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A UNIQUENESS THEOREM FOR HIGHER ORDER ELLIPTIC PARTIAL DIFFERENTIAL EQUATIONS

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Introduction.

In the by now classical paper [7] on Beppo Levi functions by J. Deny and J.-L. Lions, the following result (Theorem 5.1) is proved. Suppose f is an element of the Sobolev space $W_1^2(\Omega)$, having boundary limits zero q.e. (quasi everywhere) in the *fine topology* of potential theory. Then f can be arbitrarily well approximated by functions in $C_0^\infty(\Omega)$, that is $f \in W_{1,0}^2(\Omega)$.

We give a brief sketch of the argument. There is a function $g \in W_{1,0}^2(\Omega)$, such that $h=f-g$ is harmonic. By assumption, fine $\lim h=0$ q.e. at $\partial\Omega$, since this is trivially true for functions in $W_{1,0}^2(\Omega)$.

Since truncations operate on $W_1^2(\Omega)$, we may assume that f, hence h, is non-negative. Invoking a result due to M. Brelot, [3, Lemme 1] (or the fine minimum principle, B. Fuglede [9, p. 76]), we conclude $h \equiv 0$.

We shall give a direct proof of a result — Theorem 2 — which generalizes this. More precisely, for $p>2-1/d$ (see the remark after Theorem 4) we solve the following *approximation problem*: Given $f \in W_m^p(\Omega)$, when is it possible to approximate f by smooth compactly supported functions? In other words, when is

$$(1) \qquad f \in W_{m,0}^p(\Omega)?$$

As a corollary we get our main result, a uniqueness theorem for the Dirichlet problem (Theorem 3) of "Brelot type".

We shall also treat the following *extension problem*.

Given $f \in W_m^p(\Omega)$, define $\tilde{f}(x)=f(x)$ for $x \in \Omega$ and $\tilde{f}(x)=0$ otherwise. Find conditions on f such that

$$(2) \qquad \tilde{f} \in W_m^p(\mathbf{R}^d)\,.$$

It is easy to see that (1) $\Rightarrow$ (2), and in the case $m=1$, the converse is true for all $p \in (1,\infty)$. This follows from the characterization

$$W_{1,0}^p(\Omega) \;=\; \{g \in W_1^p(\mathbf{R}^d):\; g=0 \text{ q.e. off } \Omega\}\,.$$

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(See [12] for further references on this non-trivial fact.) When $m>1$, one has to impose restrictions on $\partial\Omega$ for the implication (2) $\Rightarrow$ (1) to hold.

To see this, consider the following example. In $\mathbf{R}^2$, let B be the open unit disk, let I be the interval $[-\frac{1}{2}, +\frac{1}{2}]$ on the x-axis and let $\Omega = B \setminus I$. If $f \in W^p_{2,0}(B) \cap W^p_2(\Omega)$ is equal to y near I, then (2) holds but (1) does not.

Among previous work in this area, we mention V. I. Burenkov's article [5] where the case $p>d$, (i.e. when the functions in question actually belong to C^{m-1}) is treated. The problem of extending a particular function — not a whole class — has also been studied by D. R. Adams [1].

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1. Preliminaries.

General references for this paragraph are V G. Maz'ja and V. P. Havin [14] and N. G. Meyers [15, 16]. For $m \in \mathbf{Z}^+$, $p \in (1,\infty)$ and Ω and open subset of $\mathbf{R}^d$, the Sobolev space $W^p_m(\Omega)$ consists of all functions $f\colon \Omega \to \mathbf{C}$ for which the distributional derivatives $D^\alpha f$, $|\alpha| \leqq m$, are in $L^p(\Omega)$. Equipped with the norm $\sum_{|\alpha|\leqq m} \|D^\alpha f\|_{L^p(\Omega)}$, $W^p_m(\Omega)$ is a Banach space. We write $W^p_m \equiv W^p_m(\mathbf{R}^d)$ and the subscript 0 is used to denote the closure in $W^p_m(\Omega)$ of functions in $C_0^\infty(\Omega)$, C^∞ functions with compact support in Ω.

