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# SQUARES IN ARITHMETICAL PROGRESSIONS II

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In an earlier paper [1] we showed that the question of the existence of a non-trivial arithmetical progression whose 1st, 3rd, $(n+1)$th and $(n+3)$rd terms are all perfect squares is equivalent to the existence of solutions in integers of the equation $z^2 = x^4 + (n^2-2)x^2y^2 + y^4$ with $xy(x^2-y^2) \neq 0$. In turn it was shown that this question was equivalent to the existence of pairwise prime integers $a$, $b$, $c$, $d$ satisfying one of a finite number of quadratic diophantine systems, which can be written down when $n$ is given. For many values of $n$ this analysis is sufficient either to find a solution or to prove that no solution exists fairly simply.

We consider here in more detail the case $n=49$ which seems to be more difficult. It is easily seen by the methods of [1] that this leads to the system

$$\begin{aligned} 51c^2 - 2401d^2 &= 2a^2 \\ c^2 - 47d^2 &= 2b^2 , \end{aligned}$$

where $a$, $b$, $c$, $d$ are all odd and pairwise coprime; all the other possibilities are easily disposed of by simple congruence considerations.

It appears that no such simple method will suffice for this system, possibly because the system does possess the solution $(a,b,c,d) = (7,1,7,1)$ which of course fails to satisfy the condition of coprimality.

THEOREM 1. *The system has no solution with $a$, $b$, $c$, $d$ pairwise coprime odd integers.*

PROOF. Suppose the contrary. Then factoring the second equation in $R[\sqrt{2}]$ we obtain

$$\begin{aligned} (c+b\sqrt{2})(c-b\sqrt{2}) &= 47d^2 \\ &= (7+\sqrt{2})(7-\sqrt{2})d^2 . \end{aligned}$$

Now $7 \nmid c$, else from the first equation $7 \mid a$ and so it easily follows that $c+b\sqrt{2}$,

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$c-b\sqrt{2}$ are coprime in $R[\sqrt{2}]$. Also if we assume, as we may without loss of generality, that $a$, $b$, $c$, $d$ are all positive then $c\pm b\sqrt{2}>0$ and so

$$c+b\sqrt{2} = (7\pm\sqrt{2})(e+f\sqrt{2})^2\varepsilon\,,$$

where $\varepsilon$ is a unit in $R[\sqrt{2}]$ which is positive and whose conjugate is also positive. Thus $\varepsilon$ must be a unit which is the square of another in $R[\sqrt{2}]$, since the fundamental unit $1+\sqrt{2}$ has norm $-1$.

Thus by modifying $e$ and $f$ as necessary we assume without loss of generality that $\varepsilon=1$. Thus

$$\begin{aligned} c+b\sqrt{2} &= (7\pm\sqrt{2})(e^2+2f^2+2ef\sqrt{2}) \\ d &= |e^2-2f^2|\,. \end{aligned}$$

Thus

$$c = 7(e^2+2f^2)\pm 4ef\,.$$

Substituting into the first equation now gives

$$\begin{aligned} 2a^2 &= 51c^2-2401d^2 \\ &= 51\{7(e^2+2f^2)\pm 4ef\}^2-2401\{e^2-2f^2\}^2 \\ &= 51.49(e^2+2f^2)^2\pm 51.56ef(e^2+2f^2)+51.16e^2f^2 \\ &\qquad -2401(e^2+2f^2)^2+2401.8e^2f^2\,. \end{aligned}$$

Thus

$$a^2 = \{7e^2\pm 102ef+14f^2\}^2-392e^2f^2\,.$$

Here $e$ and $f$ are coprime integers and neither is divisible by 7, otherwise we should find that $7\mid(a,c)$. Moreover $e$ is odd, since $d$ is. We show next that $f$ is also odd. For we have that $c=7e^2\pm 4ef+14f^2$, and if $f$ were even we should have $c\equiv 7 \pmod 8$. But then $-2a^2\equiv(49d)^2 \pmod c$ would be impossible unless $(a,c)>1$. So $f$ is odd. Thus

$$\begin{aligned} a^2+196e^2f^2 &= (7e^2\pm 102ef+14f^2)^2-196e^2f^2 \\ &= (7e^2\pm 88ef+14f^2)(7e^2\pm 116ef+14f^2)\,. \end{aligned}$$

Now since $e$ and $f$ are odd, $7e^2\pm 88ef+14f^2\equiv 5 \pmod 8$ and accordingly must be positive, since it divides the sum of two squares with no common factor. Similarly $7e^2\pm 116ef+14f^2$ must be positive, and hence so is $7e^2\pm 102ef+14f^2$, their mean. Hence

$$a^2+2(14ef)^2 = \{7e^2\pm 102ef+14f^2\}\,,$$

and factorising in $R[\sqrt{-2}]$, we obtain without difficulty

$$7e^2 \pm 102ef + 14f^2 = g^2 + 2h^2$$

where

$$7ef = gh$$

and $g$ and $h$ are suitable odd integers with no common factor. It then follows that $7 \mid g$ or $7 \mid h$. We consider these separately.

