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ON L_p-DECAY AND SCATTERING FOR NONLINEAR KLEIN–GORDON EQUATIONS

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Abstract.

In this paper we extend previous results on scattering and L_p-decay for nonlinear Klein–Gordon equations to a larger class of nonlinearities and to dimensions $n>3$. In particular, for nonlinearities of the type $|u|^{\varrho-1}u$, the existence of scattering states are proved for $1+4/n<\varrho<1+4\gamma_n/(n-2)$, where $\gamma_n=1$ for $n\leqq 10$ and $\gamma_n=n/(n+1)$ for $n>10$. No quantitative restrictions are placed on the data. Earlier results of this type were only known for $n=3$ and $8/3<\varrho<5$.

0. Introduction.

In this paper we will investigate the decay in $L_p(\mathrm{R}^n)$ as $|t| \to \infty$ of the solution of the nonlinear Klein–Gordon equation (NLKG) in $n\geqq 3$ space dimensions,

$$(0.1) \qquad \partial_t^2 u-\Delta u+m^2u+f(u) = 0, \quad u|_{t=0} = \varphi, \quad \partial_t u|_{t=0} = \psi ,$$

where $m>0$ and the data φ,ψ for simplicity are assumed to have sufficiently many derivatives in $L_1(R^n)$, and where in addition the nonlinear term $f(u)$ satisfies

(i) $f(R)\subseteqq R$ and $F(u)=\int_0^u f(v)\,dv\geqq 0$ for all $u \in R$,

(ii) $|f^{(j)}(u)|\leqq C|u|^{\varrho-j}$, $j=0,1$ and $\varrho>1$,
$|f'(u)-f'(v)|\leqq C|u-v|^{\varrho-1}$ if $\varrho\leqq 2$ and
$|f''(u)|\leqq C|u|^{\varrho-2}$ if $\varrho>2$.

(iii) $uf(u)-2F(u)\geqq \alpha F(u)$, for some $\alpha>0$ and
$F(u)\geqq \beta|u|^{\tilde\varrho+1}(1+|u|)^{-N}$ for some $\beta>0$ and some $\tilde\varrho, N$ with
$\varrho\leqq\tilde\varrho<\infty$ and $N\geqq 0$.

Let us, before we proceed, notice that ϱ in (ii) merely provides a lower bound

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for the corresponding $\varrho = \varrho_0$ valid for $|u| \leqq 1$, $|u-v| \leqq 2$, and an upper bound for the corresponding $\varrho = \varrho_\infty$ valid for $|u| > 1$ and $|u-v| > 2$.

As a consequence of the L_p-decay-estimate that we will prove (under suitable restrictions on ϱ), we get the existence and asymptotic completeness of the scattering operator associated with the NLKG. In fact, in Lemma 5.1 below we will prove, for $1+4/n < \varrho < 1+4/(n-2)$, and also for $n > 10$ if in addition $\varrho < 1 + 4n/(n-2)(n+1)$, that

$$\int_{-\infty}^{\infty} \|f(u(t))\|_{L_2} dt < \infty , \tag{0.2}$$

where u is the solution of (0.1). As is well known, (0.2) implies the stated results about the scattering operator (cf. Reed and Simon [11, Ch. XI. 13], and Strauss [12, Lemma 4.2]). In this paper we will concentrate on the L_p-decay estimates, and as one application we will prove the following result on the existence of the scattering operator as a consequence of (0.2): Let the energy norm $\|\cdot\|_e$ be defined by

$$\|v\|_e^2 = \tfrac{1}{2} \int (|\nabla v| + m^2|v| + |\partial_t v|^2)\, dx ,$$

and let a "free" solution of the Klein–Gordon equation mean a solution v of

$$\partial_t^2 - \Delta v + m^2 v = 0 , \tag{0.3}$$

with suitable initial data.

THEOREM 1. *Let $n \geqq 3$, $1+4/n < \varrho < 1+4/(n-2)$, and assume in addition that $\varrho < 1+4n/(n-2)(n+1)$ for $n > 10$. Then for each solution u of* (0.1) *there exist unique free solutions $u_\pm$, that is solutions of* (0.3), *with suitable data in $L_2^{\frac{1}{2}} \times L_2$ such that*

$$\|u(t) - u_\pm(t)\|_e \to 0 \quad as\ t \to \pm\infty .$$

Previous results on scattering for the NLKG were proved by Morawetz and Strauss [7] for $n=3$ and $f(u)=u^3$, and extended by Pecher [8], [9], still for $n=3$, for $f(u)$ satisfying (i), (ii) and (iii) with $N=0$, for $8/3 < \varrho < 5$. Notice that the lower bound for ϱ in Theorem 1 is 7/3 for $n=3$! For *small* data Theorem 1 is a consequence of the result by Marshall, Strauss and Wainger [6] who actually proved that the scattering operator in that case is in fact defined in a whole neighborhood of the origin (cf. also Strauss [13]). That Theorem 1 cannot hold for $\varrho < 1+2/n$ was proved by Glassey [5] (cf. also Strauss [12]).

The estimate (0.2), and hence the proof of Theorem 1, is based on the following L_p-decay result for solutions of (0.1):

Let $1<p\leqq 2\leqq p'<\infty$, $1/p+1/p'=1$ and $\delta=1/2-1/p'$. If $1+4\delta<\varrho$, $(n-1)\delta<1$, and $1+4/n<\varrho<1+4/(n-2)$, and $\varrho<1+4(n-3/2)/(n-2)(n-1)$ for $n>10$, then

$$\|u(t)\|_{p'} \leqq C(1+t)^{-n\delta}, \quad t\geqq 0 . \tag{0.4}$$

If in addition $\varrho<1+4n/(n-2)(n+1)$ for $n>10$, then

$$\|u(t)\|_{L^1_{p'}} \leqq C(1+t)^{-n\delta}, \quad t\geqq 0 . \tag{0.4$'$}$$

With a little care and Hölder's inequality, we will prove that (0.4), or rather (0.4)$'$, will imply (0.2).

From the L_p-estimates (0.4) and (0.4)$'$ one may also obtain maximum-norm decay results using the jacking-up process suggested by Pecher [6]; we refer to [4] for such results.

The proof of (0.4) and (0.4)$'$ is based on the following (known) basic estimates for solutions of (0.1) and (0.3): Below, $E_i(t)$, $i=0,1$, will denote the solution operators of (0.3) so that $v=E_0(t)\varphi+E_1(t)\psi$ is the solution of (0.3) with data $u|_{t=0}=\varphi$ and $\partial_t u|_{t=0}=\psi$.

A. Let $1\leqq p\leqq 2$, $1/p+1/p'=1$, $1\leqq q\leqq\infty$, and $\delta=1/2-1/p'$.
Let $B_p^{s,q}$ denote the L_p-based Besov space of order $s\geqq 0$. Then

$$\|E_1(t)g\|_{B_{p'}^{s',q}} \leqq K(t)\|g\|_{B_p^{s,q}}, \quad 0\leqq t , \tag{0.5}$$

where

$$K(t) \leqq C\begin{cases} t^{-(n-1+\theta)\delta}, & t\geqq 1 ,\\ t^{1+s-s'-2n\delta}, & 0<t<1 ,\end{cases} \tag{0.6}$$

provided $\delta(n+1+\theta)\leqq 1+s-s'$, $0\leqq\theta\leqq 1$ and $s,s'\geqq 0$.

For a definition of the Besov spaces involved, see e.g. [2], [3]. We will frequently use the following well known inclusion between Besov and Sobolev spaces:

(0.7)
$$B_p^{s,2} \supseteqq L_p^s, \quad 1<p\leqq 2 \qquad \text{and} \qquad B_{p'}^{s',2}\subseteqq L_{p'}^{s'}, \quad 2\leqq p'<\infty .$$

As norms on $B_p^{s,q}$ and on L_p^s we will use, with $s=\sigma+S$, $0<\sigma<1, S$ an integer,

$$\|u\|_{B_p^{s,q}} = \|u\|_p+\left(\int_0^1\left(t^{-\sigma}\sum_{|\alpha|=S} w_p(t,D^\alpha u)\right)^q\frac{dt}{t}\right)^{1/q},$$

$$w_p(t,v) = \sup_{|h|\leqq t}\|v_h-v\|_p, \quad v_h(x) = v(x+h)$$

and, with $\mathscr{F}u=\hat{u}$ denoting the Fouriertransform of u,

$$\|u\|_{L_p^s} \equiv \|u\|_{p,s} = \|\mathscr{F}^{-1}(\hat{u}(\xi)(1+|\xi|^2)^{s/2})\|_p$$

respectively. Notice that if s is an integer and $1<p<\infty$, then

$$\|u\|_{p,s} \sim \|u\|_p + \sum_{|\alpha|=s} \|D^\alpha u\|_p .$$

The estimate (0.5), (0.6), in the following for simplicity called estimate or inequality A, is easily proved using the method of the stationary phase as in [2] (using e.g. Theorem 3.2 in [1]). The corresponding $L_p-L_{p'}$-estimates are contained as special cases of the L_p-L_q-estimates proved by Marshall, Strauss and Wainger [6].

Next we will use the well known property of preservation of energy of the NLKG (0.1) when (i) holds:

B. Let u be a solution of (0.1). Then

$$E(u) = \|u\|_e^2 + \int F(u)\,dx = \text{constant}, \qquad t \geqq 0 .$$

Finally we will use a weak decay estimate due to Morawetz (cf. [7]):

C. Let u be a solution of the NLKG. Then

$$\iint \frac{uf(u)-2F(u)}{1+|x|}\,dxdt \leqq CE(u) .$$

Together with the integral equation for the solution of NLKG and the finite speed of propagation of supports, the estimates A, B and C above are the main tools, to be systematically used throughout this paper. In order to cover the case $\varrho \leqq 2$, i.e. space dimensions $n \geqq 6$, we will use additional Besov space inequalities for $f(u)$ of a type already employed in [3].

Conceptually this paper owes much to the papers by Morawetz and Strauss [7], Pecher [8] and [9] and Strauss [12], although the present set-up requires a technically more involved argument. The main difference is the systematic use of $L_p-L_{p'}$-estimates, and, in higher space dimensions, of the corresponding Besov space inequalities.

1. On the uniform boundedness in $L_{p'}$ of solutions to NLKG.

Let us first fix some restrictions on p', and so on $\delta=1/2-1/p' \geqq 0$, having various applications of the stimate A in mind. We introduce the following condition:

$(*)_s$ Let $1/p+1/p'=1$, $2\leqq p'<\infty$, $\delta=1/2-1/p'$, and assume that for some $\theta\in(0,1]$: $(n-1+\theta)\delta>1>(n-1-\theta)\delta$, and that for some $s,s'\in[0,1]$, $(n+1+\theta)\delta=1+s-s'$.