The *(m,p)-capacity* measures the deviation from continuity of functions in W^p_m. For a compact set K, this quantity is defined as

$$C_{m,p}(K) = \inf_{\varphi} \sum_{|\alpha|=m} \int |D^\alpha \varphi|^p \, dx ,$$

where the infimum is taken over all φ in W^p_m, supported in B, such that $\varphi \equiv 1$ on K, and where all sets considered are supposed to be subsets of the fixed large ball B. (In particular we assume that Ω is bounded. This is no restriction since in (1) and (2) we can replace f by $f\cdot\chi$, where χ is a cut-off function with support in some openball B_R, say, and then replace Ω by $\Omega \cap B_R$.) For G open we let

$$C_{m,p}(G) = \sup\{C_{m,p}(K) : \; K \subset G, \; K \text{ compact}\}$$

and finally, for an arbitrary set E, we define

$$C_{m,p}(E) = \inf\{C_{m,p}(G) : \; E \subset G, \; G \text{ open}\} .$$

(If $mp>d$, $C_{m,p}$ is of no interest to us, since then $C_{m,p}(E)=0$ if and only if $E=\emptyset$.) A property P is said to hold *(m,p)-quasi everywhere* ((m,p)-q.e. or just q.e.) if $C_{m,p}(\{P \text{ fails}\})=0$.

A function φ is said to be *(m,p)-quasicontinuous*, if for any $\varepsilon>0$ there is an open set ω with $C_{m,p}(\omega)<\varepsilon$ such that $\varphi|_{\omega^c}$ is continuous in the relative topology of ω^c, the complement of ω. Functions in W_m^p have quasicontinuous representatives which are unique up to sets of zero (m, p)-capacity and this is true also for functions in $W_m^p(\Omega)$. Whenever it makes sense, we are referring to the quasicontinuous representative when speaking of Sobolev functions.

A consequence of quasicontinuity is that for $\varphi \in W_m^p$ and for any set E with $C_{m,p}(E)>0$, the trace $\varphi|_E$ is well defined and similarly for $D^\alpha\varphi \in W_{m-|\alpha|}^p$, $|\alpha| \leqq m-1$.

We say that a closed set $F \subset \mathbf{R}^d$ admits (m, p)-*spectral synthesis* if for any function φ in W_m^p

$$(3) \qquad [D^\alpha\varphi|_F=0,\ |\alpha| \leqq m-1] \ \Leftrightarrow\ \varphi \in W_{m,0}^p(F^c)\,.$$

For $m=1$, (3) is true for any $p \in (1, \infty)$.

Recently, Hedberg [12] has proved that also for $m>1$, (3) holds provided $p>2-1/d$. It is known whether this is so for all values of p or not.

We shall use the concept of *fine* continuity. Following Meyers [16], a set E is said to be (m, p)-*thin* at a point ξ, if

$$\int_0^1 \left\{\frac{C_{m,p}(E \cap B(\xi,\delta))}{\delta^{d-mp}}\right\}^{p'-1} \frac{d\delta}{\delta} < \infty\,,$$

where $B(\xi, \delta)$ is the ball $\{x\colon |x-\xi|<\delta\}$ and $p'=p/(p-1)$. This gives rise to a topology—fine or (m, p)-fine topology—by letting U be a neighbourhood of ξ if U^c is (m, p)-thin at ξ.

