MATH. SCAND. 51 (1982), 367–381

REPRESENTATIONS OF EXCESSIVE FUNCTIONS

MURALI RAO

Introduction.

In this paper we look at the problem of representing excessive functions as potentials of measures. Various such results are known. When the process has a strong Markov dual and when the process and its dual satisfy certain strong Feller type conditions these representations are presented in Chapter 6 of [1]. Dropping duality assumptions but imposing very mild conditions on the potential kernel, such results are obtained in [4].

In the other direction, Mokobodzki has shown that every excessive function is the integral of extreme ones. However to realize this one needs to enlarge the space. In this paper we will get a representation without enlarging the space.

In this note we show that one can associate a measure with an arbitrary purely excessive function of an arbitrary transient standard Markov process satisfying hypothesis L so that the measure is the Revuz measure provided the excessive function is given by an additive functional. We shall show that with respect to this measure (provided it is finite) there is a kernel $u(\cdot,y)$ such that

1) $u(\cdot,y)$ is excessive for every y,
2) $\int u(\cdot,y)\,m(dy)=s$,
3) $u(\cdot,y)$ has support y in the sense $P_D\,u(\cdot,y)=u(\cdot,y)$ for every open set D containing y.

We show that extreme purely excessive functions have point supports. An example due to H. and U. Schirmeier is presented at the end to show that excessive functions with point supports need not be extreme.

Under certain assumptions on the excessive function we can show the following: Whenever Uf_n increases to s, $f_n(x)dx$ converges weakly to the measure associate with s.

Section 1 closely follows Meyer [3]. Notation will generally be that of [1]. Some comments and problems are presented at the end.

Received July 2, 1981.

1.

We assume that the kernel U is proper so that every excessive function is the increasing limit of a sequence Uf_n.

PROPOSITION 1. *Let s be excessive and finite almost everywhere. Then for every $\alpha > 0$, there exists a unique α-excessive function s^α such that*

$$s = s^\alpha + \alpha U^\alpha s \,. \tag{1}$$

Further for $0 < \alpha < \beta$,

$$\begin{aligned} s^\alpha &= s^\beta + (\beta - \alpha) U^\beta s^\alpha \\ &= s^\beta + (\beta - \alpha) U^\alpha s^\beta \,. \end{aligned} \tag{2}$$

In particular

$$U^\alpha s^\beta = U^\beta s^\alpha \,.$$

PROOF. Let Uf_n increase to s. The resolvent equation

$$Uf_n = U^\alpha f_n + \alpha U^\alpha U f_n \tag{3}$$

and monotone convergence theorem show that $\lim U^\alpha f_n$ exists at every point at which s is finite. Denote by s^α the regularization of the α-supermedian function

$$\liminf U^\alpha f_n \,.$$

Taking limits in (3) one gets

$$s = s^\alpha + \alpha U^\alpha s \tag{4}$$

because identity almost everywhere of α-excessive function implies equality everywhere.

Now let $\alpha < \beta$. We know $U^\alpha f_n$ converges almost everywhere. Since $U^\alpha f_n \leqq U f_n \leqq s$ and $\beta U^\beta s \leqq s$, at each point at which s is finite, we can use the resolvent equation and dominated convergence to get (2).

NOTE. It is easy to choose a density $u(x, \cdot)$ of $U(x, \cdot)$ such that for every y,

$$x \to u(x, y)$$

is excessive.

Apply Proposition 1 above to this excessive function: We get $u^\alpha(x, y)$ such that

$$u(x,y) = u^\alpha(x,y)+\alpha U^\alpha u(x,y)$$
$$= u^\alpha(x,y)+\alpha U u^\alpha(x,y) .$$

This shows that $u^\alpha(x,\cdot)$ is a density for U^α and that if we define

$$\hat{U}^\alpha g(y) \quad \text{by} \quad \int U^\alpha(x,y)g(x)\,dx ,$$

$\hat{U}^\alpha$ is a resolvent dual to U^α. This will be useful later. Note also that with respect to this dual resolvent, $y \to u(x,y)$ is co-excessive for every x.

PROPOSITION 2. *Let s and s^α be as in Proposition 1. Then $(\alpha s^\alpha, \xi)$ increases with α.*

This follows using the second part of (2) and the excessivity of ξ.