REMARK 1.1. Under assumption $(*)_s$, inequality A, i.e. (0.5) and (0.6), holds with $K(t)\in L_1(\mathbf{R}_+)$.

REMARK 1.2. Define $r=r(p',s')$ by $1/r=1/p'-s'/n$, where p',s' satisfy $(*)_s$. Then

$$1/r \;=\; (n-2(1+s)+2(1+\theta)\delta)/2n\;.$$

Thus for each $\bar{r}$ sufficiently close to r, there exist $\bar{p},\bar{s}'$ such that $(*)_s$ holds for $\bar{p}',\bar{s}'$ and such that $\bar{r}=r(\bar{p}',\bar{s}')$. Notice also that $(*)_s$ implies that $2(1+\theta)\delta>s-s'>2\delta$.

We will throughout this section assume that f satisfies (i) and (ii) in the introduction. Since we may have to use fractional derivatives, that is Besov spaces, for high dimensions, a number of arguments will split in two cases.

The following two lemmas and their corollaries will be very useful in what follows.

LEMMA 1.1. *Assume that $(*)_1$ holds, and that for some $\eta\in(0,1]$.*

$(1.1)_\varrho$ $\quad 1+4\delta-2\delta\eta\;\leqq\;\varrho\;\leqq\;\varrho_n-2\eta(1-(1+\theta)\delta)/(n-2),\qquad \varrho\geqq 2-\eta$

where $\varrho_n=(n+2(n-1-\theta)\delta)/(n-2)$. Then

$$\|f(u)\|_{p,1}\;\leqq\;C\|u\|_{2,1}^{\varrho-1+\eta}\|u\|_{p',s'}^{1-\eta}\;. \tag{1.1}$$

PROOF. Apply Hölder's inequality to

$$\|f(u)\|_{p,1}\;\leqq\;C\sum_{|\alpha|\leqq 1}\|\,|u|^{\varrho-2+\eta}|u|^{1-\eta}|D^\alpha u|\,\|_p\;.$$

The lower bound for ϱ is obtained from

$$\|\,|u|^{\varrho-2+\eta}|u|^{1-\eta}|D^\alpha u|\,\|_p\;\leqq\;C\|u\|_2^{\varrho-2+\eta}\|u\|_{p'}^{1-\eta}\|D^\alpha u\|_2$$

which holds for

$$\frac{1}{2}(\varrho-2+\eta)+\frac{1}{p'}(1-\eta)+\frac{1}{2}\;\geqq\;\frac{1}{p},\qquad \varrho-2+\eta\;\geqq\;0\;,$$

that is for $\varrho\geqq 1+4\delta-2\delta\eta$, $\varrho\geqq 2-\eta$. Similarly we obtain (1.1) for all ϱ's satisfying $(1.1)_\varrho$, using the relation $(n+1+\theta)\delta=2-s'$ and Sobolev's embedding theorem.

By chainging sign in the above argument we get:

COROLLARY 1.1. *Assume that* $(*)_1$ *holds and that for some* $\eta \geqq 0$,

$$(1.2)_\varrho \qquad 1+4\delta+2\delta\eta \leqq \varrho \leqq \varrho_n, \quad 2+\eta \leqq \varrho ,$$

where ϱ_n *is defined as above. Then*

$$(1.2) \qquad \|f(u)\|_{p,1} \leqq C\|u\|_{2,1}^{\varrho-1-\eta}\|u\|_{p',s'}^{1+\eta} .$$

Next, we treat the case of fractional derivatives:

LEMMA 1.2. *Assume that* $(*)_s$ *holds and that for some* $\eta \in (0,1]$

$$(1.3)_\varrho \quad 1+4\delta-2\delta\eta \leqq \varrho \leqq \varrho_n-2\eta\frac{1+s-(1+\theta)\delta}{n-2}-2\frac{\varrho-1-s}{n-2},$$

$$1+s-\eta \leqq \varrho \leqq 2-\eta$$

where ϱ_n *is defined as above. Then*

$$(1.3) \qquad \|f(u)\|_{B_p^{s,2}} \leqq C\|u\|_{2,1}^{\varrho-1+\eta}\|u\|_{p',s'}^{1-\eta} .$$

PROOF. In view of the norm used on $B_p^{s,q}$ it is enough to estimate

$$t^{-s}\|f(u_h)-f(u)\|_p, \quad |h| \leqq t ,$$

by

$$(1.5) \qquad t^{-s}\|f(u_h)-f(u)\|_p \leqq Ct^{-s}\|u_h-u\|_2^{\varrho-1+\eta}\|u_h-u\|_{p',s'}^{2-\varrho-\eta}\|u\|_{p',s'}^{\varrho-1} + Ct^{-s}\|u_h-u\|_2^{\varrho-1+\eta}\|u\|_{p',s'}^{1-\eta} ,$$

which holds by (i) and (ii) and Hölder's and Sobolev's inequalities provided $1-\eta \leqq \varrho \leqq 2-\eta$ and

$$\frac{1}{2}(\varrho-1+\eta)+\frac{1}{p'}(1-\eta) \geqq \frac{1}{p} \geqq \frac{1}{2}(\varrho-1+\eta)+\left(\frac{1}{p'}-\frac{s'}{n}\right)(1-\eta) ,$$

which is equivalent with $(1.3)_\varrho$. If we estimate $\|u_h-u\|_2$ by

$$\|u_h-u\|_2^{\varrho-1+\eta} \leqq Ct^{\varrho-1+\eta}\|u\|_{2,1}^{\varrho-1+\eta}, \quad |h| \leqq t ,$$

and integrate (1.5) after squaring both sides, we obtain (1.3) after having carried out the estimate corresponding to (1.5) for $f(u)$ with $s=0$, provided $\varrho-1+\eta>s$. This completes the proof of Lemma 1.2.

As above, we obtain by a change of sign:

COROLLARY 1.2. *Assume that* $(*)_s$ *holds and that for some* $\eta \geqq 0$,

$$(1.4)_\varrho \quad 1+4\delta+2\delta\eta \leqq \varrho \leqq \varrho_n - 2(\varrho-1-s)/(n-2), \qquad 1+s+\eta \leqq \varrho \leqq 2+\eta$$

where ϱ_n *is defined as aobe. Then*

$$\|f(u)\|_{B_p^{s,2}} \leqq C\|u\|_{2,1}^{\varrho-1-\eta}\|u\|_{p',s'}^{1+\eta}. \tag{1.4}$$

REMARK 1.3. In (1.1) through (1.4) we may replace the $L_{p'}^{s'}$ norm on the right hand sides of the inequalities by L_r-norms with $1/r = 1/p' - s'/n$, that is $r = r(p', s')$ defined as in Remark 1.2 above.

We now proceed to prove a boundedness result for the solution of the NLKG.

LEMMA 1.3. *Let* $n \geqq 3$, $\delta(n-1) < 1$, $1+2\delta < \varrho$, *where* $1+2/n < \varrho < 1+4/(n-2)$, *with* $\delta = 1/2 - 1/p'$, $p' \geqq 2$. *Then for any solution of the* NLKG

$$\sup_{t \geqq 0} \|u(t)\|_{p'} < \infty .$$

PROOF. It is enough to prove that if $1+2\delta < \varrho < \varrho_n$, $\delta(n-1) < 1$, and if $(*)_s$ holds either with $s=1$ or else with $s < \varrho - 1$, s close enough to $\varrho - 1$, then

$$\sup_{t \geqq 0} \|u(t)\|_{p',s'} < \infty . \tag{1.6}$$

To see this, notice that as θ runs from 1 to 0 in $(*)_s$, s choosen as above, then the set of possible choices of δ runs through the interval $(1/n, 1/(n-1))$. Thus all values of ϱ in the interval $(1+2/n,\ 1+4/(n-2))$ will be obtained by suitable choice of θ and δ in $(*)_s$. Interpolation with the uniform L_2^1-bound of u provided by the energy inequality B, now completes the proof of Lemma 1.3 from (1.6).

In order to prove (1.6) we shall prove that either $(1.1)_\varrho$ or $(1.3)_\varrho$ is satisfied. As long as $1 \geqq \eta \geqq (n-5)/(n-3)$ we will use $(1.1)_\varrho$ and $(*)_1$. If $n \leqq 5$, this covers all values of η in $(0, 1]$, and $(1.1)_\varrho$ may be satisfied by suitable choice of η. If $n=6$, we may still statisfy $(1.1)_\varrho$ for $0 < \eta < 1/3$, as simple calculations show (let θ be close to 0 and $\delta(n-1)$ close to 1 when $\eta < 1$). For $n > 6$, we will for $\eta < (n-5)/(n-3)$ instead satisfy $(1.3)_\varrho$ and $(*)_s$ with $s < \varrho - 1$ sufficiently close to $\varrho - 1$ (notice that $\varrho < 2$ for $n > 6$). Again, this is easily verified, and we omit the details.

Since $(*)_s$ will imply that inequality A will be valid with $K(t) \in L_1(\mathrm{R}_+)$, we easily obtain from (1.1) or (1.3) and the uniform L_2^1-bound B,

$$\|u(t)\|_{X^{s'}_{p'}} \leqq C(\varphi,\psi)+\int_0^t K(t-\tau)\|f(u)\|_{X^s_p}d\tau \tag{1.7}$$

$$\leqq C+C\int_0^t K(t-\tau)\|u\|^{1-\eta}_{X^{s'}_{p'}}d\tau\ ,$$

where $X^{s'}_{p'}$ is $B^{s',2}_{p'}$ if (1.3) is used and $L^{s'}_{p'}$ if (1.1) is used. The inclusions (0.7) and (1.7) also imply that

$$\sup_{\tau\geqq t}\|u(\tau)\|_{X^{s'}_{p'}} \leqq C\left(1+\sup_{\tau\leqq t}\|u(\tau)\|^{1-\eta}_{X^{s'}_{p'}}\right)$$

and so that (1.6) holds, since $\eta>0$. This completes the proof of Lemma 1.3.

REMARK 1.4. If $r=r(p',s')$, where p',s' satisfy $(*)_{s'}$ then $r>\varrho+1$. To see this, merely observe that since by Remark 1.2, $s>2\delta$,

$$\frac{1}{\varrho+1} > \frac{n-2}{2n} > \frac{n-2(1+s)+2(1+\theta)\delta}{2n} = \frac{1}{r}, \qquad \theta\leqq 1\ .$$

Thus (1.6) is an improvement of the estimate B.

