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ON THE RELATION BETWEEN THE MULTIDIMENSIONAL MOMENT PROBLEM AND THE ONE-DIMENSIONAL MOMENT PROBLEM

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Consider the multi-dimensional moment problem

$$(*) \qquad c_\alpha = \int_{\mathbf{R}^n} x^\alpha \, d\mu(x) \quad \text{for } \alpha \in \mathbf{N}_0^n .$$

Here

$$\alpha = (\alpha_1,\ldots,\alpha_n) \quad \text{is a multi-index}$$

$$\alpha_j \in \mathbf{N}_0 = \{0,1,\ldots\}$$

$$x = (x_1,\ldots,x_n) \in \mathbf{R}^n$$

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n} .$$

Under which conditions on a sequence (c_α) of real numbers, $\alpha \in \mathbf{N}_0^n$, does there exist a positive Radon measure μ on $\mathbf{R}^n$ such that (*) holds?

Generally, a measure μ is not uniquely determined by its moment sequence (c_α). In this paper we show that μ is indeed unique if each of the n coordinate projections $P_i(\mu)$ is known to be uniquely determined as a one-dimensional measure. With an example we also answer the converse question in the negative. Although the positive result was stated by Kilpi in [9], Kilpi's proof did not, in fact, settle the question.

For $x=(x_1,\ldots,x_n) \in \mathbf{R}^n$ we put $\|x\| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$ and let

$$M^*(\mathbf{R}^n) = \left\{\mu \in M^+(\mathbf{R}^n) \;\middle|\; \int_{\mathbf{R}^n} \|x\|^{2m} \, d\mu(x) < \infty \; \forall \, m \in \mathbf{N}_0 \right\},$$

where $M^+(\mathbf{R}^n)$ is the set of all positive Radon measures on $\mathbf{R}^n$.

Let $C(\mathbf{R}^n)$ denote the real vector space of continuous real-valued functions on $\mathbf{R}^n$ and $C_c(\mathbf{R}^n)$ the subspace of continuous functions with compact support. For $f \in C_c(\mathbf{R}^n)$ we define the mapping $\Phi_f \colon M^+(\mathbf{R}^n) \to \mathbf{R}$ by

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$$\Phi_f(\mu) = \int f \, d\mu \, .$$

The weakest topology on $M^+(\mathsf{R}^n)$ in which all the mappings Φ_f, $f \in C_c(\mathsf{R}^n)$, are continuous, is called the vague topology. On $M^*(\mathsf{R}^n)$ we introduce an equivalence relation $\underset{n}{\sim}$ given by

$$\mu \underset{n}{\sim} \nu \Leftrightarrow \int_{\mathsf{R}^n} x^\alpha \, d\mu(x) = \int_{\mathsf{R}} x^\alpha \, d\nu(x) \quad \forall \alpha \in \mathsf{N}_0^n \, ,$$

and the equivalence class containing μ is denoted $[\mu]_n$. The measure μ is said to be determinate if $[\mu]_n = \{\mu\}$.

Let $P_n \subseteqq C(\mathsf{R}^n)$ be the vector space of polynomials in the variables $x_1, \ldots, x_n$, with real coefficients, and let

$$P_n^+ = \{p \in P_n \mid p(x) \geqq 0 \ \forall x \in \mathsf{R}^n\} \, .$$

It is easily seen that P_n is an adapted space in the sense of Choquet [4].

A linear form T on P_n is said to be positive if $T(p) \geqq 0$ for every $p \in P_n^+$. For $\mu \in M^*(\mathsf{R}^n)$ we define $L_\mu(p) = \int p \, d\mu$, $p \in P_n$, and L_μ is a positive linear form on P_n. Conversely we have the following result, which can be found in [4] and in Haviland [8].

THEOREM 1. *To any positive linear form T on P_n there exists $\mu \in M^*(\mathsf{R}^n)$ with $T = L_\mu$.*

Given a positive linear form T on P_n, we consider the measures $\nu \in M^*(\mathsf{R}^n)$, which represent T, that is the set

$$B_T = \{\nu \in M^*(\mathsf{R}^n) \mid T = L_\nu\} \, .$$

In particular, if $\mu \in M^*(\mathsf{R}^n)$, we see that $[\mu]_n = B_{L_\mu}$ and in [4] it is shown that $[\mu]_n$ is a convex, compact subset of $M^+(\mathsf{R}^n)$. Concerning the extreme points of $[\mu]_n$ we have the following useful result, a proof of which may be found e.g. in Douglas [6].