NOTE 1. s^α decreases as α increases. This together with the second part of (2) show that either $(s^\alpha,\xi)\equiv\infty$ or finite for all $\alpha>0$.

NOTE 2. If $s^0=\lim_{\alpha\to 0} s^\alpha$ and $h=\lim_{\alpha\to 0}\alpha U^\alpha s$, then s^0 is the purely excessive part of s and h is the "regular" part. And we have for all $\alpha>0$,

$$s^0 = s^\alpha+\alpha U^\alpha s^0$$
$$= s^\alpha+\alpha U s^\alpha .$$

DEFINITION. $L(s)=\lim \alpha(s^\alpha,\xi)$.

By (5) we have $L(s)=L(s^0)$.

PROPOSITION 3. *Let s be purely excessive and $s \geqq Uf$. Then for all $\beta>0$*

$$(s^\beta,\xi) \geqq (U^\beta f,\xi) .$$

PROOF. Let $0<a<1$. Since s^α increases to s as α ends to zero, the sets

$$A_\alpha = (s^\alpha \geqq \alpha U^\alpha f)$$

have the property $\liminf A_\alpha = E$ almost everywhere. By Note 1 we may assume $(s^\alpha,\xi)<\infty$ for all α. Now $s^\alpha \geqq \alpha U^\alpha(f 1_{A_\alpha})$ everywhere. Using the first part of (2) and a similar one for $U^\alpha(f 1_{A_\alpha})$, we find

$$(s^\beta,\xi) \geqq (\alpha U^\beta(f 1_A),\xi) \quad \beta>\alpha . \tag{7}$$

Let α decrease to zero in (7) and then let a increase to 1 to get (6).

COROLLARY 4. *If s is purely excessive and $s \geqq Uf$, then $L(s) \geqq (f, \xi)$.*

This follows from the definition of $L(s)$ and the above proposition.

NOTE 3. If Uf_n increases to s, then $\lim U^\alpha f_n = s^\alpha$ almost everywhere. Hence from Proposition 3

$$\lim (|s^\alpha - U^\alpha f_n|, \xi) = 0 \tag{8}$$

provided $(s^\alpha, \xi) < \infty$.

COROLLARY 5. *Let s be purely excessive and $s \geqq t$. Then for all $\alpha > 0$*

$$(s^\alpha, \xi) \geqq (t^\alpha, \xi) \tag{9}$$

In particular $L(s) \geqq L(t)$.

PROOF. Let Ug_n increase to t. Then $Ug_n \leqq s$, and therefore $(U^\alpha g_n, \xi) \leqq (s^\alpha, \xi)$. Now use Note 3 above.

We will use the following Corollary later.

COROLLARY 6. *Let s and s_n be purely excessive and* $\lim s_n = s$ *almost everywhere. Then*

$$L(s) \leqq \liminf L(s_n) .$$

In particular if $s_n \leqq s$, and $\lim s_n = s$ *almost everywhere then*

$$\lim L(s_n) = L(s) .$$

We omit the proof.

2.

In this article we associate a measure to an excessive function. In the following s denotes a fixed excessive function. For two functions A and B, (A, B) will denote (AB, ξ). Choose a positive bounded function g so that

$$(1+s, g) < \infty . \tag{1}$$

$\hat{U}^\alpha$ denotes the dual resolvent. This always exists. We have

$$(\alpha \hat{U}^\alpha g, s+1) = (g, \alpha U^\alpha (s+1)) \leqq (g, s+1) < \infty . \tag{2}$$

So for suitable numbers

$$\left(\sum_{\alpha\in Q} \varepsilon_\alpha \alpha \hat{U}^a g, s+1\right) < \infty , \tag{3}$$

where Q denotes the set of rationals. Denote by μ the finite measure whose density is the first function in (3).