In this topology, a function φ has (finite) limit λ at ξ, if the set $\{x\colon |\varphi(x)-\lambda| \geqq \varepsilon\}$ is (m, p)-thin at ξ for all $\varepsilon>0$. This will be written

$$(m, p)-\operatorname{fine}\lim_{x\to\xi} \varphi(x) = \lambda\,.$$

The *fine exterior* of E is the set

$$e(E) = e_{m,p}(E) = \{\xi : E \text{ is } (m, p)\text{-thin at } \xi\}\,.$$

In fact, if $\tilde{E}$ denotes the fine closure of E, then $e(E)=(\tilde{E})^c$. When $p>2-m/d$, $C_{m,p}$ has the important "*Choquet property*":

Given any $\varepsilon>0$ and any set $E \subset \mathbf{R}^d$, there is an open set G such that $e(E) \subset G$ and $C_{m,p}(E \cap G)<\varepsilon$.

For a proof, see [11, Theorem 6] and the crucial Theorem 5.3 in [2]. See also Choquet's original proof in [6].

The relationship between quasi topology and fine topology is treated in

Fuglede [8], [10], and Brelot [4, Chapter IV]. From [8] (Theorem 3) we note the following result:

(m, p)-*quasicontinuous functions are* (m, p)-*finely continuous* (m, p)-q.e.

2. The quasicontinuous case.

Let $f \in W_m^p(\Omega)$. By what was said in section 1, a reasonable starting-point is to assume that f and its derivatives $D^\alpha f$, $|\alpha| \leqq m-1$, when extended by zero outside Ω, are quasicontinuous functions. For this reason we make the following

DEFINITION. A function g, defined (m, p)-q.e. in Ω, has *quasi limit* zero at $\partial\Omega$ if for any $\varepsilon > 0$, there is an open set ω with $C_{m,p}(\omega) < \varepsilon$ such that for any $\xi \in \partial\Omega \setminus \omega$, $g(x) \to 0$ as $x \to \xi$, $x \in \Omega \setminus \omega$. This will be written

$$\text{quasi}\lim g = 0 \text{ at } \partial\Omega ,$$

the (m, p) referred to being understood in each instance.

Notice that if $f \in W_m^p(\Omega)$, and if $\text{quasi}\lim f = 0$ at $\partial\Omega$, then $\tilde{f}$ is quasicontinuous in the entire space $\mathbf{R}^d$.

We have the following result.

THEOREM 1. *Suppose* $f \in W_m^p(\Omega)$, *where* $1 < p < \infty$ *and* $m \in \mathbf{Z}^+$. *Then,*

$$\text{quasi}\lim D^\alpha f = 0 \quad at \ \partial\Omega, \qquad |\alpha| \leqq m-1 , \tag{4}$$

if and only if

$$\tilde{f} \in W_m^p \quad and \quad D^\alpha \tilde{f} = (D^\alpha f)\tilde{}, \qquad |\alpha| \leqq m . \tag{5}$$

REMARK. The implication (5) $\Rightarrow$ (4) should be understood in the following sense: Suppose $F \in W_m^p$ and denote by f the function $F|_\Omega \in W_m^p(\Omega)$. If f fulfills (5), then (4) holds.

COROLLARY. *Under the hypothesis of spectral synthesis* (*hence, at least for* $m=1$, $1<p<\infty$ *or* $m>1$, $p>2-1/d$), (4) *holds if and only if* $f \in W_{m,0}^p(\Omega)$. *In particular, for* $m=1$, $1<p<\infty$,

$$f \in W_{1,0}^p(\Omega) \Leftrightarrow \text{quasi}\lim f = 0 \ at \ \partial\Omega . \tag{6}$$

PROOF. We start by proving (6). The proof is more or less the same as the proof of spectral synthesis in the case $m=1$, $1<p<\infty$; see e.g. [11, Lemma 4]. We include it for the reader's convenience.

We must prove the implication $\Leftarrow$. The converse is trivial. Choose $\varepsilon > 0$ and let ω be the corresponding exceptional set, such that f is continuous on $\Omega \setminus \omega$ and has limit zero as x in $\Omega \setminus \omega$ tends to $\xi \in \partial\Omega \setminus \omega$, and such that $C_{1,p}(\omega) < \varepsilon$.