2. On the uniform convergence to 0 at ∞ in $L^{s'}_{p'}$-norm of solutions to NLKG.

For a while we shall assume that

$(**)$ the data φ,ψ for (0.1) have compact supports contained in $|x|\leqq R_0<\infty$.

We will later remove this restriction.

LEMMA 2.1. (Morawetz and Strauss [7]). *Let $(**)$ be satisfied, and let ε_0, T, and S be positive numbers. Then there exists an S_1 depending boundedly on S, T, ε_0 and the energy $E(u)$, but not depending on R_0, and there exists an interval $I=[t^*-2T,t^*]\subseteqq[S,S_1]$ such that*

$$\iint_I F(u(x,t))\,dx\,dt < \varepsilon_0\ .$$

For a proof we refer to Morawetz and Strauss [7, Lemma 3 and pp. 17–18].

LEMMA 2.2. *Assume that $(**)$ and $(*)_s$ hold, with s chosen such that $s<\varrho-1$ and either $(1.1)_\varrho$ or $(1.3)_\varrho$ are satisfied for some $\eta\in(0,1)$. Then for ε, T and S positive numbers, there is an $S_1>S$ independent of R_0 and an interval $I=[t^*-T,t^*]\subseteqq[S,S_1]$ such that*

$$\sup_{t\in I}\|u(t)\|_{p',s'} < \varepsilon\ .$$

PROOF. Let us write the lower bound (iii) for $F(u)$ as

$$F(u) \geqq \beta|u|^{\tilde{\varrho}+1}(1+|u|)^{-N(\tilde{\varrho}+1)}.$$

Let $\sigma>0$ (small and let $r=r(p',s')$, $1/r=1/p'-s'/n$, where p' and s' satisfy $(*)_{s}$. Then

$$\begin{aligned}\|u\|_r^r &= \int (1+|u|)^{-N}\sigma|u|^{\sigma}|u|^{r-\sigma}(1+|u|)^{N\sigma}\,dx \\ &\leqq \left(\int (1+|u|)^{-N(\tilde{\varrho}+1)}|u|^{\tilde{\varrho}+1}\,dx\right)^{\sigma/(\tilde{\varrho}+1)}\left(\int (1+|u|)^{N\sigma q}|u|^{(r-\sigma)q}\,dx\right)^{1/q},\end{aligned}$$

where $1/q=1-\sigma/(\tilde{\varrho}+1)$ is chosen close to 1. In particular, we want $N\varrho q+(r-\sigma)q\leqq\bar{r}$ and $2\leqq(r-\sigma)q$, where also $\bar{r}$ is of the form $\bar{r}(\bar{p}',\bar{s}')$ with $\bar{p}'$ and $\bar{s}'$ satisfying $(*)_{s}$. This is accomplished by choosing $\sigma>0$ small enough. If we now apply (iii) and (1.6), that is the fact that $\sup\|u(t)\|_r<\infty$ for $2\leqq r\leqq\bar{r}$, we obtain

$$\|u(t)\|_r \leqq C\left(\int F(u(x,t))\,dx\right)^{\alpha}, \tag{2.1}$$

where C is independent of t and where $\alpha=\sigma/r(\tilde{\varrho}+1)>0$ is as small as we want by suitable choice of σ.

If we now apply (1.1) or (1.3), using L_r-norms instead of $L_{p'}^{s'}$-norms (as suggested in Remark 1.3) we obtain by using inequality A and the uniform boundedness in L_2^1 (i.e. the energy estimate B) of $u(t)$,

$$\begin{aligned}\|u(t)\|_{X_{p'}^{s'}} &\leqq C(1+t)^{-n\delta}+C\int_0^{t-T}+C\int_{t-T}^{t}K(t-\tau)\|u(t)\|_r^{1-\eta}\,d\tau \\ &\leqq C(1+t)^{-n\delta}+CT^{1-(n-1+\theta)\delta}+C\int_{t-T}^{t}K(t-\tau)\|u(\tau)\|_r^{1-\eta}\,d\tau,\end{aligned}$$

where $X_{p'}^{s'}$ is either $L_{p'}^{s'}$ (if $s=1$) or $B_{p'}^{s',2}$ (if $s<1$). In both cases $X_{p'}^{s'}\subseteqq L_{p'}^{s'}$ by (0.7). Next, since $r=r(p',s')$, where as above p',s' satisfy $(*)_{s'}$ we have by (2.1) that

$$\|u(\tau)\|_r^{1-\eta} \leqq C\left(\int F(u)\,dx\right)^{\alpha(1-\eta)}$$

and so

$$\begin{aligned}\|u(t)\|_{p',s'} \leqq C\|u(t)\|_{X_{p'}^{s'}} &\leqq C(1+t)^{-n\delta}+CT^{1-(n-1+\theta)\delta} \\ &+C\left(\int_0^T K(\tau)^q\,d\tau\right)^{1/q}\left(\int_{t-T}^{t}\int F(u)\,dx\,dt\right)^{\alpha(1-\eta)},\end{aligned}$$

where $1/q=1-\alpha(1-\eta)$ is chosen so close to 1 that $K(t)\in L_q(0,T)$. Let now

$t^* \geqq 2T+S$ in Lemma 2.1. Then, for $t^*-T<t<t^*$ we find that $[t-T,t] \subseteqq [t^*-2T,t^*]$ and hence for t and T large enough

$$\|u(t)\|_{p',s'} \leqq C\varepsilon_0 + C\varepsilon_0^{\alpha(1-\eta)} < \varepsilon$$

for sufficiently small ε_0. This completes the proof of Lemma 2.2.

Next, we shall prove that we actually have uniform convergence of $\|u(t)\|_{p',s'}$ to 0 as $t \to \infty$. In order to do so, we have to sharpen the lower bound for ϱ: Instead of allowing ϱ's in the range $1+2/n<\varrho<1+4/(n-2)$, we will have a new lower bound $1+4/n<\varrho$.

LEMMA 2.3. *Let $(*)_s$ hold with s chosen such that either $(1.2)_\varrho$ or $(1.4)_\varrho$ holds for $\eta>0$ small enough. Then*

$$\lim_{t\to\infty} \|u(t)\|_{p',s'} = 0 .$$

REMARK 2.1. If

$$(2.2) \qquad 1+4\delta < \varrho, \quad \delta(n-1) < 1 \quad \text{and} \quad 1+4/n < \varrho < 1+4/(n-2),$$

then we can find p',s' satisfying $(*)_s$ (and $s=1$ or $s<\varrho-1$ sufficiently close to $\varrho-1$) such that $(1.2)_\varrho$ or $(1.4)_\varrho$ will hold for $\eta>0$ small enough. The verification is simple: Choose $\eta>0$ sufficiently small and let θ run from 1 to 0 in $(*)_s$. Then any δ in $(1/n, 1/(n-1))$ can be obtained, and so any ϱ satisfying (2.2) will satisfy either $(1.2)_\varrho$ or $(1.4)_\varrho$ for suitable δ and s.

PROOF OF LEMMA 2.3. Let $\varepsilon>0$ be so small that, with C_0 given by (2.3) below

$$\varepsilon^\eta C_0 \int_0^\infty K(\tau)\,d\tau < 1/3 .$$

Determine t^* by Lemma 2.2 such that

$$\|u(t)\|_{p',s'} < \varepsilon \quad \text{for } t^*-T \leqq t \leqq t^*$$

and define

$$t^{**} = \sup\{\tau;\ \|u(t)\|_{p',s'} < \varepsilon \quad \text{in } [t^*-T,\tau]\} .$$

If $t^{**}=\infty$ there is nothing to prove. Thus, assume that $t^{**}<\infty$. For ε_1 small (to be specified below), let $t^{**}<t=t^{**}+\varepsilon_1$. Then by the estimate A and (0.7)

$$\|u(t)\|_{p',s'} \leqq C\|u(t)\|_{X^{s'}_{p'}} \leqq C(1+t)^{-n\delta} + \int_0^{t-T} + \int_{t-T}^{t^*}$$

$$+\int_{t^*}^{t} K(t-\tau)\|f(u)\|_{X_p^s}\,d\tau\ ,$$

where as above X_p^s is L_p^1 if $s=1$ and $B_p^{s,2}$ if $s<1$, and with the corresponding choice of $X_{p'}^s$.

The first term is estimated by $c(1+T)^{-n\delta}$, since $t>t^{**}\geqq T$. For the estimates of the following terms we apply (1.2) for $s=1$ and (1.4) for $s<1$:

$$\|f(u)\|_{X_p^s} \leqq C_0\|u\|_{p',s'}^{1+\eta}\ , \tag{2.3}$$

where once more the uniform boundedness of the L_2^1-norm has been used.

We find by inequality A and Lemma 1.3 (or, rather, (1.6)) that the second term is estimated by

$$C\int_0^T (t-\tau)^{-(n-1+\theta)}\|u\|_{p',s'}^{1+\eta}\,d\tau \leqq CT^{1-(n-1+\theta)\delta}$$

and by (2.3) and the integrability of $K(t)$, and our choice of $\varepsilon>0$,

$$\int_{t-T}^{t^{**}} K(t-\tau)\|f(u)\|_{X_p^s}\,d\tau$$

$$\leqq C_0\int_0^\infty K(\tau)\,d\tau \sup\{\|u(\tau)\|_{p',s'}^{1+\eta};\ t-T\leqq t\leqq t^*\} < \varepsilon/3\ ,$$

applying the definition of t^{**}. Finally (1.6) and inequality A imply that

$$\int_{t^{**}}^{t} K(t-\tau)\|f(u)\|_{X_p^s}\,d\tau \leqq C\int_{t^{**}}^{t} (t-\tau)^{-(n-1-\theta)\delta}\|u\|_{p',s'}^{1+\eta}\,d\tau$$

$$\leqq C(t-t^{**})^{1-(n-1-\theta)\delta} \leqq C\varepsilon_1^{1-(n-1-\theta)\delta}\ .$$

Adding these estimates we have

$$\|u(t)\|_{p',s'} \leqq C(1+T)^{-n\delta}+CT^{1-(n-1+\theta)\delta}+\tfrac{1}{3}\varepsilon+C\varepsilon_1^{1-(n-1-\theta)\delta}\ . \tag{2.4}$$