Let now $Uf_n \leqq s$ with limit s. Since $(s,\mu)<\infty$, Uf_n is uniformly integrable relative to μ. For any continuous bounded function φ the sequence $U(f_n\varphi)$ is also uniformly integrable. Denote by Y the space of continuous functions on the one-point compactification of the state space. Y is separable. By choosing a subsequence of necessary we may assume that

$$\lim_n \left(U(f_n\varphi)\varrho, \mu\right) = (\varrho s_\varphi, \mu) \tag{4}$$

for a function s_φ all bounded functions ϱ and all $\varphi \in Y$. For each $\alpha \in Q$ and each $0 \leqq \varrho \leqq 1$, the measure $\varepsilon_\alpha(\alpha\hat{U}^\alpha(g\varrho))d\xi$ is less or equal to μ. From (4)

$$\lim_n \varepsilon_\alpha \int \alpha\hat{U}^\alpha(g\varrho)U(f_n\varphi)\,d\xi = \varepsilon_\alpha \int \alpha\hat{U}^\alpha(g\varrho)s_\varphi\,d\xi . \tag{5}$$

Let φ be non-negative. We can rewrite (5) as

$$\begin{aligned}(\alpha U^\alpha s_\varphi, g\varrho) &= \lim_n \left(\alpha U^\alpha U(f_n\varphi), g\varrho\right) \\ &\leqq \lim_n \left(U(f_n\varphi), g\varrho\right) = (s_\varphi, g\varrho) .\end{aligned}$$

The last equality following because the measure $g\varrho\, d\xi$ is dominated by μ. One concludes that $\alpha U^\alpha s_\varphi \leqq s_\varphi$ almost surely for all rational α. Thus s_φ is equal almost everywhere to an excessive function. This we again denote by s_φ. Clearly we have $s_\varphi + s_{1-\varphi} = s$. We have proved

PROPOSITION 1. *To every b non-negative continuous function φ on the one point compactification of E corresponds an excessive function s_φ. The msp $\varphi \to s_\varphi$ is linear and $s_1 = s$.*

If $s(x)<\infty$, so is $s_\varphi(x)$ for all $\varphi \in Y$. For each such x, corresponds a measure $L(x,dy)$ on $E \cup \infty$ such that

$$s_\varphi(x) = \int \varphi(y)L(x,dy) \tag{6}$$

Let

$$A = \{x : s(x)<\infty\} .$$

We proceed to define s_f for each positive measurable function f on $E \cup \infty$. First let f be lower semi continuous. Define

$$s_f = \sup_{\varphi \leqq f} s_\varphi, \qquad \varphi \in Y.$$

Clearly s_f is excessive. If $x \in A$ and f is lower semi continuous, we must have

$$s_f(x) = \int f(y)L(x,dy).$$

If g is non-negative and measurable, define

$$s_g = \overline{\inf_{f \geqq g} s_f} \qquad \text{and } f \text{ l.s.c.},$$

the top bar indicating regularization. Hypothesis (L) makes things work. If $x \in A$ we must have

$$\inf_{f \geqq g} s_f = \int g(y)L(x,dy),$$

so that

$$s_g = \int g(y)L(x,dy) \qquad \text{almost all } x.$$

It follows that $s_{f+g} = s_f + s_g$ almost everywhere and hence identically. Also if $b_n \uparrow b$, so does s_{b_n} increase to s_b.

The map

$$A \to s_{1_A}(x) = s(A,x)$$

determines a kernel such that

$$s(A,\cdot) \qquad \text{is excessive for every } A,$$

$$s(\cdot,x) \qquad \text{is a measure for every } x,$$

$$s(E \cup \infty, x) = s.$$

We have thus associated a kernel with every excessive function which is finite almost everywhere. This kernel gives rise to a measure in the obvious way:

Define for each $A \subset E \cup \infty$

$$\nu(A) = L(s_A),$$

where L is defined in section 1. It is obvious that ν is a measure. In case s is the potential of a natural additive functional, ν is simply the Revuz measure of the natural additive functional.

NOTE. If s is purely excessive, the definition of $L(s)$ shows that $L(s) = 0$, iff

$s=0$. If s is purely excessive, so are $s(A, \cdot)$ for all sets A. $\nu(A)=0$ implies then that $s(A, \cdot)=0$. But if $s(x)<\infty$

$$s(A,x) = L(x,A)$$

That is to say: The measure $L(x, \cdot)$ is absolutely continuous relative to ν if $s(x)<0$.

3.

We show in this article that when ever $Uf_n \leqq a$ purely excessive function s with limit s, then for a subsequence

$$\lim U(f_n\varphi)(x) = s_\varphi(x)$$

exists everywhere along the subsequence at every point at which $s(x)<\infty$ and for every function φ which is between 0 and 1 and is continuous on the one point compactification $E \cup \infty$ of E.

