, where $\psi \in \mathscr{D}(\Omega)^p$, and where $-\Delta_0 w + w = \operatorname{div} \psi$, $w \in H_0^1(\Omega)$. Equivalently,

$$(\nabla w \mid \nabla \varphi)_0 + (w \mid \varphi)_0 = (\operatorname{div} \psi \mid \varphi)_0 \tag{3.2}$$

for every $\varphi \in H_0^1(\Omega)$. We use the identity

$$\|\nabla u\|_0^2 = \|\nabla(\psi + \nabla w)\|_0^2 = \|\nabla \psi\|_0^2 + 2(\nabla \psi \mid \nabla^2 w)_0 + \|\nabla^2 w\|_0^2 , \tag{3.3}$$

where further by $\psi \in \mathscr{D}(\Omega)^p$ holds

$$\begin{aligned} (\nabla \psi \mid \nabla^2 w)_0 &= -(\Delta \psi \mid \nabla w)_0 = (\operatorname{curl}(\operatorname{curl} \psi) \mid \nabla w)_0 - (\nabla \operatorname{div} \psi \mid \nabla w)_0 \\ &= -(\nabla \operatorname{div} \psi \mid \nabla w)_0 . \end{aligned} \tag{3.4}$$

Choosing $\varphi = \operatorname{div} \psi$ in (3.2), we find by (3.4)

$$(\nabla \psi \mid \nabla^2 w)_0 = -\|\operatorname{div} \psi\|_0^2 + (w \mid \operatorname{div} \psi)_0 . \tag{3.5}$$

The essential trick is to use the sharp estimate (2.6) for the last term in (3.3). This yields

$$\begin{aligned} \|\nabla^2 w\|_0^2 &\leqq \|\Delta w\|_0^2 = \|w - \operatorname{div} \psi\|_0^2 \\ &= \|w\|_0^2 - 2(w \mid \operatorname{div} \psi)_0 + \|\operatorname{div} \psi\|_0^2 . \end{aligned} \tag{3.6}$$

Since, on the other hand,

$$\|\nabla \psi\|_0^2 = \|\operatorname{curl} \psi\|_0^2 + \|\operatorname{div} \psi\|_0^2 = \|\operatorname{curl} u\|_0^2 + \|\operatorname{div} \psi\|_0^2 ,$$

$$\|\operatorname{div} u\|_0^2 = \|\operatorname{div} \psi + \Delta w\|_0^2 = \|w\|_0^2 ,$$

formulae (3.3), (3.5), and (3.6) imply estimate (3.1).

REMARK 3.2. Inequality (3.1) instead of the equality is essentially due to the "curvature" of the boundary. Namely, if Ω is a smooth bounded domain, then we have the imbedding $X(1) \subset H^1(\Omega)^p$ with

$$\|\nabla u\|_0^2 = \|\operatorname{curl} u\|_0^2 + \|\operatorname{div} u\|_0^2 - \int_\Gamma \operatorname{div} n(x)|u|^2 \, do . \tag{3.7}$$

Here $\operatorname{div} n(x)|_\Gamma = -2H(x)$, where $H(x)$ is the mean curvature. If Ω is convex, then especially holds $\operatorname{div} n(x)|_\Gamma \geqq 0$, cf. Saranen [14, Lemma 1.3 and Satz 1.4], the proof.

REMARK 3.3. If Ω is a bounded convex polyhedron, then we have

$$\|\nabla u\|_0^2 = \|\operatorname{curl} u\|_0^2 + \|\operatorname{div} u\|_0^2, \qquad u \in X(1) . \tag{3.8}$$

This can be seen from the proof of Theorem 3.1, if one applies the equality

$$\|\nabla^2 w\|_0 = \|\Delta w\|_0 \tag{3.9}$$

(cf. Grisvard [6, p. 363]) instead of inequality (2.6) used above.

Next we consider the case where ε is constant. In the following β' denotes the adjoint of the matrix β; $\beta\beta' = \det \beta \cdot \delta$, where $\delta = 1$ is the identity matrix.