THEOREM 2. *Given $\mu \in M^*(\mathsf{R}^n)$. Then $\nu \in [\mu]_n$ is an extreme point of $[\mu]_n$, if and only if P_n is dense in $L_1(\mathsf{R}^n, \nu)$.*

For $i = 1, \ldots, n$ we define $\varphi_i \colon \mathsf{R}^n \to \mathsf{R}$ by $\varphi_i(x_1, \ldots, x_n) = x_i$. To $\mu \in M^*(\mathsf{R}^n)$ we associate the image measures $\varphi_i(\mu) \in M^*(\mathsf{R})$, $i = 1, \ldots, n$, given by

$$\int_{\mathsf{R}} f \, d\varphi_i(\mu) = \int_{\mathsf{R}^n} f \circ \varphi_i \, d\mu \quad \forall f \in C_c(\mathsf{R}) \, .$$

We can now state our main result:

THEOREM 3. *A measure $\mu \in M^*(\mathsf{R}^n)$ is determinate if the projections $\varphi_i(\mu)$ are determinate for $i=1,\ldots,n$.*

PROOF. Let $\sigma \in [\mu]_n$. For $i=1,\ldots,n$ and $m \in \mathsf{N}_0$ we have

$$\int_{\mathsf{R}} t^m \, d\varphi_i(\sigma)(t) = \int_{\mathsf{R}^n} x_i^m \, d\sigma(x_1,\ldots,x_n)$$
$$= \int_{\mathsf{R}^n} x_i^m \, d\mu(x_1,\ldots,x_n) = \int_{\mathsf{R}} t^m \, d\varphi_i(\mu)(t)$$

so by hypothesis $\varphi_i(\sigma)=\varphi_i(\mu)$.

For $f_1,\ldots,f_n \in C(\mathsf{R})$ let $f=f_1\otimes\ldots\otimes f_n$ denote the function

$$f(x_1,\ldots,x_n) = f_1(x_1)\ldots f_n(x_n),$$

where $(x_1,\ldots,x_n) \in \mathsf{R}^n$. For $f_1,\ldots,f_n \in C_c(\mathsf{R})$ and $p_1,\ldots,p_n \in P_1$ we now have

$$\int_{\mathsf{R}^n} |f_1\otimes\ldots\otimes f_n - p_1\otimes\ldots\otimes p_n| \, d\sigma$$
$$= \int_{\mathsf{R}^n} |(f_1-p_1)\otimes f_2\ldots\otimes f_n + p_1\otimes(f_2-p_2)\otimes f_3\ldots f_n + \ldots$$
$$\ldots + p_1\otimes p_2\ldots\otimes p_{n-1}\otimes(f_n-p_n)| \, d\sigma$$
$$\leqq \int_{\mathsf{R}^n} |(f_1-p_1)\otimes f_2\ldots\otimes f_n| \, d\sigma + \ldots + \int_{\mathsf{R}^n} |p_1\otimes\ldots\otimes p_{n-1}\otimes(f_n-p_n)| \, d\sigma$$
$$\leqq \|f_1\circ\varphi_1 - p_1\circ\varphi_1\|_2 \|1\otimes f_2\ldots\otimes f_n\|^2 + \ldots$$
$$\ldots + \|p_1\otimes\ldots p_{n-1}\otimes 1\|_2 \|f_n\circ\varphi_n - p_n\circ\varphi_n\|_2,$$

where $\|\cdot\|_2$ is the 2-norm with respect to σ. For $i=1,\ldots,n$ we have

$$\|f_i\circ\varphi_i - p_i\circ\varphi_i\|_2^2 = \int_{\mathsf{R}^n} |f_i\circ\varphi_i(x) - p_i\circ\varphi_i(x)|^2 \, d\sigma(x)$$
$$= \int_{\mathsf{R}} |f_i(t)-p_i(t)|^2 \, d\varphi_i(\sigma)(t)$$
$$= \int_{\mathsf{R}} |f_i(t)-p_i(t)|^2 \, d\varphi_i(\mu)(t).$$