By condition $(*)_{s'}$ $(n-1+\theta)\delta>1>(n-1-\theta)\delta$, and so we may choose T such that the first two terms in (2.4) together are less than $\varepsilon/3$, and we may choose $\varepsilon_1>0$ so small that also the last term in (2.4) is less than $\varepsilon/3$. Altogether, we have

$$\|u(t)\|_{p',s'} < \varepsilon \quad \text{for } t^{**} < t \leqq t^{**}+\varepsilon_1, \quad \varepsilon_1 > 0\ ,$$

which contradicts the maximality of t^{**}. Thus $t^{**}=\infty$, and Lemma 2.3 is proved assuming $(**)$. Let us finally remove this assumption:

Notice first that S_1 in Lemma 2.2, and so t^* in the proof above, can be chosen independently of the supports of the data. Let $v=u_\nu$ be a solution of NLKG with compactly supported data φ_ν, ψ_ν. Assume that φ_ν and ψ_ν

approximate φ and ψ in L_1^k, for k large enough. Choose t^* independently of ν such that $\|u_\nu(t)\|_{p',s'} < \varepsilon$ for $t \geqq t^*$, which can be done by the above proof. Then

$$\|u(t)\|_{p',s'} \leqq \|u_\nu(t)\|_{p',s'} + \|u(t)-u_\nu(t)\|_{p',s'} < \varepsilon + \|u(t)-u_\nu(t)\|_{p',s'} .$$

Lemma 2.3 now follows in general if we can prove that

$$\|u(t)-u_\nu(t)\|_{p',s'} \to 0 \quad \text{as } \nu \to \infty, \text{ each } t \geqq 0 . \tag{2.5}$$

But a simple computation, given in the appendix show that, with $u_\nu = v$,

$$\|u-v\|_{2,1} + \|u-v\|_{p',s'} \leqq C\{\|\varphi-\varphi_\nu\|_{1,k} + \|\psi-\psi_\nu\|_{1,k}\} + C\int_0^t (1+K(t-\tau))\{\|u-v\|_{p',s'} + \|u-v\|_{2,1}\}\,d\tau , \tag{2.6}$$

and so by the standard estimates for Volterra integral equations,

$$\|u-u_\nu(t)\|_{p',s'} \leqq \{\|\varphi-\varphi_\nu\|_{1,k} + \|\psi-\psi_\nu\|_{1,k}\}C(t), \qquad C(t) \in L_\infty^{\text{loc}} ,$$

and so (2.5) follows.

3. Proof of the $L_{p'}$-estimate (0.4).

We begin by proving a well known variant of Gronwall's lemma.

LEMMA 3.1. *Let $K \geqq 0$, $K \in L_1(\mathsf{R}_+)$ and assume that for some $\varepsilon > 0$,*

$$K(\tau) \leqq C(1+\tau)^{-1-\varepsilon}, \qquad \tau \geqq 1 . \tag{3.1}$$

Assume also that $0 \leqq h(\tau) \leqq C$ for $\tau \geqq 0$ and that $h(\tau) \to 0$ as $\tau \to \infty$ and let $1+\varepsilon \geqq \varkappa > 0$. If $f \geqq 0$ is uniformly bounded for $\tau \geqq 0$ and if

$$f(t) \leqq C(1+t)^{-\varkappa} + \int_0^t K(t-\tau)h(\tau)f(\tau)\,d\tau , \tag{3.2}$$

then

$$f(t) \leqq C(1+t)^{-\varkappa}, \qquad t \geqq 0 . \tag{3.3}$$

PROOF. We may assume that $\varkappa = \varepsilon k$ for some integer $k \geqq 1$. Let us first prove that if for some $j \geqq 1$,

$$\int_0^{t/2} K(t-\tau)h(\tau)f(\tau)\,d\tau \leqq C(1+t)^{-\varepsilon j} \tag{α}$$

then, for $j\varepsilon \leqq \varkappa$

$$f(t) \leqq C(1+t)^{-j\varepsilon}, \qquad t \geqq 0 . \tag{β}$$

Once this assertion is proved the lemma follows: To see this, notice that (β) and (3.1) together imply (α) with j replaced by $j+1$. Thus (3.3) follows after a finite number of steps, since (α) certainly holds for $j=1$ by (3.1) and the boundedness of f.

Assume now that (α) holds. Then by (3.2)

$$f(t) \leqq C(1+t)^{-\varkappa}+\int_0^{t/2}+\int_{t/2}^t K(t-\tau)h(\tau)f(\tau)\,d\tau\,.$$

Let $m(t)=\sup_{\tau\geqq t}(1+\tau)^{\varepsilon j}f(\tau)$. Then, since for $\varepsilon j\leqq\varkappa$,

$$(1+t)^{\varepsilon j}f(t) \leqq C+\int_{t/2}^t K(t-\tau)h(\tau)\,d\tau\cdot m(t)\,.$$

Now, since $h(\tau)\to 0$ as $\tau\to\infty$ and $K\in L_1$,

$$\int_{t/2}^t K(t-\tau)h(\tau)\,d\tau \to 0 \qquad \text{as } t\to\infty\,,$$

and so, for some $t_0<\infty$,

$$(1+t)^{\varepsilon j}f(t) \leqq C+\tfrac{1}{2}m(t), \qquad t \geqq t_0\,.$$

Thus,

$$m(t) \leqq C'+\tfrac{1}{2}m(t), \qquad t\geqq 0\,,$$

and hence $m(t)$ is bounded and (β) follows. This completes the proof of the lemma.

We are now in position to give a proof of the decay estimate (0.4), that is: If $\delta=1/2-1/p'\geqq 0$, $\delta(n-1)<1$ and $1+4\delta<\varrho$, with $1+4/n<\varrho<1+4/(n-2)$, then the solution of the NLKG (0.1) satisfies the decay estimate

$$\|u(t)\|_{p'} \leqq C(1+t)^{-(n-1+\theta)\delta} \tag{3.4}$$

where $\theta=1$ if

$$\varrho < 1+\frac{4}{n-2}\cdot\frac{n-\frac{3}{2}}{n-1}\,,$$

and $0<\theta\leqq 1$ for

$$1+\frac{4}{n-2}\cdot\frac{n-\frac{3}{2}}{n-1} \leqq \varrho < 1+\frac{4}{n-1}\cdot\frac{n-1-\frac{1}{2}\theta}{n-1}\,.$$

Proof of (3.4). First, let $\eta>0$ be sufficiently small, such that for the ϱ in question, either $(1.2)_\varrho$ and $(*)_1$ or $(1.4)_\varrho$ and $(*)_{s'}$ with $s<\varrho-1$ sufficiently close to $\varrho-1$, hold. As shown above, such a choice of $\eta>0$, δ (and p',s') is possible. From inequality A and either (1.4) or (1.2) we obtain

$$\|u(t)\|_{p',s'} \leqq C(1+t)^{-n\delta}+C\int_0^t K(t-\tau)\|u(\tau)\|_{p',s'}^{1+\eta}\,d\tau\,.$$

With $f(t)=\|u(t)\|_{p',s'}$ and $h(t)=\|u(t)\|_{p',s'}^{\eta}$, and noticing that $K\in L_1$ satisfies (3.1) for $(n-1+\theta)\delta-1=\varepsilon>0$ by $(*)_{s'}$ we find by Lemma 2.3 that the assumptions in Lemma 3.1 are satisfied with $\varkappa=(n-1+\theta)\delta$. The conclusion is (3.4), provided δ satisfies the above restrictions. The general case now follows by interpolation with the uniform bound for $\|u(t)\|_{2,1}$ which follows from the energy estimate B. This completes the proof of (3.4), that is of the $L_{p'}$-estimate (0.4) in the introduction.

4. Proof fo the $L_{p'}^1$-estimate (0.4)′.

In this section we will prove (0.4)′, that is: If $\delta=1/2-1/p'\geqq 0$, $\delta(n-1)<1$, $1+4\delta<\varrho$, and if $1+4/n<\varrho<1+4/(n-2)$, and in addition $\varrho<1+4n/(n-2)(n+1)$ for $n>10$, then

$$\|u(t)\|_{p',1} \leqq C(1+t)^{-n\delta} \tag{4.1}$$

for the solution u of the NLKG (0.1).

The proof of (4.1) is much more complicated then that of (3.4), in particular for the case $\varrho<2$. The argument in that case is effectively a boot-strapping argument, carried out though several lemmas. The arguments may also be used to prove uniform (that is L_∞) decay estimates as in [4].

LEMMA 4.1. *Let $1/q+1/q'\geqq 2$, and $\bar\delta=1/2-1/q'$ with $\bar\delta(n+1)=1-\varepsilon$, $0\leqq\varepsilon\leqq 1$. If $1+2\bar\delta r\leqq\varrho\leqq 1+(2\bar\delta+\varepsilon/n)r$, then*

$$\|f(u)\|_{q,1} \leqq C\|u\|_r^{\varrho-1}\|u\|_{q',1+\varepsilon}\,.$$

REMARK 4.1. If we in addition assume that $1/r\geqq 1/p'-s'/n$, where p',s' satisfy $(*)_s$ with either $s=1$ or $s<\varrho-1$ sufficiently close to $\varrho-1$, then the above limitations on r and ϱ imply that we may use any ϱ with $\varrho<(n+2)/(n-2)$ for $n\leqq 10$ and with $\varrho<1+4n/(n-2)(n+1)$ for $n>10$. The corresponding computations are identical with those of Lemma 6.3 in [2] for $n\leqq 5$, and Lemma III.5 in [3] for $n\geqq 6$.

In order to shorten some staements below we introduce the notation

$$\varrho_n' = 1+\begin{cases}4/(n-2)\,, & n\leqq 10\,,\\ 4n/(n-2)(n+1), & n>10\,.\end{cases}$$

PROOF OF LEMMA 4.1. By Hölder's inequality and (ii),

$$\|f(u)\|_{q,1} \leqq C\|u\|_r^{\varrho-1}\|u\|_{q',1+\bar{\varepsilon}}$$

with

$$\frac{\varrho-1}{r} = 2\bar{\delta}+\frac{\bar{\varepsilon}}{n}.$$

If we let $\bar{\varepsilon}=0$ we obtain the lower bound for ϱ, and if we let $\bar{\varepsilon}=\varepsilon$, the corresponding upper bound. This completes the proof of the lemma.

The next lemma will be crucial for the proof of (4.1), and will be the starting point of the boot-strapping argument.