THEOREM 3.4. *Let $\Omega \subset \mathbf{R}^p$, $p=3$ or 2 be a bounded convex domain and let ε be a symmetric positive definite constant matrix. Then holds $X(\varepsilon) \subset H^1(\Omega)^p$, where the imbedding is continuous such that*

$$m(\varepsilon)^2 \|\nabla u\|_0^2 \leqq M(\varepsilon')\|\operatorname{curl} u\|_0^2 + \|\operatorname{div} \varepsilon u\|_0^2 \tag{3.10}$$

for all $u \in X(\varepsilon)$.

PROOF. This assertion follows from Theorem 3.1 by using a suitable transformation. Since some notations are different for $p=3$ and $p=2$, we write the formulae only for $p=3$.

Because ε is symmetric and positive definite then there exists a symmetric positive definite matrix γ such that $\varepsilon = \gamma^2$, $\det \gamma > 0$. Denote $\alpha = \gamma^{-1}$ and consider the mapping $x \to y = \alpha x \colon \Omega \to \alpha(\Omega)$. Again, it suffices by Lemma 2.1 and Theorem 2.2 to assume that $u \in H_0(\operatorname{curl}, \Omega) \cap H^1(\Omega)^3$.

We define in the domain $\alpha(\Omega)$ the field v by

$$v(y) = \gamma u(\gamma y), \qquad y \in \alpha(\Omega) . \tag{3.11}$$

Then we have $v \in H^1(\alpha(\Omega))^3$ and (summation convention)

$$\frac{\partial}{\partial y_\nu} v_i(y) = \gamma_{i\mu}\frac{\partial}{\partial y_\nu} u_\mu(\gamma y) = \gamma_{i\mu}\gamma_{\sigma\nu}\frac{\partial}{\partial x_\sigma} u_\mu(x)\,. \tag{3.12}$$

Accordingly,

$$\operatorname{div} v = \frac{\partial}{\partial y_i} v_i(y) = \gamma_{i\mu}\gamma_{\sigma i}\frac{\partial}{\partial x_\sigma} u_\mu(x) = \varepsilon_{\sigma\mu}\frac{\partial}{\partial x_\sigma} u_\mu = \operatorname{div}(\varepsilon u)\,. \tag{3.13}$$

Furthermore, if $i, j, k \in \{1,2,3\}$ are the indices in the usual cyclic order $i<j<k$, then

$$(\operatorname{curl} v)_i = \frac{\partial}{\partial y_j} v_k - \frac{\partial}{\partial y_k} v_j = (\gamma_{k\mu}\gamma_{\sigma j} - \gamma_{j\mu}\gamma_{\sigma k})\partial_\sigma u_\mu\,. \tag{3.14}$$

One obtains from (3.14) by the symmetry of γ after a reordering of the terms the formula

$$(\operatorname{curl} v)_i = (\gamma' \operatorname{curl} u)_i\,.$$

Therefore holds

$$\operatorname{curl} v = \gamma' \operatorname{curl} u\,. \tag{3.15}$$

The mapping $u \to v$ preserves the boundary condition $n \wedge u|_\Gamma = 0$. If namely $\psi(y) = \gamma\varphi(\gamma y)$, then one concludes that $\psi \in H(\operatorname{curl}, \alpha(\Omega))$ iff $\varphi \in H(\operatorname{curl}, \Omega)$ and then $\operatorname{curl}\psi = \gamma' \operatorname{curl}\varphi$. For all $\psi \in H(\operatorname{curl}, \alpha(\Omega))$ thus holds

$$\begin{aligned}(v \mid \operatorname{curl}\psi)_{0,\alpha(\Omega)} &= \det\alpha(v(\alpha x) \mid \gamma' \operatorname{curl}\varphi)_{0,\Omega} \\ &= (\gamma^{-1} v(\alpha x) \mid \operatorname{curl}\varphi)_{0,\Omega} = (u \mid \operatorname{curl}\varphi)_{0,\Omega} \\ &= (\operatorname{curl} u \mid \varphi)_{0,\Omega} = (\operatorname{curl} v \mid \psi)_{0,\alpha(\Omega)}\,,\end{aligned} \tag{3.16}$$