Since each $\varphi_i(\mu)$ is determinate, P_1 is dense in $L_2(\mathsf{R},\varphi_i(\mu))$ by the theorem of Riesz, cf. Riesz [11] or Akhiezer [1]. Given any $\varepsilon>0$, we can find $p_1 \in P_1$ so that

$$\|f_1 \circ \varphi_1 - p_1 \circ \varphi_1\|_2 \leqq \frac{\varepsilon}{n\|1 \otimes f_2 \otimes \ldots \otimes f_n\|_2}.$$

We can now find $p_2 \in P_1$ so that

$$\|f_2 \circ \varphi_2 - p_2 \circ \varphi_2\|_2 \leqq \frac{\varepsilon}{n\|p_1 \otimes 1 \otimes f_3 \ldots \otimes f_n\|_2}.$$

Continuing in this way we end up having n polynomials $p_1, \ldots, p_n \in P_1$ so that

$$(**) \qquad \int_{\mathbf{R}^n} |f_1 \otimes \ldots \otimes f_n - p_1 \otimes \ldots \otimes p_n| \, d\sigma < \varepsilon .$$

The set $\{f_1 \otimes \ldots \otimes f_n \mid f_1, \ldots, f_n \in C_c(\mathbf{R})\}$ is dense in $L_1(\sigma)$, and since $p_1 \otimes \ldots \otimes p_n \in P_n$, we see from (**) that P_n is dense in $L_1(\sigma)$, and therefore σ is an extreme point of $[\mu]_n$. Since this is true for any $\sigma \in [\mu]_n$, the set $[\mu]_n$ must be a singleton.

Using Hölder's inequality instead of the Cauchy–Schwarz-inequality the proof of Theorem 3 can be modified to give the following density result:

PROPOSITION. *Given $\mu \in M^*(\mathbf{R}^n)$. If P_1 is dense in $L_p(\mathbf{R}, \varphi_i(\mu))$ for $i = 1, \ldots, n$, then P_n is dense in $L_r(\mathbf{R}^n, \mu)$ for any $1 \leqq r < p$.*

EXAMPLE. That the converse of Theorem 3 is not true, can be seen from the following example: Let $\mu_1, \mu_2 \in M^*(\mathbf{R})$ be two determinate measures with $\mu_1 + \mu_2$ being indeterminate (to see that this is possible, let $\mu = \sum_{n=0}^{\infty} a_n \varepsilon_{x_n}$ be an indeterminate N-extremal measure $M^*(\mathbf{R})$. Then $\mu' = \sum_{n=1}^{\infty} a_n \varepsilon_{x_n}$ is determinate, cf. [1, p. 115] or [2, Theorem 7]). Put $v = \mu_1 \otimes \varepsilon_0 + \mu_2 \otimes \varepsilon_1$. Then $v \in M^*(\mathbf{R}^2)$ with

$$\operatorname{supp}(v) \subseteqq \mathbf{R} \times \{0\} \cup \mathbf{R} \times \{1\} .$$

Taking $\tau \in [v]_2$ we have

$$\int p \, d\tau = \int p \, dv = 0 ,$$

where $p(x_1, x_2) = x_2^2(1 - x_2)^2$. Thus

$$\operatorname{supp}(\tau) \subseteqq \mathbf{R} \times \{0\} \cup \mathbf{R} \times \{1\} .$$