We keep the notation of section 2. Let $Uf_n \leqq s$ with limit s. It was shown in section 2 that by choosing a subsequence if necessary we may assume

$$\text{(1)} \qquad \lim_n \int U(f_n\varphi)\,\varrho d\mu = \int \varrho s_\varphi \, d\mu$$

with s_φ excessive, for every bounded measurable ϱ. Recall that μ is the measure with density

$$\sum_{\alpha \in Q} \varepsilon_\alpha \alpha \hat{U}^\alpha g \, .$$

Let $0 \leqq \varrho \leqq 1$. For each $\alpha \in Q$, $\hat{U}^\alpha(g\varrho)d\xi \leqq d\mu$.

Therefore

$$\text{(2)} \qquad \lim \int U(f_n\varphi)\hat{U}^\alpha(g\varrho)\, d\xi = \int s_\varphi \hat{U}^\alpha(g\varrho)\, d\xi \, .$$

which is the same as

$$\text{(3)} \qquad \lim \int U^\alpha U(f_n\varphi) g\varrho \, d\xi = \int U^\alpha(s_\varphi) g\varrho \, d\xi \, .$$

Recall that there is a unique α-excessive function s_φ^α such that

$$\text{(4)} \qquad s_\varphi = s_\varphi^\alpha + \alpha U^\alpha s_\varphi$$

using (3) and (4), for all $\alpha \in Q$:

$$\text{(5)} \qquad \lim \int U^\alpha(f_n\varphi) g\varrho \, d\xi = \int s_\varphi^\alpha \cdot g\varrho \, d\xi \, .$$

It is possible to choose a version of the density u of U such that for every x_0

$$y \to u(x_0, y) = q(y)$$

is co-excessive, namely

$$q = \text{increasing limit as } \alpha \text{ tends to infinity of } \alpha \hat{U}^\alpha q .$$

For any n and $\alpha \in Q$,

$$(qf_n\varphi, \xi) \geqq (f_n\varphi\alpha\hat{U}^\alpha q, \xi) = \alpha(q, U^\alpha(f_n\varphi))$$
$$\geqq \alpha(q \wedge Ng, U^\alpha(f_n\varphi)) .$$

Since $q \wedge Ng$ is of the form $g\varrho$ with $0 \leqq \varrho \leqq N$ we see from what we have already proved that the right side of last inequality tends to $\alpha(q \wedge Ng, s_\varphi)$. Namely

$$\liminf (qf_n\varphi, \xi) \geqq \alpha(s_\varphi^\alpha, q \wedge Ng) .$$

Since $g > 0$, letting N increase to infinity we obtain

$$\liminf (qf_n\varphi, \xi) \geqq \alpha(s_\varphi^\alpha, q)$$
$$= \alpha U(s_\varphi^\alpha)(x_0) \quad \text{by (5) of section 1 .}$$
$$= \alpha(U^\alpha s_\varphi)(x_0)$$

because s being purely excessive, so is s_φ. Now let α tend to ∞, note that s_φ is excessive and recall the definition of q to get

$$\liminf U(f_n\varphi)(x_0) \geqq s_\varphi(x_0) .$$

Because Uf_n tends to $s = s_\varphi + s_{1-\varphi}$, the last must be an equality whenever $s(x_0) < \infty$. Thus we have proved

THEOREM 1. *Let $Uf_n \leqq$ a purely excessive function s with limit s. There exists a subsequence such that*

$$\lim U(f_n\varphi)(x) = s_\varphi(x) \tag{6}$$

along the subsequence for all non-negative continuous functions φ on the point compactification of E and all x at which $s(x) < \infty$.

Theorem 1 permits another look at the measure ν of a purely excessive function.