LEMMA 4.2. *Assume that* $1+4/n<\varrho<\varrho_n'$. *If* $3\leqq n\leqq 5$ *and* $2<\varrho$, *and if* $\bar{\delta}(n+1)=1-\varepsilon$, $0\leqq\varepsilon\leqq 1$, *with* $\bar{\delta}=1/2-1/q'$, *then*

$$\|u(t)\|_{q',1+\varepsilon} \leqq C(1+t)^{-n\bar{\delta}}.$$

If $n\geqq 6$ *or if* $\varrho\leqq 2$, *the same conclusion is valid for* $\varepsilon=0$, *that is for* $\bar{\delta}(n+1)=1$.

COROLLARY 4.2. *Let* $3\leqq n\leqq 5$, *and* $1+4/n<\varrho<1+4/(n-2)$ *and* $\varrho>2$. *Then*

$$\sup_t \|u(t)\|_{2,2} < \infty .$$

The corollary is an immediate consequence of Lemma 4.2 with $\varepsilon=1$.

PROOF OF LEMMA 4.2. Let the assumptions of Lemma 4.1 be satisfied. Then by inequality A

$$\text{(4.2)}\quad \begin{aligned}\|u(t)\|_{q',1+\varepsilon} &\leqq C(1+t)^{-n\bar{\delta}}+C\int_0^t (t-\tau)^{-(n-1)\bar{\delta}}\|f(u)\|_{q,1}\,d\tau \\ &\leqq C(1+t)^{-n\bar{\delta}}+C\int_0^t (t-\tau)^{-(n-1)\bar{\delta}}\|u(\tau)\|_r^{\varrho-1}\|u(\tau)\|_{q',1+\varepsilon}\,d\tau .\end{aligned}$$

We may now apply (3.4),

$$\text{(4.3)}\qquad \|u(t)\|_r \leqq C(1+t)^{-(n-1+\theta)\delta},$$

provided we also satisfy the limitations

$$(n-1)\delta < 1, \quad 1+4\delta < \varrho, \quad 1+4/n < \varrho < \varrho_n',$$
$$(2n\bar{\delta}+\varepsilon)/n(\varrho-1) \geqq 1/r \geqq 2n\bar{\delta}/n(\varrho-1),$$

under which we will prove that we may choose δ such that

$$(4.4) \qquad \bar{\delta}(n-1)+(\varrho-1)(n-1+\theta)\delta \;=\; \beta \;>\; 1\;.$$

We will choose $\theta=1$ if $\varrho\leqq 1+4/(n-1)$, $\theta>0$ close to 0 if $1+4/(n-1)<\varrho<\varrho_n'$. We also let $1/r=1/2-\delta-\sigma/n$, where $\delta\leqq\delta'=1/2-1/p'$ and $\sigma\leqq s'$, and $(*)_s$ holds for p', s' with $s=1$ if $\varrho>2$, and with $s<\varrho-1$ close enough to $\varrho-1$ for $\varrho<2$. Let us prove (4.4) afterwards, first showing how (4.2), (4.3), and (4.4) will imply the statement of the lemma.

Now, if (4.3) holds we have for $\bar{\delta}>0$

$$(4.5) \quad \int_0^t (t-\tau)^{-(n-1)\bar{\delta}}\|u(t)\|_r^{\varrho-1}\,d\tau$$

$$C\int_0^t (t-\tau)^{-(n-1)\bar{\delta}}(1+\tau)^{-(n-1+\theta)\delta(\varrho-1)}\,d\tau$$

$$= C\left[\int_0^{t/2}+\int_{t/2}^t\right](t-\tau)^{-(n-1)\bar{\delta}}(1+\tau)^{-(n-1+\theta)\delta(\varrho-1)}\,d\tau \;\leqq\; C(1+t)^{1-\beta'}\,,$$

where $\beta'=\beta$ if $(\varrho-1)(n-1+\theta)\delta<1$, $1+(n-1)\bar{\delta}\geqq\beta'>1$ otherwise. If $\bar{\delta}=0$, $\|u\|_r^{\varrho-1}$ is integrable by (4.3) and (4.4), and so $\|u\|_{2,2}=\|u\|_{q',1+\varepsilon}$ is bounded by Gronwall's inequality. If $\bar{\delta}>0$, we have by (4.5)

$$\|u(t)\|_{q',1+\varepsilon} \;\leqq\; C(1+t)^{-n\bar{\delta}}+C(1+t)^{1-\beta'}\sup_{0\leqq\tau\leqq t}\|u(\tau)\|_{q',1+\varepsilon}\,.$$

Since (4.4) holds, and since (4.2) and (4.3) imply that $\|u(t)\|_{q',1+\varepsilon}\in L_\infty^{\mathrm{loc}}(\mathsf{R}_+)$ this gives by (4.2) and (4.3) that
by (4.2) and (4.3) that

$$\|u(t)\|_{q',1+\varepsilon} \;\leqq\; C(1+t)^{-n\bar{\delta}}, \qquad \bar{\delta}(n+1) \;=\; 1-\varepsilon\;.$$

It remains to check (4.4): As above, let $1/r=1/2-\delta-\sigma/n$, where $\delta\leqq\delta'=1/2-1/p'$, $\sigma\leqq s'$, where p' and s' satisfy $(*)_1$ for $\varrho>2$ and $(*)_s$ with $s<\varrho-1$ sufficiently close to $\varrho-1$ for $\varrho<2$. Since $\delta<(\varrho-1)/4$ we may choose s in this way in $(1.4)_\varrho$ for $\eta>0$ sufficiently small.

Let first $\varepsilon=0$, that is $\bar{\delta}(n+1)=1$. Then (4.4) means that if $\varrho\leqq 1+4/(n-1)$, $\delta>2/(n+1)(\varrho-1)n$, and if $\varrho>1+4/(n-1)$, $\delta>2/(n+1)(\varrho-1)(n-1)$. Since $\varrho>1+4/n$, we take $\delta>1/2(n+1)$. Notice that

$$2n\bar{\delta}/n(\varrho-1) \;\leqq\; 1/2-1/2(n+1) \qquad \text{for } \varrho \;>\; 1+4/n\;,$$

while by the remark after Lemma 4.1, we can find p', s' satisfying $(*)_s$ and $\delta'=1/2-1/p'<\min[(\varrho-1)/4, 1/(n-1)]$ such that $1/\tilde{r}=1/p'-s'/n$ is at most $2n\bar{\delta}/n(\varrho-1)$ for $\varrho<\varrho_n'$. Thus we may choose $1/2(n+1)<\delta<\delta'$, $0\leqq\sigma\leqq s'$ such that $1/r\leqq 2n\bar{\delta}/n(\varrho-1)$, $1+4/n<\varrho<\varrho_n'$, $\delta<(\varrho-1)/4$ *and* such that (4.4) holds.

This completes the proof of (4.4) under the given limitations for all $n \geqq 3$, when $\varepsilon = 0$, i.e. $\bar{\delta}(n+1) = 1$.

Next, consider the case $\varrho > 2$ and $n \leqq 5$. As in the proof above, it is enough to find a $\delta_0 \leqq \delta'$ such that $1/2 - \delta_0 \geqq 2n\bar{\delta}/(n-1+\theta)(\varrho-1)$ and $\bar{\delta}(n-1) + (\varrho-1)n\delta_0 = \beta > 1$, i.e. such that

$$(\varrho-1)(n-1+\theta)\delta_0 + (n-1)\bar{\delta} \leqq \tfrac{1}{2}(\varrho-1)n - (n+1)\bar{\delta} ,$$

$$\bar{\delta}(n-1) = (\varrho-1)(n-1+\theta)\delta_0 > 1 .$$

These two inequalities hold for $\varrho > 1 + 4/n$, $\bar{\delta}(n+1) \leqq 1$, provided $(n-1+\theta)\delta_0$ $(\varrho-1)$ is close enough to $1 - \bar{\delta}(n-1)$. Thus it remains to show that $\delta' \geqq \delta_0 > (1 - \bar{\delta}(n-1))/(n-1+\theta)(\varrho-1)$. But for $\varrho > 2$, $\delta(n+1) = 1 - \varepsilon \leqq 1$, the largest value this lower bound may take is $1/n$, and so this inequality holds for all δ' for which $(*)_1$ holds. This completes the proof of (4.4) in case $\varrho > 2$, and $0 \leqq \varepsilon \leqq 1$, and so of Lemma 4.2.

We next prove (4.1) using Corollary 4.2 and the following Lemma.

LEMMA 4.3. *Let* $3 \leqq n \leqq 5$, $2 < \varrho$, $1 + 4/n < \varrho < (n+2)/(n-2)$ *and assume, as before that* $\delta < (\varrho-1)/4$. *Then for* $\eta > 0$ *sufficiently small and* $2 < r \leqq p'$

$$\|f(u)\|_{p,2} \leqq C\|u\|_{2,2}^{\varrho-1-\eta}\|u\|_{p',1+s'}\|u\|_r^{\eta} .$$

PROOF. By assumption we have

$$|\partial_i\partial_j f(u)| \leqq C(|u|^{\varrho-2-\eta}|\partial_i u||\partial_j u||u|^{\eta} + |u|^{\varrho-2-\eta}|u||\partial_i\partial_j u||u|^{\eta}) .$$

Let $0 \leqq \alpha \leqq 2$, $0 \leqq \beta \leqq 1$ and $0 \leqq s \leqq s'$. Then by Hölder's inequality

$$\|f(u)\|_{p,2} \leqq C\|u\|_{2,\alpha}^{\varrho-2-\eta}\|u\|_{2,1+\beta}\|u\|_{p',1+s}\|u\|_r^{\eta}$$
$$+ C\|u\|_{2,\alpha}^{\varrho-2-\eta}\|u\|_{p',\beta+s}\|u\|_{2,2}\|u\|_r^{\eta} ,$$

valid for

$$\frac{1}{p} = (\varrho-2-\eta)\left(\frac{1}{2} - \frac{\alpha}{n}\right) + \frac{1}{2} - \frac{\beta}{n} + \frac{1}{p'} - \frac{s}{n} + \frac{\eta}{r} .$$

A lower bound for ϱ is obtained by letting $\alpha = \beta = s = 0$:

$$\frac{1}{p} = \frac{\varrho-1-\eta}{2} + \frac{1}{p'} + \frac{\eta}{r} .$$

For r close to 2 and $\eta > 0$ small enough, any $\delta < (\varrho-1)/4$ will do. Next, an upper bound for ϱ (and correspondingly a lower bound for δ) is obtained by letting $\alpha = 2$, $\beta = 1$ and $s = s'$,

$$\frac{1}{p} = (\varrho-2-\eta)\max\left(0,\frac{n-4}{2n}\right)+\frac{1}{2}+\frac{1}{p'}-\frac{1+s'}{n}+\frac{\eta}{r}.$$

Thus by Remark 1.2, we find this to imply the bound

$$\delta \geqq (\varrho-2)\left(\frac{n-4}{2n}\right)_+ +\frac{n-6+2\delta(1+\theta)}{2n}+\eta\left(\frac{1}{r}-\left(\frac{n-4}{2n}\right)_+\right).$$

For $\eta>0$ small enough and $n=3$, 4 or 5, this holds trivially for any with $2<\varrho<(n+2)/(n-2)$ and any δ with $0\leqq\delta<\frac{1}{2}$. This completes the proof of the lemma.