since $u \in H_0(\operatorname{curl}, \Omega)$. By (3.16) holds $v \in H_0(\operatorname{curl}, \alpha(\Omega))$. As the domain $\alpha(\Omega)$ is bounded and convex and as $v \in H_0(\operatorname{curl}, \alpha(\Omega)) \cap H^1(\alpha(\Omega))^p$, we have by Theorem 3.1

$$\|\nabla v\|^2_{0,\alpha(\Omega)} \leqq \|\operatorname{curl} v\|^2_{0,\alpha(\Omega)} + \|\operatorname{div} v\|^2_{0,\alpha(\Omega)}\,. \tag{3.17}$$

Writing (3.17) by means of the field u we obtain

$$(\varepsilon_{\sigma l}\partial_\sigma(\gamma u)_i \mid \partial_l(\gamma u)_i)_0 \leqq \|\gamma' \operatorname{curl} u\|^2_0 + \|\operatorname{div} \varepsilon u\|^2_0\,. \tag{3.18}$$

Observing the estimate

$$\|\gamma' \operatorname{curl} u\|^2_0 = (\gamma'\gamma' \operatorname{curl} u \mid \operatorname{curl} u)_0 \leqq M(\varepsilon')\|\operatorname{curl} u\|^2_0 \tag{3.19}$$

and

$$(3.20) \qquad (\varepsilon_{\sigma l}\partial_\sigma(\gamma u)_i \,|\, \partial_l(\gamma u)_i)_0 \geqq m(\varepsilon) \sum_{i=1}^{p} \|\nabla(\gamma u)_i\|_0^2$$
$$= m(\varepsilon)(\varepsilon_{\lambda\mu}\partial_\sigma u_\lambda \,|\, \partial_\sigma u_\mu)_0$$
$$\geqq m(\varepsilon)^2 \sum_{\sigma,\mu}^{p} \|\partial_\sigma u_\mu\|_0^2$$
$$= m(\varepsilon)^2 \|\nabla u\|_0^2$$

we find the desired inequality by (3.18)–(3.20).

As the main result of this section we achieve the Friedrichs inequality (2.1) for Lipschitz continuous matrices ε.

A function $\varphi\colon \Omega \to \mathbf{R}$ is called Lipschitz continuous if the norm

$$[\varphi]_{1,\Omega} = \sup\left\{\frac{|\varphi(x)-\varphi(y)|}{|x-y|} \;\middle|\; x,y \in \Omega,\ x \neq y\right\}$$

is finite. Let $C^{0,1}(\Omega)$ be the corresponding function space. For $\varphi \in C^{0,1}(\Omega)$ holds

$$|\varphi(x)-\varphi(y)| \leqq [\varphi]_{1,\Omega}|x-y|, \qquad x,y \in \Omega\,.$$

THEOREM 3.5. *Let $\Omega \subset \mathbf{R}^p$, $p=3$ or 2 be a bounded convex domain and let ε be a symmetric positive definite matrix function such that $\varepsilon_{ij} \in C^{0,1}(\Omega)$. Then holds $X(\varepsilon) \subset H^1(\Omega)^p$, where the imbedding is continuous, that is*

$$(3.21) \qquad \|u\|_{1,\Omega} \leqq c(\|\operatorname{curl} u\|_{0,\Omega} + \|\operatorname{div} \varepsilon u\|_{0,\Omega} + \|u\|_{0,\Omega})$$

for all $u \in X(\varepsilon)$.