Suppose $L(s) < \infty$. Then $\varphi \to L(s_\varphi)$ defined a measure, which we called ν in section 2. Let Ug_n and Uh_n increase to the excessive functions s_φ and $s_{1-\varphi}$. Since $\lim U(f_n\varphi) = s_\varphi$, we see from Corollary 6 of section 1

(7) $$\begin{cases} \lim (g_n, \xi) = L(s_\varphi) \leqq \liminf (f_n\varphi, \xi) \\ \lim (h_n, \xi) = L(s_{1-\varphi}) \leqq \liminf (f_n(1-\varphi), \xi) \end{cases}$$

and

$$\lim (g_n + h_n, \xi) = L(s) = \lim (f_n, \xi) .$$

We therefore have

$$\nu(\varphi) = L(s_\varphi) = \lim (f_n\varphi, \xi) .$$

4.

One property of s_f (see section 2 for its definition) is the following: If D is open and f vanishes off D, then

(1) $$p_D s_f = s_f .$$

To see this it is sufficient to assume $f = 1_D$. We know

(2) $$s_D = \sup s_\varphi$$

where the sup is overall $0 \leqq \varphi \leqq 1$ continuous on $E \cup \infty$ and vanishing off D. For each such φ,

$$\lim U(f_n\varphi)(x) = s_\varphi(x) \quad \text{if } s(x) < \infty$$

and

$$p_D(Uf_n\varphi) = U(f_n\varphi) .$$

Since $Uf_n\varphi \leqq s$, by dominated convergence, at each point x at which $s(x) < \infty$, we have

(3) $$p_D s_\varphi = s_\varphi ,$$

and hence (3) holds identically. (2) and (3) imply that (1) holds when f in the indicator of D.

DEFINITION. An excessive function s is called a potential if $p_{K^c}s$ decreases to zero almost everywhere as the compacts K increase to the state space E. It is said to have point support y if there is a point $y \in E$ such that

$$p_D s = s$$

for each openset D containing y.

THEOREM 1. *Let s be a purely excessive potential for which $L(s) < \infty$. Then there is a measure ν and a kernel v such that*

$$s = \int v(\cdot, y)\, \nu(dy) \tag{4}$$

with $v(\cdot, y)$ excessive having point support y for every y.

PROOF. Recall the notation of section 2. $s(\infty, \cdot)$ is excessive and for each compact set K,

$$p_{K^c} s(\infty, \cdot) = s(\infty, \cdot) \tag{5}$$

s being potential, (5) shows that $s(\infty, \cdot) \equiv 0$.

ν is a finite measure on $E \cup \infty$ and it cannot have any mass at ∞ since $s(\infty, \cdot) \equiv 0$. Thus ν is a measure on E. For each x, $s(\cdot, x)$ is absolutely continuous relative to ν. Using (1) and standard arguments the proof is completed.

THEOREM 2. *Let s be purely excessive and extreme. Then either s has point support or is harmonic* i.e. *$p_{K^c} s = s$ for each compact set K.*

PROOF. Suppose s is not harmonic. Then there is a compact set K such that $s \neq p_{K^c} s$. Now

$$s = s_K + s_{K^c}$$

because s is extreme, s_{K^c} is a constant multiple of s. Because $p_{K^c} s_{K^c} = s_{K^c}$, the same will hold with s_{K^c} replaced by s if this constant were not zero. Namely $s_{K^c} \equiv 0$ and $s = s_K$.

We claim there is a $y \in K$ such that $p_D s = s$ for all open D containing y.

If this were not the case, for each $y \in K$ we can find an open set containing y such that $p_D s \neq s$. Using compactness, we can thus choose open sets $D_1, \ldots, D_n$ covering K such that $p_{D_j} s \neq s$. But

$$\begin{aligned} s = s_K &= s_{D_1 \cup \ldots \cup D_n} \\ &= s_{D_1} + s_{D_2 \setminus D_1} + \ldots \\ &= s_1 + s_2 + \ldots + s_n, \text{ say} . \end{aligned}$$

Since s is extreme each of these is a multiple of s and at least one of these is not zero. If,

$$s_i = \alpha_i s \quad \text{with} \quad \alpha_i \neq 0$$

we have

$$\alpha_i s = s_i = p_{D_i} s_i = \alpha_i p_{D_i} s$$

implying

$$p_{D_t} s = s$$

which is a contradiction. This proves the Theorem.