PROOF OF (4.1) FOR $3\leqq n\leqq 5$ AND $\varrho>2$. Since by our previous results $\|u(t)\|_{p'} \to 0$ as $t \to \infty$ and since by Corollary 4.2, $\|u(t)\|_{2,2}$ is uniformly bounded for $t\geqq 0$, we may use Lemma 4.3 and then take $h(t)=\|u(t)\|_r^\eta$ and $f(t)=\|u(t)\|_{p',1+s'}$ and use the argument which proved (3.4) from Lemma 3.1. Since, by interpolation $\|u(t)\|_r \to 0$ as $t \to \infty$ for $2<r\leqq p'$, the assumptions of Lemma 3.1 are satisfied, and so (4.1) is proved in case $3\leqq n\leqq 5$ and $\varrho>2$.

We now proceed to discuss the case $\varrho\leqq 2$. We may then only use the second part of the conclusion of Lemma 4.2, that is

$$\|u(t)\|_{q',1} \leqq C(1+t)^{-n\bar{\delta}} \tag{4.5}$$

for $\bar{\delta}(n+1)=1$, $\bar{\delta}=1/2-1/q'\geqq 0$.

LEMMA 4.4. *Let $\bar{\delta}(n+1)=1$, $\delta_r=\bar{\delta}/2$, with $1/r=1/2-\delta_r$, and $s=s'<\varrho-1$. Then for $1+4/n<\varrho<\varrho_n'$ and $|\eta|>0$ small enough*

$$\|f(u)\|_{B_q^{1+s,2}} \leqq C\|u\|_{r,1}^{\varrho-1-\eta}\|u\|_{B_{q'}^{1+s',2}}^{1+\eta},$$

where in addtion $\delta_r(\varrho-1-\eta)n+\bar{\delta}(n-1) > 1$.

COROLLARY 4.4 (Iteration argument). *Let $\bar{\delta}(n+1)=1$ and $s'<\varrho-1$. Then*

$$\|u(t)\|_{q',1+s'} \leqq C(1+t)^{-n\bar{\delta}} \tag{4.6}$$

LEMMA 4.5. *The conclusion of Corollary 4.4 holds for all $\bar{\delta}$ with $0\leqq\bar{\delta}(n+1)\leqq 1$.*

PROOF OF LEMMA 4.5. It is enough to prove (4.6) for $q'=2$ (and $\bar{\delta}=0$). The statement then follows in general from interpolation.

If $\bar{\delta}(n+1)=1$, $1/q'=1/2-\bar{\delta}$, then

$$\|f(u)\|_{2,\sigma} \leqq C\|f(u)\|_{B_q^{1+s,2}},$$

if $1/q-(s+1)/n\leqq 1/2-\sigma/n$, $q<2$, that is if $\sigma\leqq s+1-n\bar{\delta}=s+1/(n+1)$. By Lemma 4.4,

$$\begin{aligned}\|f(u)\|_{B_q^{1+s,2}} &\leqq C\|u\|_{r,1}^{\varrho-1}\|u\|_{B_{q'}^{1+s',2}} \leqq C(1+t)^{-\delta_r(\varrho-1)}(1+t)^{-n\bar{\delta}}\\ &= C(1+t)^{-\gamma}\end{aligned}$$

and, since $\gamma = n\delta_r(\varrho-1)+n\bar{\delta} > 1+\bar{\delta}$,

$$\|u\|_{2,1+\sigma} \leqq C+\int_0^t C\|f(u)\|_{2,\sigma}\,d\tau \leqq C+C\int_0^t (1+t)^{-\gamma}\,d\tau \leqq C < \infty$$

by which

$$\sup_t \|u(t)\|_{2,1+\sigma} \leqq C < \infty ,$$

for any $\sigma < \varrho-1+1/(n+1)$. This proves the lemma.

Proof of Lemma 4.4. Let $|\eta|>0$ be small. For $|\alpha|\leqq 1$ we have

$$\begin{aligned}(4.7)\qquad t^{-s}\|D^\alpha f(u_h)-D^\alpha f(u)\|_q &\leqq Ct^{-s}\|D^\alpha u - D^\alpha u_h\|_{q'}\|u\|_{q',1+s'}^{\eta}\|u\|_{r,1}^{\varrho-1-\eta}\\ &\quad + Ct^{-s}\|u-u_h\|_r^{\varrho-1-\eta}\|D^\alpha u\|_{q',s'}^{1+\eta}\end{aligned}$$

provided $|\eta|>0$ is small enough, and

$$(4.7\text{a})\qquad (1+\eta)\frac{1}{q'}+(\varrho-1-\eta)\frac{1}{r} \geqq \frac{1}{q} \geqq \frac{1}{q'}+\eta\left(\frac{1}{q'}-\frac{1+s'}{n}\right)+(\varrho-1-\eta)\left(\frac{1}{r}-\frac{1}{n}\right)$$

and

$$(4.7\text{b})\qquad (1+\eta)\frac{1}{q'}+(\varrho-1-\eta)\frac{1}{r} \geqq \frac{1}{q} \geqq (1+\eta)\left(\frac{1}{q'}-\frac{s'}{n}\right)+(\varrho-1-\eta)\frac{1}{r} .$$

First, both (4.7a) and (4.7b) will give the same lower bound for the admissible values of ϱ:

$$\begin{aligned}\varrho \geqq 1+2r\bar{\delta}+r\eta\bar{\delta} &= 1+(2\bar{\delta}+\eta\bar{\delta})/(1/2-\delta_r) = 1+(4+2\eta)\frac{\bar{\delta}}{1-\bar{\delta}}\\ &= 1+(4+2\eta)/n ,\end{aligned}$$

that is each $\varrho > 1+4/n$ will be admitted in (4.7) provided $|\eta| = |\eta(\varrho)| > 0$ is chosen small enough. If we let $s = s'$ be sufficiently close to $\varrho-1$, then (4.7b) gives an upper bound for ϱ of the form

$$(\varrho-1)\left(\frac{1}{r}-\frac{1}{n}\right) \leqq 2\bar{\delta}+\eta\left(\frac{1}{r}-\frac{1}{p'}+\frac{s'}{n}\right)-\frac{1}{n}(\varrho-1-s') .$$

The last two terms may be choosen arbitrarily small by suitable choice of $|\eta| > 0$ (small) and $s' < \varrho-1$. Thus we only have to check that

$$2\bar{\delta}\cdot\frac{nr}{n-r}+1 \geqq \varrho_n' .$$

But with our choice of r and $\bar{\delta}$ the right hand sie of this inequality becomes

$$(2\bar{\delta}n+n/r-1)/(n/r-1) \;=\; 1+4n\bar{\delta}/(n-2-\bar{\delta}n) \;=\; 1+\frac{4}{n-2}\,\frac{n\bar{\delta}}{1-n\bar{\delta}/(n-2)}$$

where $n\bar{\delta}>1-n\bar{\delta}/(n-2)$, that is $n\bar{\delta}(1+1/(n-2))>1$, since $\bar{\delta}(n+1)=1$ and $n(n-1)/(n-2)>n+1$ for $n\geqq 3$. Thus

$$2\bar{\delta}\frac{nr}{n-r}+1 \;>\; 1+\frac{4}{n-2} \;\geqq\; \varrho_n' \,,$$

gives an upper bound for those ϱ that are admitted in (4.7b). Notice that (4.7a) gives the same upper bound for ϱ as (4.7b), but without the $(\varrho-1-s')/n$-term, that is an even larger upper bound for ϱ. Thus (4.7) is valid for $s=s'<\varrho-1$ close to $\varrho-1$, and $|\eta|>0$ small enough. Squaring and integrating (4.7) against dt/t over $(0,1)$, we obtain the desired estimate, provided $\varrho-1-\eta>s=s'$, which holds for $|\eta|>0$ small enough.

It remains to verify that

$$\delta_r(\varrho-1-\eta)n+\bar{\delta}(n-1) \;=\; \beta \;>\; 1\,.$$

But by definition

$$\beta \;=\; (\tfrac{1}{2}(\varrho-1-\eta)n+(n-1))\bar{\delta} \;=\; \left((\varrho-1-\eta)\frac{n}{2}-2\right)\Big/(n+1)+1 \;>\; 1\,,$$

if $\varrho>1+4/n+\eta$, which is the case for $|\eta|>0$ small and $\varrho>1+4/n$. This completes the proof of Lemma 4.4.

PROOF OF COROLLARY 4.4. Since (4.6) is already known for $s'=0$ by Lemma 4.2, it is enough to consider the case when s' is close to $\varrho-1$. Let us then apply inequality A and Lemma 4.4:

$$\text{(4.8)}\qquad \|u(t)\|_{B_{q'}^{1+s',2}} \;\leqq\; C(1+t)^{-n\bar{\delta}}+C\int_0^t (t-\tau)^{-(n-1)\bar{\delta}}\|f(u)\|_{B_q^{1+s,2}}\,d\tau$$
$$\leqq\; C(1+t)^{-n\bar{\delta}}+C\int_0^t (t-\tau)^{-(n-1)\bar{\delta}}\|u\|_{r,1}^{\varrho-1-\eta}\|u\|_{B_{q'}^{1+s',2}}^{1+\eta}\,d\tau\,.$$

Now

$$\|u(t)\|_{r,1}^{\varrho-1-\eta} \;\leqq\; C(1+t)^{-n\delta_r(\varrho-1-\eta)}$$

by Lemma 4.2 ($\delta_r=\frac{1}{2}\bar{\delta}\leqq\bar{\delta}$!) and $(n-1)\bar{\delta}+n\delta_r(\varrho-1-\eta)=\beta>1$. With η small and negative, we find that, as before, since

$$\int_0^t (t-\tau)^{-(n-1)\bar{\delta}}(1+\tau)^{n\delta_r(\varrho-1-\eta)}\,d\tau$$

is uniformly bounded, also $\|u(t)\|_{B_{q'}^{1+s',2}}$ is uniformly bounded. Next, (4.8) will then imply that $\|u(t)\|_{B_{q'}^{1+s',2}} \to 0$ as $t \to \infty$, with a rate of at least $(1+t)^{1-\beta}$, where we now have used $\eta>0$ (and small). An obvious iteration argument in (4.8) will now imply (4.6) for s' sufficiently close to $\varrho-1$. As mentioned above this completes the proof of the corollary.