PROOF. Again, it suffices to consider the fields $u \in H_0(\operatorname{curl}) \cap H^1(\Omega)^p$. Denote

$$B(y,r) = \{x \in \mathbf{R}^p \mid |x-y| < r\}$$

and let $\mathbf{Z}$ be the set of all integers. The family $\{B(\varrho q, \varrho) \mid q \in \mathbf{Z}^p\}$ covers all of the space $\mathbf{R}^p$ for every $\varrho > 0$. We abbreviate $U = U(y,r) = B(y,r) \cap \Omega$, if the intersection is nonvoid. If η is any function belonging to $\mathscr{D}(U)$, then holds $\eta\varphi_k \in H_0(\operatorname{curl}, U)$, because $\eta\varphi_k \in \mathscr{D}(U)^p$ with

$$\|\operatorname{curl}(\eta\varphi_k - \eta u)\|_{0,U} + \|\eta\varphi_k - \eta u\|_{0,U} \leqq c\|\varphi_k - u\|_{c,\Omega} \to 0\,,$$

when $\varphi_k \to u$ in $H_0(\operatorname{curl}, \Omega)$, $\varphi_k \in \mathscr{D}(\Omega)^p$.

We fix for every ball $B(\varrho q, 3\varrho)$ a test function $\xi \in \mathscr{D}(B(\varrho q, 3\varrho))$ (depending on ϱ and q) such that $0 \leqq \xi(x) \leqq 1$ and that

$$\xi(x) = \begin{cases} 1 & \text{if } |x-\varrho q| \leqq \varrho\,, \\ 0 & \text{if } |x-\varrho q| \geqq 2\varrho\,. \end{cases}$$

There exist the points $x_{\varrho,q} \in U(\varrho q, \varrho)$ and it holds

$$m(\varepsilon) \leqq m(\varepsilon(x_{\varrho,q})), \qquad M(\varepsilon'(x_{\varrho,q})) \leqq M(\varepsilon')\,. \tag{3.22}$$

Since $\xi u \in H_0(\text{curl}, U(\varrho q, 3\varrho)) \cap H^1(U(\varrho q, 3\varrho))^p$, and since the domain $U(\varrho q, 3\varrho)$ is convex, we find by (3.22) and by Theorem 3.4

$$\begin{aligned} m(\varepsilon)^2 \|\nabla u\|_{0,\Omega}^2 &\leqq m(\varepsilon)^2 \textstyle\sum' \|\nabla u\|_{0,U(\varrho q,\varrho)}^2 \\ &\leqq \textstyle\sum' m(\varepsilon(x_{\varrho,q}))^2 \|\nabla(\xi u)\|_{0,U(\varrho q,3\varrho)}^2 \\ &\leqq M(\varepsilon') \textstyle\sum' (\|\text{curl}\,\xi u\|_{0,U(\varrho q,3\varrho)}^2 \\ &\qquad + \|\text{div}\,\xi\varepsilon(x_{\varrho,q})u\|_{0,U(\varrho q,3\varrho)}^2)\,, \end{aligned} \tag{3.23}$$

where $\sum'$ means the summation over all $q \in \mathbf{Z}^p$ satisfying $B(\varrho q, \varrho) \cap \Omega \neq \varnothing$. Furthermore holds $(U' = U(\varrho q, 3\varrho))$

$$\begin{aligned} \|\text{curl}\,\xi u\|_{0,U'}^2 &= \|\xi\,\text{curl}\,u + (\nabla\xi) \wedge u\|_{0,U'}^2 \\ &\leqq 2(\|\text{curl}\,u\|_{0,U'}^2 + c\|u\|_{0,U'}^2) \end{aligned} \tag{3.24}$$

and similarly

$$\|\text{div}\,\xi\varepsilon(x_{\varrho,q})u\|_{0,U'}^2 \leqq 2(\|\text{div}\,\varepsilon(x_{\varrho,q})u\|_{0,U'}^2 + c\|u\|_{0,U'}^2)\,. \tag{3.25}$$

The last term in (3.23) is the crucial one. Denoting by c_0 a generic constant independent on q and the radius ϱ we obtain by (3.23)–(3.25)

$$\begin{aligned} \|\nabla u\|_{0,\Omega}^2 &\leqq c(\|\text{curl}\,u\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2) \\ &\qquad + c_0 \textstyle\sum' \|\text{div}\,\varepsilon(x_{\varrho,q})u\|_{0,U(\varrho q,3\varrho)}^2\,. \end{aligned} \tag{3.26}$$