Then one can find $\varphi = \varphi_\varepsilon \in W_1^p$ with $\int |\nabla\varphi|^p < \varepsilon$, such that $0 \leqq \varphi \leqq 1$ everywhere and $\varphi \equiv 1$ on ω. Since $W_1^p(\Omega)$ is closed under truncation [7, p. 316], we may assume that $0 \leqq f \leqq M < \infty$. For $\delta > 0$, let

$$f' = f'_\delta = (f - \delta)^+ \in W_1^p(\Omega).$$

Then $F = F_{\varepsilon,\delta} = f'(1-\varphi)$ is in $W_1^p(\Omega)$ and F has compact support in Ω, so $F \in W_{1,0}^p(\Omega)$.

By assumption, φ_ε tends to zero in measure. Thus, the inequality

$$\|\nabla(f'\varphi)\|_p \leqq M\|\nabla\varphi\|_p + \|\varphi \cdot \nabla f\|_p$$

shows that $f'\varphi_\varepsilon \to 0$ in $W_1^p(\Omega)$ as $\varepsilon \to 0$, by dominated convergence.

Consequently, by first letting ε, then δ, tend to zero we get $f - F_{\varepsilon,\delta} = f - f'_\delta + f'_\delta\varphi_\varepsilon \to 0$ in $W_1^p(\Omega)$, and (6) follows.

Now we turn to the general case. Let $m > 1$ and assume (4). We shall prove that (5) holds. It is easy to see that if $j > 1$, then $C_{1,p}(\cdot) \leqq \text{const} \cdot C_{j,p}(\cdot)$, so $(1,p)-\text{quasi}\lim g = 0$ at $\partial\Omega$ if $(j,p)-\text{quasi}\lim g = 0$ at $\partial\Omega$. Hence from the case $m = 1$, we see that $D^\alpha f \in W_{1,0}^p(\Omega)$ for $|\alpha| \leqq m-1$.

Choose $\psi_n \in C_0^\infty(\Omega)$ such that $\|f - \psi_n\|_{W_1^p(\Omega)} \to 0$, $n \to \infty$.

For $i = 1, 2, \ldots, d$ and for any $\varphi \in C_0^\infty(\mathbf{R}^d)$, we get

$$\int \tilde{f} D_i\varphi = \int_\Omega f D_i\varphi = \lim_n \int_\Omega \psi_n D_i\varphi$$

$$= \lim_n \left(- \int_\Omega D_i\psi_n\varphi \right) = - \int_\Omega D_i f\varphi = - \int (D_i f)^\sim \varphi,$$

so $D_i\tilde{f} = (D_i f)^\sim$ a.e. Similarly, $D_i D_j \tilde{f} = D_i[(D_j f)^\sim] = (D_i D_j f)^\sim$ and proceeding inductively, one gets $D^\alpha\tilde{f} = (D^\alpha f)^\sim$ a.e. for any multiindex α with $|\alpha| \leqq m$. This proves that f has distributional derivatives which for $|\alpha| \leqq m$ all belong to L^p. Hence $\tilde{f} \in W_m^p$. By assumption all the functions $(D^\alpha f)^\sim$, $|\alpha| \leqq m-1$, are quasicontinuous, so by uniqueness of quasicontinuous representatives, we conclude $D^\alpha\tilde{f} = (D^\alpha f)^\sim$ for all $|\alpha| \leqq m-1$, which is (5).

That (5) $\Rightarrow$ (4) as described in the remark above is clear. Finally, the corollary follows by spectral synthesis.

3. The finely continuous case.

In this section, the rôle of quasicontinuity is taken over by fine continuity. Thus the condition on our given function $f \in W^p_m(\Omega)$ will be:

$$(m-|\alpha|,p)-\operatorname*{fine\,lim}_{\Omega \ni x \to \xi} D^\alpha f(x) = 0, \qquad (m-|\alpha|,p)-\text{q.e. } \xi \in \partial\Omega,\ |\alpha| \leqq m-1\ ,$$

abbreviated "$\operatorname{fine\,lim} D^\alpha f = 0$ q.e. at $\partial\Omega$, $|\alpha| \leqq m-1$".