LEMMA 4.6. *Under the above assumptions, if $(*)_s$ holds for $\varrho-1>s$, then with $\delta_q(n+1)=1$,*

$$\|u(t)\|_{p',1+s'} \leqq C(1+t)^{-n\delta_q}\,.$$

PROOF. It follows from $(*)_s$ and the above assumption that is enough to prove the estimate for $s'<\varrho-1-2\delta$. But Sobolev's embedding theorem implies that for any $s'<\varrho-1-2\delta$,

$$\|u(t)\|_{p',1+s'} \leqq C\|u(t)\|_{q',1+\sigma'} \tag{4.9}$$

provided

$$\frac{1}{p'}-\frac{s'}{n} > \frac{1}{q'}-\frac{\sigma'}{n}$$

that is if

$$\sigma' > s'+n(\delta-\delta_q) = s'+\frac{1}{n+1}\,.$$

Since the upper bound for s' is $\varrho-1-2\delta$ and $\delta>1/n$ by $(*)_{s'}$, (4.9) holds for any $\sigma'<\varrho-1$ which is sufficiently close to $\varrho-1$. But for such σ' we have the estimate (4.6) for the right hand side of (4.9), which proves Lemma 4.6.

We are now in position to prove (4.1) also for $\varrho\leqq 2$, that is (0.4)′ in the introduction, using the following lemma:

LEMMA 4.7. *Assume that $\varrho\leqq 2$, that $(*)_s$ holds with $s<\varrho-1$ sufficiently close to $\varrho-1$, and that $\delta(n-1)<1$, $1+4\delta<\varrho$ and $1+4/n<\varrho<\varrho_n'$. Then*

$$\|u(t)\|_{p',1+s'} \leqq C(1+t)^{-n\delta}\,.$$

PROOF. Let $\varrho-1>\sigma>s$. As above we have

$$\|f(u)\|_{B_p^{1+s,2}} \leqq C\|u\|_{q',1+\sigma}\|u\|^{\varrho+1}_{B_{p'}^{1+s',2}}+C\|u\|^{\varrho-1}_{r,1+\sigma}\|u\|\|u\|_{B_{p'}^{1+s',2}} \tag{4.10}$$

provided $\bar{\delta}(n+1) \leqq 1$, $\delta_r(n+1) \leqq 1$ and

(4.10a) $$\frac{1}{q'} + (\varrho - 1)\frac{1}{p'} \geqq \frac{1}{p} \geqq \frac{1}{q'} + (\varrho - 1)\left(\frac{1}{p'} - \frac{s'}{n}\right)$$

(4.10b) $$\frac{1}{r}(\varrho - 1) + \frac{1}{p'} \geqq \frac{1}{p} \geqq (\varrho - 1)\left(\frac{1}{r} - \frac{\sigma}{n}\right) + \frac{1}{p'} - \frac{s'}{n}.$$

From (4.10b) we get the following upper bound on ϱ:

$$\varrho \leqq 1 + \frac{2\delta n + s'}{n} \cdot \frac{rn}{n - r\sigma}$$

and as

$$\frac{r}{n - r\sigma} > \frac{2}{n - 8/n}$$

a straightforward computation gives

$$\varrho \leqq \{(n + 2\delta(n-1))/(n-2) - 2(\varrho - 1 - s)/(n-2)\}$$

while (4.10a) gives correspondingly, with $\bar{\delta} = 1/2 - 1/q'$,

$$(\varrho - 1)\left(\frac{1}{2} - \delta - \frac{s'}{n}\right) \leqq \delta + \bar{\delta},$$

by which

$$\varrho \leqq \frac{n + 2 - 2/(n+1)}{n - 2 - 2(s - (1+\theta)\delta)} = \frac{n+2}{n-2}\gamma_n.$$

It is not difficult to check that if s is sufficiently close to $\varrho - 1 > 4\delta$, then

$$\gamma_n = \frac{1 - 2/(n+2)(n+1)}{1 - 2(s - (1+\theta)\delta)/(n-2)} \geqq 1,$$

since

$$2\delta > \frac{2}{n} \geqq \frac{n-2}{n+2}\frac{1}{n+1}$$

and so, as ϱ is assumed $> 1 + 4\delta$ and $\theta \leqq 1$,

$$\varrho - 1 - (1+\theta)\delta > \frac{n-2}{n+2}\frac{1}{n+1}.$$

Thus the upper bound in both (4.10a) and (4.10b) for the admissible values of ϱ is at most ϱ_n'.

Next, we have to check the lower bounds for the ϱ's appearing in (4.10a) and (4.10b): From (4.10b) we merely get $\varrho \geqq 1+2r\delta$, so that any $r>2$ sufficiently close to 2 will do. Further, (4.10a) requires $\varrho > 1+p'(\delta+\bar{\delta})$, that is

$$(4.11) \qquad (\varrho-1)(\tfrac{1}{2}-\delta) \geqq \delta+\bar{\delta}, \qquad \bar{\delta}(n+1) \leqq 1 .$$

We let $\varepsilon>0$ be such that

$$n\bar{\delta}\varrho \geqq n\delta+\varepsilon, \qquad n\bar{\delta}+(\varrho-1)\delta_r \geqq n\bar{\delta}+\varepsilon ,$$

which is possible since $\varrho>1$ and $\delta_r>0$. We then obtain by Lemmas 4.5 and 4.6 and (4.10) that, since $(*)_s$ holds, and the upper bounds of ϱ above are independent of θ,

$$\|u(t)\|_{B^{s'+1,2}_{p'}} \leqq C(1+t)^{-n\delta}+C\int_0^t K(t-\tau)(1+\tau)^{-n\bar{\delta}-\varepsilon}\,d\tau$$

$$\leqq C(1+t)^{-n\delta}+\int_0^{t/2}(1-\tau)^{-n\delta}(1+\tau)^{-n\bar{\delta}-\varepsilon}\,d\tau+\int_{t/2}^t K(t-\tau)(1+\tau)^{-n\bar{\delta}-\varepsilon}\,d\tau$$

so that

$$\begin{aligned}\|u(t)\|_{B^{s'+1,2}_{p'}} &\leqq C(1+t)^{-n\delta}+C(1+t)^{-n\bar{\delta}-\varepsilon}+(1+t)^{-n\bar{\delta}+1-\varepsilon-(n)}\\ &\leqq C(1+t)^{-n\delta}+C(1+t)^{-n\bar{\delta}-\varepsilon},\end{aligned}$$

where $n\delta-1>0$ by $(*)_s$. We now iterate this procedure, replacing the estimate of $\|u(t)\|_{B^{1+s',2}_{p'}}$ in Lemma 4.6 by the bound

$$\|u(\tau)\|_{B^{1+s',2}_{p'}} \leqq C(1+\tau)^{-n\bar{\delta}-j\varepsilon}+C(1+\tau)^{-n\delta}$$

for $j=1,2,\ldots$.

It remains to satisfy (4.11). But since $\delta<\frac{1}{2}$ we may simply choose $\bar{\delta}$ as close to $(\varrho-1)(1/2-\delta)-\delta$ as possible, satisfying $\bar{\delta}(n+1)\leqq 1$. (Notice that $(\varrho-1)(\frac{1}{2}-\delta)-\delta>\delta(1-4\delta)>0$ for $n\geqq 5$.)

This completes the proof of the lemma.

The proof of (4.1) is now completed by interpolation between the estimate given in Lemma 4.7 and the bound for the L^1_2-norm given by the energy estimate B.

5. Proof of Theorem 1.

We will finally use (0.4) and (0.4)′ to prove Theorem 1. In fact, since (0.4)′ is the more powerful estimate, we will use this result rather than (0.4), which however can be used for dimensions $n=3$ and 4 (cf. [4]).

First a lemma, which connects the $L_{p'}$-estimates with the existence of scattering states:

LEMMA 5.1. *Let u be a solution of* (0.1) *and assume that for some* $r \geqq n(\varrho-\varkappa)/(\varkappa+\frac{1}{2}n(1-\varkappa))$, *where* $0 \leqq \varkappa \leqq 1$,

$$\|u(t)\|_r^{\varrho-\varkappa} \in L_1(\mathbf{R}_+) .$$

Then there exists a unique solution u_+ *of the corresponding linear equation* (0.3) *with data in* $L_2^1 \times L_2$ *such that*

$$\|u(t)-u_+(t)\|_e \leqq C \sup_{\tau \geqq 0} \|u(\tau)\|_e^{\varkappa} \int_t^{\infty} \|u(\tau)\|_r^{\varrho-\varkappa}\, d\tau .$$

In particular,

$$\|u(t)-u_+(t)\|_e \to 0 \quad \text{as } t \to \infty .$$

PROOF. We first notice that

$$\|f(u)\|_2 \leqq C\|u\|_{2,\varepsilon}^{\varkappa}\|u\|_r^{\varrho-\varkappa} \tag{5.1}$$

for $r \geqq n(\varrho-\varkappa)/(\varepsilon\varkappa+\frac{1}{2}n(1-\varkappa))$, $0 \leqq \varepsilon \leqq 1$. We conclude that

$$v(t) = \int_0^{\infty} E_1(t-\tau)f(u(\tau))\, d\tau$$

is a solution of (0.3) with data in $L_2^1 \times L_2$. Denote by u_0 the solution of (0.3) which has the same data as $u(t)$. Then $u_+(t) = u_0(t) - v(t)$ is a solution of (0.3) with data in $L_2^1 \times L_2$ such that

$$u(t)-u_+(t) = \int_t^{\infty} E_1(t-\tau)f(u(\tau))\, d\tau .$$

By the well known energy estimates for (0.3),

$$\|u(t)-u_+(t)\|_e \leqq C \int_t^{\infty} \|f(u(\tau))\|_2\, d\tau$$

which by (5.1) proves the lemma.