We use next the fact (cf. Gilbarg and Trudinger [3, pp. 142–144]) that if $\varphi \in C^{0,1}(\Omega)$, then the weak derivatives $\partial_k\varphi$ exist and coincide almost everywhere with the partial derivatives defined for φ as an absolute continuous function. Accordingly

$$|\partial_k\varphi(x)| \leqq [\varphi]_{1,\Omega}$$

for almost all $x \in \Omega$. If in addition $w \in H^1(\Omega)$, then the weak derivatives $\partial_k(\varphi w)$ exist and satisfy $\partial_k(\varphi w) = (\partial_k\varphi)w + \varphi\partial_k w$.

Applying these remarks to the functions $(\varepsilon_{ij}(x) - \varepsilon_{ij}(x_{\varrho,q}))u_j$ and noting that $|x - x_{\varrho,q}| \leqq 6\varrho$, $x \in U'$ we obtain

$$\|\text{div}\,\varepsilon(x_{\varrho,q})u\|_{0,U'} \leqq \|\text{div}\,\varepsilon u\|_{0,U'} + \|\text{div}\,(\varepsilon(x) - \varepsilon(x_{\varrho,q}))u\|_{0,U'} \tag{3.27}$$

$$\leqq \|\operatorname{div} \varepsilon u\|_{0,U'} + \|(\partial_i \varepsilon_{ij}) u_j\|_{0,U'} + \|(\varepsilon_{ij}(x) - \varepsilon_{ij}(x_{\varrho,q}))\partial_i u_j\|_{0,U'}$$

$$\leqq \|\operatorname{div} \varepsilon u\|_{0,U'} + c\|u\|_{0,U'} + c_0 \varrho \|\nabla u\|_{0,U'} .$$

By (3.26) and (3.27) follows

$$\|\nabla u\|_{0,\Omega}^2 \leqq c(\|\operatorname{curl} u\|_{0,\Omega}^2 + \|\operatorname{div} \varepsilon u\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2) + c_0 \varrho^2 \sum{}' \|\nabla u\|_{0,U(\varrho q, 3\varrho)}^2 . \tag{3.28}$$

Now we use the characteristic property of the chosen family of the balls. If namely

$$k(p) := \operatorname{kard}\{q \in \mathbf{Z}^p \mid |q| \leqq 6\} ,$$

then for every q holds

$$\operatorname{kard}\{q' \in \mathbf{Z}^p \mid U(\varrho q, 3\varrho) \cap U(\varrho q', 3\varrho) \neq \emptyset\} \leqq k(p) \tag{3.29}$$

for all $\varrho > 0$. Inequality (3.28) implies by (3.29)

$$\|\nabla u\|_{0,\Omega}^2 \leqq c(\|\operatorname{curl} u\|_{0,\Omega}^2 + \|\operatorname{div} \varepsilon u\|_{0,\Omega}^2 + \|u\|_{0,\Omega}^2) + c_0 k(p) \varrho^2 \|\nabla u\|_{0,\Omega}^2 ,$$

which yields the desired estimate if ϱ is small enough.

REMARK 3.6. If the domain Ω is nonconvex, then the Friedrichs inequality need not be valid. It is namely well-known that the necessary condition described by means of the second order elliptic problem in section 2 does not hold for all bounded open sets. For domains with conical boundary points we refer to Kondrat'ev [10] and for polyhedral domains to Grisvard [4]–[7] as well as to Hanna and Smith [8]. On the other hand, the compactness of the imbedding $X(\varepsilon) \subset L^2(\Omega)^p$ is true for these types of domains, Weber [16] and Weck [17]. Note also that for the compactness no regularity for the matrix ε is needed; it suffices that ε is bounded and measurable [16].

4. The refinement.

We turn to the refinement (0.6) announced in the introduction. In the following result it suffices that the matrix ε is bounded positive definite and measurable.