We now look at a condition which guarantees that the measure of section 2 does not depend on the particular sequence Uf_n which increases to s. For this let us assume $L(s) < \infty$. Define the stopping time T (depending on r) by

$$T = \inf(t : |x_t - x_0| \geqq r)$$

and let T_n, $n \geqq 1$ denote the successive iterates of T. We shall call s a pure potential, if $p_{T_n} s$ decreases to zero almost everywhere as n increases to infinity. We then have

THEOREM 1. *If s is a pure potential whenever Uf_n increases to s the sequence of measures $f_n(x)\,dx$ converges weakly to v.*

A proof can be modelled using Theorem 8 of 2.

5. Complements.

We assume below that we have a Hunt process and that points are polar.

DEFINITION. An excessive function s is said to be of class (D) if $P_{T_n} s$ decreases to zero almost everywhere. Here T_n is the hitting time to the set $(s > n)$.

LEMMA 1. *Let s be excessive and finite almost everywhere. We can write*

$$s = g + h \tag{1}$$

where g is of class (D) *and h satisfies*

$$P_O h = h$$

where O is any set of the form

$$O = (h > n) \quad \text{for some } n.$$

PROOF. Let O_n denote the set $(s > n)$. Let $a = \lim P_{O_n} s$. By Exercise 3.20 p. 84 of [1], a is excessive at every point x at which $a(x) < \infty$. Let h denote the excessive regularization of the supermedian function a. It can be shown that

$$P_{O_n} h = h .$$

Indeed $P_{O_n} a(x)$ equals $a(x)$ wherever the latter is finite. Thus if $T_n = T_{O_n}$ and $a(x)$ is finite $h(x) = a(x)$ and $h(x_{T_n}) = a(x_{T_n})$, Px – a.s. Therefore

(3) $$P_{O_n}h(x) = h(x) \quad \text{if } a(x)<\infty$$

and hence identically.

Now let

$$b(x) = s(x)-h(x) \quad \text{if } s(x) < \infty$$
$$= \infty \quad \text{if } s(x)=\infty .$$

An argument similar to that in [2] shows that b is supermedian and that its excessive regularization g satisfies

(4) $$s = g+h .$$

(3) and the defintion of h show that $P_{O_n}g$ decreases to zero almost everywhere. It is the clear that g is a class (D) potential.

Now apply the argument to h to write

(5) $$h = g_1+h_1$$

with g_1 class (D) and

(6) $$h_1 = P_{(h>n)}h_1 \quad \text{all } n .$$

From (3) and (5)

(7) $$P_{O_n}g_1 = g_1 \quad \text{all } n .$$

But g_1 being class (D), (7) forces $g_1=0$. We find

(8) $$P_{(h>n)}h = h \quad \text{all } n .$$

This proves the Lemma.

COROLLARY 2. *Let the excessive function s have point support. Then*

$$s = P_{(s>n)}s \quad \textit{for every } n .$$

The proof is clear from Lemma 1.

Let s be purely excessive, let A be a thin compact set, and let T_n be the successive hitting times to A. Suppose $P_{T_n}s$ decreases to zero a.e. then

$$P_As \quad \text{is of class D} .$$

PROOF. Let S_k be stopping times increasing to ∞. Let $s(x)<\infty$ and $\varepsilon>0$ be given. Choose n so that $P_ns(x)$ and $P_{T_n}s(x)\leqq\varepsilon$. Then for any k,

$$E^x[P_Ts(x_{S_k}):\ S_k\geqq T_n\wedge n] \leqq P_{T_n}s(x)+P_ns(x) \leqq 2\varepsilon$$

$$E^x[P_T s(x_{S_k}) : S_k < T_n \wedge n]$$

$$= \sum_{i=1}^{n} E^x[s(x_{S_k+T(\theta_{s_k})}) : T_{i-1} \leqq S_k < T_i,\ S_k < n] \qquad (T_0=0,\ T_1=T)$$

$$= \sum_{i=1}^{n} E^x[s(x_{T_i}) : T_{i-1} \leqq S_k < T_i,\ S_k < n]$$

(because on the set $T_{i-1} \leqq S < T_i$, $S_k + T(\theta_{S_k})$ is just the first hit to A after T_{i-1}, which is T_i)

$$\leqq \sum_{i=1}^{n} E^x[s(x_{T_i}) : S_k < n] .$$

Since n is fixed, as k gets large, this sum is small, since $\sum_1^n s(x_{T_i})$ is integrable. Thus

$$\lim E^x[P_T s(x_{S_k})] \to 0 .$$

Remark. That $P_t s$ is class D is proved similarly. Indeed, choose n so that $P_{nt} s(x) < \varepsilon$. Then

$$E^x[s(x_{S_k+t})] = E^x[s(x_{S_k+t}) : S_k \leqq n] + E^x[s(x_{S_k+t}) : S_k > n]$$
$$\leqq E^x[s(x_t) : S_k \leqq n] + E^x[s(x_n)] \to 0 .$$

We can say more when the process is strong Feller:

$P_t S$ is a regular excessive function for every excessive s if the semigroup is strong Feller.