If we put $\tau_1 = \varphi_1(\tau|_{\mathbf{R} \times \{0\}})$ and $\tau_2 = \varphi_1(\tau|_{\mathbf{R} \times \{1\}})$, where $\varphi_1 \colon \mathbf{R}^2 \to \mathbf{R}$ is the projection $\varphi_1(x_1, x_2) = x_1$, it is easy to see that $\tau = \tau_1 \otimes \varepsilon_0 + \tau_2 \otimes \varepsilon_1$. We now have for every $m \in \mathbf{N}_0$,

$$\int_{\mathbf{R}} t^m \, d\tau_1(t) = \int_{\mathbf{R}^2} x_1^m (1 - x_2) \, d\tau(x_1, x_2)$$

$$= \int_{R^2} x_1^m (1-x_2)\, dv(x_1,x_2) = \int_R t^m\, d\mu_1(t)\,,$$

$$\int_R t^m\, d\tau_2(t) = \int_{R^2} x_1^m x_2\, d\tau(x_1,x_2) = \int_{R^2} x_1^m x_2\, dv(x_1,x_2) = \int_R t^m\, d\mu_2(t)\,,$$

and therefore $\tau_1=\mu_1$ and $\tau_2=\mu_2$. Hence $\tau=v$ and v is determinate, but $\varphi_1(v) =\mu_1+\mu_2$ is indeterminate.

In the case of product measures, Theorem 3 may be sharpened:

THEOREM 4. *Given $\mu_1,\ldots,\mu_n \in M^*(R)$ and $p \geqq 1$. Then the product measure $\mu =\mu_1\otimes\ldots\otimes\mu_n$ is determinate if and only if μ_i is determinate for every $i=1,\ldots,n$, and P_n is dense in $L_p(R^n,\mu)$, if and only if P_1 is dense in $L_p(R,\mu_i)$ for $i=1,\ldots,n$.*

PROOF. If $\mu_1 \underset{1}{\sim} v_1$, then $\mu_1\otimes\ldots\otimes\mu_n \underset{n}{\sim} v_1\otimes\ldots\otimes\mu_n$, which shows that μ is indeterminate if μ_1 is indeterminate.

Assume that P_1 is dense in $L_p(R,\mu_i)$ and let $f_1,\ldots,f_n \in C_c(R)$ and $p_1,\ldots,p_n \in P_1$. As in Theorem 3 we have

$$\left(\int |f_1\otimes\ldots\otimes f_n - p_1\otimes\ldots\otimes p_n|^p\, d\mu\right)^{1/p}$$

$$\leqq \|f_1-p_1\|_{\mu_1,p}\|f_2\|_{\mu_2,p}\cdots\|f_n\|_{\mu_n,p}+\ldots+$$

$$+\|p_1\|_{\mu_1,p}\|p_2\|_{\mu_2,p}\cdots\|f_n-p_n\|_{\mu_n,p}$$

where $\|\cdot\|_{\mu_i,p}$ is the p-norm with respect to μ_i. Since

$$\{f_1\otimes\ldots\otimes f_n \mid f_1,\ldots,f_n \in C_c(R)\}$$

is dense in $L_p(R^n,\mu)$, it follows that P_n is dense in $L_p(R^n,\mu)$.

We can without any restriction assume that $\mu_1,\ldots,\mu_n$ are all probability measures, and we will prove that P_1 is dense in $L_p(R,\mu_1)$, if P_n is dense in $L_p(R^n,\mu)$. Let $f \in C_c(R)$ and $\varepsilon>0$. Thus there exists $p \in P_n$ so that

$$\|f\circ\varphi_1 - p\|_{\mu,p} < \varepsilon\,,$$

where $\|\cdot\|_{\mu,p}$ is the p-norm with respect to μ. Setting

$$q(x_1) = \int_{R^{n-1}} p(x_1,\ldots,x_n)\, d(\mu_2\otimes\ldots\otimes\mu_n)(x_2,\ldots,x_n)\,,$$

we have that q is a polynomial in the variable x_1 and from Hölder's inequality we get

$$\|f-q\|_{\mu_1,p}^p = \int_R \left|\int_{R^{n-1}} f(x_1)-p(x_1,\ldots,x_n)\, d\mu_2\otimes\ldots\otimes\mu_n(x_2,\ldots,x_n)\right|^p d\mu_1(x_1)$$

$$\leq \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} |f(x_1)-p(x_1,\ldots,x_n)|^p \, d\mu_2 \otimes \ldots \otimes \mu_n(x_2,\ldots,x_n) \, d\mu_1(x_1)$$

$$= \|f \circ \varphi_1 - p\|_{\mu,p}^p < \varepsilon^p .$$

Hence P_1 is dense in $L_p(\mathbb{R}, \mu_1)$.

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