THEOREM 4.1. *Let $\Omega \subset \mathbf{R}^p$, $p = 3$ or 2 be a bounded convex domain. Then the problem $u \in X(\varepsilon)$*

$$\begin{cases} \operatorname{curl} u = 0 , \\ \operatorname{div} \varepsilon u = 0 \end{cases} \tag{4.1}$$

has only the trivial solution $u = 0$.

PROOF. Let v be the extension of u in $\mathbf{R}^p$ defined by $v(x)=0$ $x \notin \Omega$. Since $u \in H_0(\operatorname{curl}, \Omega)$ it holds for every $\varphi \in \mathscr{D}(\mathbf{R}^p)^p$ by (1.4)

$$(v \mid \operatorname{curl} \varphi)_{0,\mathbf{R}^p} = (u \mid \operatorname{curl} \varphi)_{0,\Omega} = (\operatorname{curl} u \mid \varphi)_{0,\Omega} = 0 .$$

Accordingly we have $v \in L^2(\mathbf{R}^p)^p$ with $\operatorname{curl} v = 0$.

Let us first consider the case $p=3$. According to Weber [16, Lemma 3.6], there exists a function $w \in L^2(\mathbf{R}^3)$ such that $\nabla w = v$ and $w(x)=0$, $x \in \mathbf{R}^3 \setminus \Omega$. Since Ω has the segment property it holds by [16, Lemma 3.2] that $w|_\Omega \in H_0^1(\Omega)$. Since $w|_\Omega$ is a solution of the Dirichlet problem $\operatorname{div}(\varepsilon\nabla(w|_\Omega))=0$, it follows that $w=0$ and consequently $u=0$.

In the case $p=2$ the argument of [16, Lemma 3.6] can be modified as follows. From $v \in L^2(\mathbf{R}^2)^2$, $\operatorname{curl} v=0$ one obtains for the Fourier transform $\hat{v} = (\hat{v}_1, \hat{v}_2)$ that $x_1\hat{v}_2 - x_2\hat{v}_1 = 0$. Accordingly $\hat{v} = (\hat{v}\cdot x)|x|^{-1}x$.

Since $\hat{v}$ is bounded and $\hat{v} \in L^2(\mathbf{R}^2)^2$, we can define the function w as the inverse Fourier transform of the tempered distribution $-i(\hat{v}\cdot x)|x|^{-2}$. By using $\hat{v}_j = (\hat{v}\cdot x)|x|^{-1}x_j$ one verifies that $\nabla w = v$, where $w \in L^2_{\mathrm{loc}}(\mathbf{R}^2)$. Since $\nabla w = 0$ in $\mathbf{R}^2 \setminus \Omega$, there exists a constant c such that $w' = w - c$ satisfies $\nabla w' = v$ with $w'(x) = 0$, $x \in \mathbf{R}^2 \setminus \Omega$, $w' \in L^2(\mathbf{R}^2)$. The rest follows as above.

Finally, we obtain the refinement of Theorem 3.5

THEOREM 4.2. *Let $\Omega \subset \mathbf{R}^p$, $p=3$ or 2 be a bounded convex domain and let ε be a symmetric positive definite Lipschitz continuous matrix function in Ω. Then holds $X(\varepsilon) \subset H^1(\Omega)^p$, where the imbedding is continuous such that*

$$\|u\|_{1,\Omega} \leqq c(\|\operatorname{curl} u\|_{0,\Omega} + \|\operatorname{div} \varepsilon u\|_{0,\Omega}) \tag{4.2}$$

for all $u \in X(\varepsilon)$.

PROOF. Assume that (4.2) is not true. Then there exists a sequence $u_k \in X(\varepsilon)$ with the properties

$$\|u_k\|_1 \equiv 1 , \tag{4.3}$$

$$\|\operatorname{curl} u_k\|_0 + \|\operatorname{div} \varepsilon u_k\|_0 \to 0,\ k \to \infty . \tag{4.4}$$

The imbedding $H^1(\Omega)^p \subset L^2(\Omega)^p$ being compact we can by (4.3) choose a subsequence $u'_\nu = u_{k_\nu}$ such that

$$\|u'_\nu - u\|_0 \to 0, \quad \nu \to \infty , \tag{4.5}$$

for a field $u \in L^2(\Omega)^p$.