The proof is very simple $P_{t/2} s$ is of class (D). So

$$P_{t/s} s = \sum s_n$$

where s_n are bounded excessive. So

$$P_t s = \sum P_{t/s} s_n$$

$P_{t/s} s_n$ is continuous, because s_n is bounded. Hence it is regular. This implies that, $P_t s$ is regular.

It can be shown that every class (D) purely excessive function can be written as a sum $\sum S_i$ with $L(S_i) < \infty$. The following example shows that it is in general not possible to do so, if the restriction class (D) is dropped.

Example. Let D be the unit ball in three spaces and $S=1$. It is clearly sufficient to show that $L(S)=\infty$. Indeed, if Gf_n increases to 1, G denoting the Green function we can find N such that

$$\int Gf_n(x)\,dx \geqq \tfrac{1}{2}M(D), \qquad n \geqq N\,,$$

where $M(D)$ is the volume of D.

$$\int Gf_n(x)\,dx = \int f_n(y)g(y)\,dy\,,$$

where $g(y)=\int g(x,y)\,dx$ tends to zero uniformly as $|y|$ tends to 1. Also $f_n(x)dx$ tends to zero vaguely. It must then be true that $\int f_n(y)\,dy$ tends to infinity i.e. $L(1)=\infty$.

The following standard counterexample (communicated for instance by H. and U. Schirmeirer) shows that excessive functions with point supports need not be extreme.

EXAMPLE. Consider the Brownian motion on $(-1,1)$ with reflection to the right at 0. The functions

$$P_1(x) = \begin{cases} 1+x & x<0 \\ 0 & x\geqq 0 \end{cases}$$

$$P_2(x) = \begin{cases} 1+x & x<0 \\ 1-x & x\geqq 0 \end{cases}$$

and $\frac{1}{2}(P_1+P_2)$ are excessive with support zero.

Here are some questions whose answers would have greatly improved this note.

1. Is it possible that an excessive function can have two different point supports? If this cannot happen one can conclude the following: If $L(S)<\infty$ there is a unique measure ν such that whenever f_n increases to s, $f_n(x)dx$ converges weakly to ν.

2. Is the set of those points y which are supports of at least two linearly independent excessive functions of measure zero?

3. Suppose the measure in section 2 is finite with a density f. Is it true that $s=Uf$?

Positive answers to 2 or 3 would imply the following: If s is excessive with $L(S) < \infty$, it is possible to choose a density $u(x, y)$ for U and a measure ν such that $s = U\nu$.

4. Let K be a compact thin set. Can there exist an excessive function s satisfying

$$s = P_K S?$$

REFERENCES

1. R. M. Blumenthal and R. K. Getoor, *Markov Processes and Potential theory* (Pure and Applied Mathematics 29), Academic Press, New York, London, 1968.
2. K. L. Chung and M. Rao, *On existence of a dual process*, Preprint Series 1978, No. 25, Mathematics Department, Aarhus University.
3. P. A. Meyer, *Note sur l'interprétation des measures d'equilibre* in *Séminaire de Probabilités* VII, 1971–72, eds. M. Karoubi and P. A. Meyer, Université de Strasbourg (Lecture Notes in Mathematics 321), pp. 210–216, Springer-Verlag, Berlin - Heidelberg - New York, 1973.
4. K. L. Chung and M. Rao, *Representations as potentials of measures*, Preprint Series 1979/80, No. 28, Mathematics Department, Aarhus University.

UNIVERSITY OF FLORIDA
DEPARTMENT OF MATHEMATICS
201 WALKER HALL
GAINESVILLE, FLORIDA 32611
U.S.A.