We have the following

THEOREM 2. *Let* $f \in W^p_m(\Omega)$, *where* $p > 2-1/d$ *and* $m \in \mathbf{Z}^+$. *Then all conditions below are equivalent*:

(7) $\operatorname{fine\,lim} D^\alpha f = 0$ q.e. *at* $\partial\Omega$, $|\alpha| \leqq m-1$,

(8) $\operatorname{quasi\,lim} D^\alpha f = 0$ *at* $\partial\Omega$, $|\alpha| \leqq m-1$,

(9) $\tilde{f} \in W^p_m$ *and* $D^\alpha \tilde{f} = (D^\alpha f)^{\sim}$, $|\alpha| \leqq m$,

(10) $f \in W^p_{m,0}(\Omega)$.

PROOF. In view of Theorem 1 and its corollary, we need only prove that for $|\alpha| \leqq m-1$, $g = D^\alpha f$ has quasilimit zero at $\partial\Omega$ as soon as (7) holds. (Recall that quasicontinuity implies fine continuity q.e.)

We extend g by zero outside Ω and denote again the resulting function by g. In this way we get a function $\mathbf{R}^d \to \mathbf{C}$ which is finely continuous q.e. For the values of p that we are considering, $C_{j,p}$ has the Choquet property for all $j \in \mathbf{Z}^+$. Hence the following lemma implies the theorem.

LEMMA. ([4]) *Suppose that* $g: \mathbf{R}^d \to \mathbf{C}$ *is* (s,q)*-finely continuous* q.e. *and that* $C_{s,q}$ *has the Choquet property. Then* g *is* (s,q)*-quasicontinuous.*

PROOF. We may assume that g is finely continuous everywhere, since we can replace $\mathbf{R}^d$ by $\mathbf{R}^d \setminus N$, where $C(N) = C_{s,q}(N) = 0$, if necessary. If $\{B_n,\ n \in \mathbf{N}\}$ is a countable basis for the topology in $\mathbf{C}$, let $\{F_n,\ n \in \mathbf{N}\}$ denote all complementary sets: $F_n = B_n^c$. For a fixed $n \in \mathbf{N}$, define $K_n = g^{-1}(F_n)$. Then K_n is finely closed. Furthermore, $e(K_n) = K_n^c$.

Choose $\varepsilon > 0$. By the Choquet property there is an open set (in the usual topology) G_n such that $K_n^c \subset G_n$ and $C(G_n \cap K_n) < \varepsilon \cdot 2^{-n}$. Let $\omega_n = G_n \cap K_n$ and $\omega = \bigcup \omega_n$. Then $C(\omega) < \varepsilon$ and g is continuous on ω^c because $K_n \setminus \omega$, $n \in \mathbf{N}$, are closed with respect to ω^c.

REMARK. Thinness can be defined in several ways and some of them are known to be different, when $p<2-m/d$; see Meyers [16, p. 164]. It is clear that any definition of thinness such that fine continuity is weaker than quasicontinuity and such that the Choquet property holds is sufficient for our purposes.

4. Application to the Dirichlet problem.

We shall use Theorem 2 to prove a uniqueness theorem for the equation $\Delta^m u = 0$, $u \in W_m^2(\Omega)$. Here $\Delta = \sum D_i^2$ is the Laplace operator and Δ^m denotes its mth power. Among possible generalizations, for instance via Gårding's inequality, we only mention the following. Suppose

$$P(x,D) = \sum_{|\alpha|,|\beta| \leqq m} (-1)^{|\alpha|} D^\alpha (a_{\alpha\beta}(x) D^\beta)$$

determines the same topology on $W_m^2(\Omega)$ as Δ^m do, i.e. suppose that for some constant A and for all v in $W_m^2(\Omega)$,

$$\frac{1}{A} \int_\Omega |\nabla^m v|^2 \leqq \sum_{\alpha,\beta} \int_\Omega a_{\alpha,\beta}(x) D^\beta v(x) \overline{D^\alpha v(x)}\, dx \leqq A \int_\Omega |\nabla^m v|^2 .$$

Then the theorem below holds with Δ^m replaced by $P(x,D)$.