By (4.4) and (4.5) follows $u \in X(\varepsilon)$, $\operatorname{curl} u = 0$, $\operatorname{div} \varepsilon u = 0$. Hence $u=0$ by Theorem 4.1.

On the other hand, inequality (3.21) with (4.4), (4.5) yields that u'_ν is a Cauchy sequence in $H^1(\Omega)^p$. Necessarily $u'_\nu \to u=0$ also in $H^1(\Omega)^p$, which contradicts (4.3).

BIBLIOGRAPHY

1. S. Agmon, *Lectures on elliptic boundary value problems*, (Van Nostrand Mathematical Studies 2), D. Van Nostrand Company, Inc., New York - Toronto - London - Melbourne, 1965.
2. K. O. Friedrichs, *Differential forms on Riemannian manifolds*, Comm. Pure Appl. Math. 8 (1955), 551–590.
3. D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order* (Grundlehren der mathematischen Wissenschaften 224), Springer-Verlag, Berlin - Heidelberg - New York, 1977.
4. P. Grisvard, *Problème de Dirichlet dans un cone*, Ricerche Mat. 20 (1971), 175–192.
5. P. Grisvard, *Alternative de Fredholm relative au problème de Dirichlet dans un polygone ou un polyèdre*, Boll. Un. Mat. Ital. (4), 5 (1972), 132–164.
6. P. Grisvard, *Alternative de Fredholm relative au problème de Dirichlet dans un polyèdre*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 2 (1975), 359–388.
7. P. Grisvard, *Behavior of the solutions of an elliptic boundary value problem in a polygonal or polyhedral domain*, in *Numerical Solution of Partial Differential Equations*. III, ed. B. Hubbard, (Proc. Third Symp., Univ. Maryland, College Park, Md, 1975), pp. 207–274. Academic Press, Inc., New York - London, 1976.
8. M. S. Hanna and K. I. Smith, *Some remarks on the Dirichlet problem in piecewise smooth domains*, Comm. Pure Appl. Math. 20 (1967), 575–593.
9. J. Kadlec, *The regularity of the solution of the Poisson problem in a domain whose boundary is similar to that of a convex domain* (Russian). Czechoslovak Math. J. 89 (1964), 386–393.
10. V. A. Kondrat'ev, *Boundary problems for elliptic equations in domains with conical or angular points*, Trans. Moscow Math. Soc. 16 (1967), 227-313.
11. R. Leis, *Zur Theorie elektromagnetischer Schwingungen in anisotropen inhomogenen Medien*, Math. Z. 106 (1968), 213–224.
12. R. Leis, *Zur Theorie der zeitunabhängigen Maxwellschen Gleichungen*, Berichte der Gesellschaft für Mathematik und Datenverarbeitung, Bonn, No. 50, 1971.
13. P. Neittaanmäki and J. Saranen, *Finite element approximation of electromagnetic fields in the three dimensional space*, Numer. Funct. Anal. Optim. 2 (6) (1980), 487–506.
14. J. Saranen, *Über das Verhalten der Lösungen der Maxwellschen Randwertaufgabe in einigen Gebieten mit Kegelspitzen*, Math. Methods. Appl. Sci. 2 (1980), 235–250.
15. J. Saranen, *Über die Approximation der Lösungen der Maxwellschen Randwertaufgabe mit der Methode der finiten Elemente*, Applicable Anal. 10 (1980), 15–30.
16. Ch. Weber, *A local compactness theorem for Maxwell's equations*, Math. Methods Appl. Sci. 2 (1980), 12–25.
17. N. Weck, *Maxwell's boundary value problem on Riemannian manifolds with nonsmooth boundaries*, J. Math. Anal. Appl. 46 (1974), 410–437.

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