PROOF OF THEOREM 1. Let us choose $\varkappa \in [0,1]$ as close to $\varrho-1$ as possible, so that $\varkappa=1$ for $\varrho \geqq 2$ and $\varkappa=\varrho-1$ for $\varrho<2$. If $n\delta(\varrho-\varkappa)>1$ we may use (0.4)′ with $1/r=1/2-\delta-1/n$, and Sobolev's inequality in Lemma 5.1 to conclude that Theorem 1 holds (as $t \to +\infty$; the argument is the same as $t \to -\infty$). Now, choose $\delta>1/n$ close to $(\varrho-1)/4$ as long as $(\varrho-1)/4<1/(n-1)$ and close to $1/(n-1)$ otherwise. Since $1+4/n<\varrho$, $1+4\delta<\varrho$, this choice of δ is possible.

Then, in particular, $n\delta(\varrho-\varkappa)\geqq n\delta>1$. In order to be able to apply Lemma 5.1, it only remains to check the condition

$$r \geqq \frac{n(\varrho-\varkappa)}{\varkappa+\frac{n}{2}(1-\varkappa)}.$$

First, let $\varrho\leqq 2$, so that $\varkappa=\varrho-1$. Then we require

$$r \geqq n\Big/\left(\varrho-1+\frac{n}{2}-\frac{n}{2}(\varrho-1)\right)$$

that is, that

$$(\varrho-1)\left(1-\frac{n}{2}\right)+\frac{n}{2} \geqq \frac{n}{r}. \tag{5.2}$$

As above, choose $1/r=1/2-\delta-1/n$. Then (5.2) follows if

$$(\varrho-1)\left(\frac{1}{2}-\frac{1}{n}\right) < \frac{1}{2}-\frac{1}{r} = \frac{1}{n}+\delta$$

and hence, since $\delta>1/n$, if

$$\varrho < 1+4/(n-2).$$

This last inequality is satisfied for all $\varrho<\varrho_n'$, however. Thus Theorem 1 is proved for $\varrho\leqq 2$.

If $\varrho>2$ (and so $n\leqq 5$), we take $\varkappa=1$, and now have to verify that $r\geqq n(\varrho-1)$. Once more, choose $1/r=1/2-\delta-1/n$. Then the condition becomes

$$n(\varrho-1)\left(\frac{1}{2}-\delta-\frac{1}{n}\right) \leqq 1,$$

that is

$$(\varrho-1)\left(\frac{1}{2}-\frac{1}{n}\right)-\delta(\varrho-1) \leqq \frac{1}{n}.$$

If $\varrho-1<4/(n-1)$ we take δ close to $(\varrho-1)/4$ and so we only have to check that

$$(\varrho-1)\frac{n-2}{2n}-\frac{1}{4}(\varrho-1)^2 < \frac{1}{n}.$$

Let $x=(\varrho-1)/2$ so that this inequality becomes

$$-x^2+x\frac{n-2}{n} < \frac{1}{n}, \quad x = \tfrac{1}{2}(\varrho-1) > \tfrac{1}{2}. \tag{5.3}$$

But the left hand side of (5.3) takes its maximum at $x=1/2-1/n<1/2$, and so is decreasing for $x\geqq 1/2$. Since for $x=1/2$, the left hand side of (5.3) takes the value $1/4-1/n<1/n$ for $n=3$, 4 or 5, this completes the proof of (5.3), and so of the condition $r\geqq n(\varrho-1)$ for $n\leqq 5$, $\varrho>2$ and $(\varrho-1)/4<1/(n-1)$. In case $4/(n-1)\leqq\varrho-1\leqq 4/(n-2)$ (say), we choose δ close to $1/(n-1)$ and now the required estimate is given by

$$(\varrho-1)\left(\frac{1}{2}-\delta-\frac{1}{n}\right) \leqq \frac{1}{n}$$

which is satisfied provided

$$\varrho-1 < \frac{2(n-1)}{n^2-5n+2}\,. \tag{5.4}$$

Since the right hand side of (5.4) is at least $4/(n-2)$ for $n\leqq 6$, this proves that $r\geqq n(\varrho-1)$ also in case $\varrho>2$. The proof of Theorem 1 is complete.

Appendix.

PROOF OF (2.6) IN LEMMA 2.3. By the energy inequality

$$\|u-v\|_{2,1} \leqq C(\|\varphi-\varphi_\nu\|_{2,1}+\|\psi-\psi_\nu\|_{2,1})+C\int_0^t \|f(u)-f(v)\|_2\,d\tau\,,$$

and writing

$$f(u)-f(v) = (u-v)\int_0^1 f'(w(s))\,ds, \qquad w(s) = v+s(u-v)\,, \tag{A1}$$

we have by (ii) and Hölder's inequality that

$$\|f(u)-f(v)\|_2 \leqq C\|u-v\|_{2,1}\int_0^1 \|w(s)\|_r^{\varrho-1}\,d\tau$$

provided $r\geqq n(\varrho-1)$. By Lemma 1.3, and Remark 1.3, $\|w\|_r$ is uniformly bounded for $1/2\geqq 1/r\leqq 1/p'-s'/n$, where p',s' satisfy $((*))_1$ for $n\leqq 5$. Now, for $\varrho<(n+2)/(n-2)$ this holds with $1/r=(n-4+2(1+\theta)\delta)/2n$, with θ and δ suitably chosen satisfying $(*)_1$. Hence

$$\|u-v\|_{2,1} \leqq C(\|\varphi-\varphi_\nu\|_{2,1}+\|\psi-\psi_\nu\|_2)+C\int_0^t \|u-v\|_{2,1}\,d\tau\,.$$

Next by the inequality A, we have for large enough k,

$$\|u-v\|_{p',s'} \leqq C\{\|\varphi-\varphi_\nu\|_{1,k}+\|\psi-\psi_\nu\|_{1,k}\}+\int_0^t K(t-\tau)\|f(u)-f(v)\|_{p,1}\,d\tau\,.$$

If $r\delta\geqq\varrho-1$, then (A1) gives

$$\|f'(w)(u-v)\|_{p,1} \leqq C \sum_{|\alpha|\leqq 1} \{\|w^{\varrho-1}D^\alpha(u-v)\|_p + \|w^{\varrho-2}(D^\alpha w)(u-v)\|_p\}$$
$$\leqq C \sum_{|\alpha|\leqq 1} \{\|w\|_r^{\varrho-1}\|D^\alpha(u-v)\|_2 + C\|w\|_2^{\varrho-1}\|D^\alpha w\|_2\|u-v\|_{p',s'}\}\,.$$

By the above $\delta > 1/n$ and $r \geqq n(\varrho-1)$, and so $r\delta \geqq \varrho - 1$. We have thus proved

$$\text{(A2)}\qquad \|u-v\|_{p',s'} \leqq C\{\|\varphi-\varphi_v\|_{1,k} + \|\psi-\psi_v\|_{1,k}\}$$
$$+\int_0^t K(t-\tau)\{\|u-v\|_{2,1} + \|u-v\|_{p',s'}\}\{\|u\|_{p'}^{\varrho-1} + \|u\|_{2,1}^{\varrho-1} + \|v\|_{p'}^{\varrho-1} + \|v\|_{2,1}^{\varrho-1}\}\,d\tau$$

and with the above estimate for $\|u-v\|_{2,1}$ this completes the proof of (2.6) for $n \leqq 5$.

It remains to consider $n \geqq 6$, that is $\varrho \leqq 2$. As in the proof of Lemma 1.2 (or, rather, of (1.4))

$$t^{-s}\|f'(w_h)(u-v)_h - f'(w)(u-v)\|_p$$
$$\leqq Ct^{-s}\|\,|w_h-w|^{\varrho-1}|u-v|\,\|_p + t^{-s}\|\,|w_h|^{\varrho-1}|u-v|^{\varrho-1}|\dot{u}-v|^{2-\varrho}\|_p$$
$$\leqq C\|w\|_{2,1}^{\varrho-1}\|u-v\|_{p',s'} + C\|w\|_{p'}^{\varrho-1}\|u-v\|_{2,1}^{\varrho-1}\|u-v\|_{p'}^{2-\varrho}$$

provided

$$1+4\delta \leqq \varrho \leqq \varrho_n - 2(\varrho-1-s)/(n-2)$$

where p', s' satisfy $(*)_s$ with s close to $\varrho-1$. Altogether this implies that

$$\text{(A3)}\qquad \|f(u)-f(v)\|_{B_p^{s,2}} \leqq C\{\|u\|_{p',s'}^{\varrho-1} + \|u\|_{2,1}^{\varrho-1} + \|v\|_{p',s'}^{\varrho-1} + \|v\|_{2,1}^{\varrho-1}\}$$
$$\{\|u-v\|_{p',s'} + \|u-v\|_{p',s'}^{2-\varrho}\|u-v\|_{2,1}^{\varrho-1}\}\,.$$

In addition, with $\varkappa = \varrho - 1$, and $r \geqq n/(\varrho - 1 + \frac{1}{2}n(2-\varrho))$, we also have

$$\|f(u)-f(v)\|_2 \leqq C\|u-v\|_{2,1}^{\varkappa}\|u-v\|_r^{1-\varkappa}\int_0^1 \|w(s)\|_r^{\varrho-1}\,ds\,.$$

If $s < \varrho - 1$ is sufficiently close to $\varrho - 1$, we have once more the possibility to choose $1/r \geqq 1/p' - s'/n$, where p', s' satisfy $(*)_{s'}$. Thus, by Sobolev's inequality and the energy inequality, and the uniform bound (1.6),

(A4)

$$\|u-v\|_{2,1} \leqq C\{\|\varphi-\varphi_v\|_{2,1} + \|\psi-\psi_v\|_2\} + C\int_0^t \|u-v\|_{2,1}^{\varrho-1}\|u-v\|_{p',s'}^{2-\varrho}\,d\tau\,.$$

If we now apply inequality A to (A3) and add that result to (A4) we obtain, for k large enough

$$\|u-v\|_{2,1}+\|u-v\|_{p',s'} \leqq C\{\|\varphi-\varphi_\nu\|_{1,k}+\|\psi-\psi_\nu\|_{1,k}\}$$
$$+C\int_0^t (1+K(t-\tau))(\|u-v\|_{p',s'}+\|u-v\|_{2,1}^{\varrho-1}\|u-v\|_{p',s'}^{2-\varrho}\,d\tau\ .$$

Since, for $1<\varrho\leqq 2$,

(A5) $$\|u-v\|_{2,1}^{\varrho-1}\|u-v\|_{p',s'}^{2-\varrho} \leqq (\varrho-1)\|u-v\|_{2,1}+(2-\varrho)\|u-v\|_{p',s'}\ ,$$

we once more get an estimate (A2), and as before, inequality (2.6) follows from this and (A4) combined with (A5), now also for $n\geqq 6$. This completes the proof of (2.6) in general.

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