THEOREM 3. *Suppose* $f \in W_m^2(\Omega)$, $m \in \mathbf{Z}^+$ *and* $\Delta^m f = 0$ *in the open and bounded set* Ω. *If* fine lim $D^\alpha f = 0$ q.e. *at* $\partial\Omega$ *for* $|\alpha| \leqq m-1$, *then* $f \equiv 0$.

REMARKS. 1) In Hedberg [12] it is shown that in W_m^2, $D^\alpha u|_{\partial\Omega} = 0$, $|\alpha| \leqq m-1$, and $\Delta^m u = 0$ in Ω is satisfied only by the function $u \equiv 0$, so the above theorem can be seen as an amplification of this result.

2) When $p \neq 2$, $p > 2 - 1/d$, we get "maximum principles" for certain nonlinear equations; e.g. suppose $(1,p)-$fine lim $u = 0$ q.e. on $\partial\Omega$, $u \in W_1^p(\Omega)$, and suppose

$$\operatorname{div}(\nabla u |\nabla u|^{p-2}) = 0 .$$

Then $u \equiv 0$. Cf. Maz'ja [13]. This can be generalized to higher order equations.

PROOF OF THE THEOREM: We can decompose $W_m^2(\Omega)$ as an orthogonal sum, see [7, p. 367],

$$W_m^2(\Omega) = W_{m,0}^2(\Omega) \oplus H_m(\Omega) , \tag{11}$$

where $H_m(\Omega)$ is the subspace of $W_m^2(\Omega)$ consisting of function u satisfying $\Delta^m u = 0$ in Ω. From Theorem 2, f belongs to $W_{m,0}^2(\Omega)$ and by our assumption, to $H_m(\Omega)$. Hence $f \equiv 0$ since $W_{m,0}^2(\Omega) \cap H_m(\Omega) = \{0\}$.

5. A stronger result in the plane.

It would be desirable to extend Theorem 2 to the case $m=1$, $1<p<\infty$, since the spectral synthesis theorem holds in this case. Unfortunately, we have not succeeded with this. We have, however, the following partial result.

THEOREM 4. *Assume* $f \in W_m^p(\Omega)$, $1<p<\infty$, $m \in \mathbf{Z}^+$ *and* $\Omega \subset \mathbf{R}^2$. *Then* (7) *implies* (9). *In particular, for* $m=1$,

$$\text{fine}\lim f = 0 \text{ q.e. } on\ \partial\Omega \iff f \in W_{1,0}^p(\Omega)\,. \tag{12}$$

PROOF. We start by observing that if (7) holds, then $D^\alpha f$, $|\alpha| \leqq m-2$, must be continuous. Thus it suffices to prove the case $m=1$, i.e. we must prove the implication $\Rightarrow$ in (12). We can also assume that $p<2$ and $f \geqq 0$.

As is well-known, a function u belongs to $W_{1,\mathrm{loc}}^p(G)$, $G \subset \mathbf{R}^2$ an open set, if and only if u is absolutely continuous on almost every (with respect to one-dimensional Lebesgue measure) line segment in G — parallel to some co-ordinate axis — and if the partial derivatives $D_1 u$ and $D_2 u$ are in L_{loc}^p. See [7, p. 315]. Moreover, given such a line segment, u tends to a limit as the endpoints tend to points on the boundary of G. See Morrey [17, p. 66].

