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ON BROOKS' THEOREM AND SOME RELATED RESULTS

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1. Brooks' theorem.

Let G be a simple, connected and finite graph, and let $k \geqq 3$ be the maximal degree of the vertices. Assume, too, that G is not K_{k+1}, the complete graph on $k+1$ vertices. Brooks' theorem [1] says that the chromatic number of G is then at most k. Its form strongly suggests a proof procedure: Show that for some set B of independent vertices, each component of the induced subgraph on $V(G)-B$ satisfies Brooks' conditions with k replaced by $k-1$ if $k>3$, and has chromatic number at most 2 if $k=3$.

The procedure suggested has been used by Catlin [2] and Gerenczér [5], but they do not arrive at simple proofs. The proof that follows is quite simple. Another simple proof is the one by Lovász [6].

The proof needs the concept of a k-tree, which was introduced by Gallai [4], but occurs implicitly already in Dirac [3]. One of the results in [3], which was expressed in terms of k-trees and reproved in [4], will get yet another proof here. The possibility of such a proof was pointed out by the referee, who also made me aware of [3] and [4]. For this, as well as for some other useful suggestions, I'm very grateful to him/her. Catlin also uses k-trees and there'll be a new proof of one of his results, too. But let's start with Brooks' theorem and the definition of k-trees.

If $k>3$, K_k is the only k-tree on k vertices, and a k-tree on kr vertices is any graph which can be obtained from a k-tree on $k(r-1)$ vertices and a disjoint K_k by adding an edge between a vertex of the former, of degree $k-1$, and a vertex of the latter. A 3-tree is either an odd cycle, or a graph built up from odd cycles (of maybe varying lengths) in the same way as a k-tree is built up from K_k's. We shall misuse the language and let K_3 denote an odd cycle. Note that if a k-tree is not just a single K_k, then it has at least two K_k's with only one vertex of degree k. Note also that any k-tree has chromatic number k. In view of this, Brook's theorem is an immediate consequence of the following

LEMMA. *If H satisfies Brooks' conditions, but is not a k-tree, then there is a vertex b of degree k such that no component of $H-b$ is a k-tree.*

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For applying the lemma to a graph G, then to components of $G-b$, etc. one arrives finally at a graph $G-b_1 \ldots -b_m$ with no vertex of degree k. Each component of this graph satisfies Brooks' conditions, with k replaced by $k-1$, if $k>3$, and is an even cycle or a path if $k=3$. The set $B=\{b_1, \ldots b_m\}$ is clearly independent and so is what we want.

PROOF OF THE LEMMA. If H does not contain an induced K_k, there is nothing to prove, so let K be one. If K has only one vertex, b, of degree k, then the components of $H-b$ is a K_{k-1} (or a path if $k=3$) and a graph which cannot be a k-tree, as then H would be one, too. We may thus assume that each induced K_k in H has at least two vertices, $b, b', \ldots$ of degree k and we choose if possible one, K, which has also a vertex c of degree $k-1$. If $k>3$, b and c are adjacent, and if $k=3$, they may be assumed to be.

In the component of $H-b$ which contains $K-b$, c has degree $k-2$, so we are through, unless the component of $H-b$ which contains the one neighbour of b outside K is different from the former. But if the latter is a k-tree, it clearly contains a K_k with only one vertex of degree k (in H), against assumption.

Finally, if each vertex of each induced K_k has degree k, let b be any vertex of degree k, and L a component of $H-b$. If L is a K_k, each vertex of L must be adjacent to b, but this gives a K_{k+1} in H. If L is a larger k-tree, then at least two of its K_k's have only one vertex of degree k (in L), so that at least $2(k-1)$ of its vertices are adjacent to b (in H). But $2(k-1)>k$, as $k \geqq 3$.

In the case $k>3$ it is possible to avoid the use of the Lemma as follows: Consider, for the given k, a counter example G with minimal number of edges. Choose an edge which is in no K_k (easy), delete it and colour the new graph. Let B' be all the vertices of some fixed colour not used on the endpoints of the chosen edge. Then B' is an independent set meeting each K_k and extending it to a maximal independent set B we get what we need for a reduction from the case k to the case $k-1$.

The lemma is, however, not much more difficult to prove in the general form and, moreover, we need it below.

2. The Dirac–Gallai theorem.

This theorem (theorems 4 and 10 of [3], theorem 3.3 of [4]) can be expressed as follows. *Let x be a vertex of a simple, connected and finite graph G. Assume that each vertex of G, except possibly x, has degree $\leqq k$ $(k \geqq 3)$ and that no component of $G-x$ is a k-tree. Then G is k-colourable.*

The theorem follows easily from our proof of Brooks' theorem. For let the components of $G-x$ be $G_1, \ldots, G_r$. As they are not k-trees our Lemma produces independent vertex sets $B_1, \ldots, B_r$ so that $B_i \subset V(G_i)$ and $G_i - B_i$ is $(k-1)$-colourable, $i = 1, \ldots r$. Thus $G - x$ can be k-coloured with, say, the blue set being $B = B_1 \cup \ldots \cup B_r$. But each vertex in B has degree k in $G - x$ and so is not adjacent to x in G. Thus x can be coloured blue, too.

3. Catlin's result.

In [2] Catlin studies Brooks' theorem in some detail, proving for instance that among the colourings of G, there is at least one in which the number of, say, blue vertices equals the maximal number of independent vertices in G. The most difficult part of [2] is the proof that (essentially), if G satisfies the conditions of our Lemma, then there is a *superstable* set S such that $\chi(G-S) \leqq k-1$. (Let V_k be the set of vertices of G of degree k, and let s be the maximal number of independent vertices in V_k. Then the superstable sets for G are the sets of s independent vertices in V_k).

Let a *super* vertex be one which belongs to at least one superstable set. To get the result above it will clearly be enough to show that the vertex b, occurring in the Lemma, can be taken to be super. Now any induced K_k, K, in H has at least one super vertex. For let S be a superstable set, containing no vertex of K, and let c be a vertex of K, of degree k. Then c has $k-1$ neighbours in K, and hence outside S, and one, d, in S. The set $S-d+c$ will then be superstable.

Using this observation one finds, examining the proof of the Lemma, that the vertex b can always be taken to be super, except possibly, in the case $k=3$, if no K_3 has only one vertex of degree 3, but some K_3, K, has at least one vertex of degree 2. Picking a super vertex from K, we shall be all right unless both its neighbours in K are of degree 3. Thus we shall be all right unless each induced K_3 has at least 3 vertices of degree 3. But in the latter case it is easy to see that, whenever b is a super vertex, no component of $H-b$ is a 3-tree.

Note that in Catlin's definition of Brook's trees (as he calls the k-trees), the condition $\Delta(T) \leqq h$ should be omitted in the case $h=3$.

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