CASE 1. $7 \mid g$. Then $ef = (\frac{1}{7}g)h$ and if $(e, \frac{1}{7}g) = \alpha$ we obtain $e = \alpha\beta$, $\frac{1}{7}g = \alpha\gamma$, say with $(\beta, \gamma) = 1$. Then $\beta f = \gamma h$, and so $\beta \mid h$. Thus $h = \beta\delta$, say, with $f = \gamma\delta$. Thus

$$7\alpha^2\beta^2 \pm 102\alpha\beta\gamma\delta + 14\gamma^2\delta^2 = 49\alpha^2\gamma^2 + 2\beta^2\delta^2$$

i.e.

$$\alpha^2(7\beta^2 - 49\gamma^2) \pm 102\alpha\beta\gamma\delta + \delta^2(14\gamma^2 - 2\beta^2) = 0\,.$$

Regarding this as a quadratic in the rational ratio $(\alpha/\delta)$, it follows that for some integer $\xi$,

$$\xi^2 = 51^2\beta^2\gamma^2 + 14(\beta^2 - 7\gamma^2)^2\,.$$

But then

$$\xi^2 \equiv -3(\beta^2 - 7\gamma^2)^2 \pmod{17}\,,$$

and since $(-3 \mid 17) = -1$, this could occur only if $\beta^2 \equiv 7\gamma^2 \pmod{17}$. But $(7 \mid 17) = -1$ and so this is impossible as $(\beta, \gamma) = 1$.

CASE 2. $7 \mid h$ is entirely similar with now $e = \alpha\beta$, $f = \gamma\delta$, $g = \alpha\gamma$, and $h = 7\beta\delta$. Then

$$7\alpha^2\beta^2 \pm 102\alpha\beta\gamma\delta + 14\gamma^2\delta^2 = \alpha^2\gamma^2 + 98\beta^2\delta^2$$

$$\alpha^2(7\beta^2 - \gamma^2) \pm 102\alpha\beta\gamma\delta + \delta^2(14\gamma^2 - 98\beta^2) = 0$$

$$\xi^2 = 51^2\beta^2\gamma^2 + 14(\gamma^2 - 7\beta^2)^2$$

and again $17 \mid (\gamma^2 - 7\beta^2)$, which is impossible.

THEOREM 2. *The most general solution of the system is*

$$(a, b, c, d) = (7k, \pm k, \pm 7k, \pm k)$$

*with $k$ an integer.*

PROOF. If $abcd = 0$, then $a = b = c = d = 0$. Suppose then that $abcd \neq 0$; then

$$(2ab)^2 = 51c^4 - 4798c^2d^2 + 112847d^4$$

and so

$$0 = 51\left(\frac{c^2}{d^2}\right)^2 - \left(\frac{c^2}{d^2}\right)\left(4798 + \frac{4a^2b^2}{c^2d^2}\right) + 112847$$

whence

$$\left(2399 + \frac{2a^2b^2}{c^2d^2}\right)^2 - 51.112847 = \frac{k^2}{c^4d^4}$$

for some integer $k$. Simplifying

$$(\tfrac{1}{2}k)^2 = (cd)^4 + (ab)^2(cd)^2.2399 + (ab)^4 .$$

But by the last theorem and the remarks in the second paragraph this equation has no solutions with $abcd(c^2d^2 - a^2b^2) \neq 0$. Since we assume that $abcd \neq 0$, $(cd)^2 = (ab)^2$. Then

$$0 = 51c^4 - 4802c^2d^2 + 112847d^4$$

$$0 = (c^2 - 49d^2)(51c^2 - 2303d^2)$$

and so $c^2 = 49d^2$. Then $a^2 = c^2$, $b^2 = d^2$, which concludes the proof.

## REFERENCE

1. J. H. E. Cohn, *Squares in arithmetical progressions* I, Math. Scand. 52 (1983), 5–19.

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