Let $I = \bigcup_{-\infty}^{\infty} I_\nu$, $I_\nu = (a_\nu, b_\nu)$, be the intersection of Ω with the x-axis $\mathbf{R}$, and assume that $f(\cdot, 0)$ is absolutely continuous on every interval I_ν. Define $\varphi(t) = D_1 f(t,0)$ for $t \in I$ and $\varphi(t)=0$ elsewhere. Then $g(t) = \int_{-\infty}^t \varphi(\tau)\,d\tau$ is absolutely continuous on $\mathbf{R}$. We shall show that $g(t) = \tilde{f}(t,0)$. Clearly this proves the theorem. Let $\Gamma = \partial\Omega \setminus \bigcup \bar{I}_\nu$ and suppose $t \in \mathbf{R} \setminus \Gamma$. We can enumerate the I_ν's such that $b_\nu \leqq t$ for all $\nu < N$. (The enumeration will of course depend on t, but this does not matter.) Denoting by 1_E the indicator function of a set E (1_E equals 1 on E and 0 on E^c) we get

$$\begin{aligned} g(t) &= \sum_{\nu<N} \int_{I_\nu} \varphi(\tau)\,d\tau + \left\{\int_{a_N}^t \varphi(\tau)\,d\tau\right\} \cdot 1_{\{t \in I_N\}} \\ &= \sum_{\nu<N} \{f(b_\nu, 0) - f(a_\nu, 0)\} + \{f(t,0) - f(a_N, 0)\} \cdot 1_{\{t \in I_N\}}\,. \end{aligned} \tag{13}$$

Let us assume for a moment that

$$\forall \nu:\ f(a_\nu, 0) = f(b_\nu, 0) = 0\,. \tag{14}$$

Then (13) yields $g(t) = \tilde{f}(t,0)$ for $t \in \mathbf{R} \setminus \Gamma$ and we shall prove that this is true also for $t \in \Gamma$. We have two alternatives: either t is an isolated boundary point of Ω in which case

$$g(t) = \lim_{\tau \to t} g(\tau) = 0 = \tilde{f}(t,0) \qquad (\tau \in \mathbf{R} \setminus \bar{\Omega}),$$

or $t = \lim t_n$, where $t_n \in \partial I_n$. In this case,

$$g(t) = \lim g(t_n) = 0 = \tilde{f}(t,0),$$

os the theorem follows from (14).

To prove (14), assume that the converse is true for

$$a_\nu = 0: \; f(\delta,0) \geqq \text{const.} > 0, \qquad \text{when } \delta \leqq \delta_0 ,$$

whereas the set $E_\varepsilon = \{f \geqq \varepsilon\}$ is $(1,p)$-thin at 0 for every $\varepsilon > 0$. Then, if ε is small enough and if $\delta \leqq \delta_0$, $E_\varepsilon \cap B(0,\delta)$ contains the interval $(0,\delta)$ on the x-axis. Let $J = (0,1)$, so $(0,\delta) = \delta J$. It follows by comparing $C_{1,p}$ with Hausdorff measure — see [14; Theorem 1, p. 133] — that $C_{1,p}(J) \neq 0$. By a change of variables in the integral defining capacity, one sees that $C_{1,p}(\delta J) = C_{1,p}(J) \cdot \delta^{2-p}$, whence

$$\int_0^{\delta_0} \{\delta^{p-2} C_{1,p}(E_\varepsilon \cap B(0,\delta))\}^{p'-1} \, d\delta/\delta = \infty ,$$

which is a contradiction.

Added in proof. Since this work was completed, Hedberg and Wolff [18] have extended the results on spectral synthesis to arbitrary $p \in (1,\infty)$. Moreover, the Choquet property is valid for all $p \in (1,\infty)$. Hence, in our Theorem 2, the condition $p > 2 - 1/d$ is superfluos and the result is valid for all $1 < p < \infty$.

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