MATH. SCAND. 52 (1983), 41–57

CARDINALITY OF GENERATING SETS FOR MODULES OVER A COMMUTATIVE RING

ROBERT GILMER[1] and WILLIAM HEINZER[2]

Let M be a unitary module over a commutative ring R with identity, and let α be an infinite cardinal. Following the terminology of universal algebra [5], [4], we call M a *Jónsson α-generated R-module* if there exists a set of generators for M of cardinality α and no such generating set of cardinality less than α, while each proper submodule of M has a set of generators of cardinality less than α. (We remark that there is an analogous notion that considers the cardinality of the module itself, rather than its generating sets. We call an infinite module M a *Jónsson α-module* if $|M| = \alpha$ while $|N| < \alpha$ for each proper submodule N of M. This analogous notion is considered in [10].) We are interested in the structure of rings R and modules M such that M is a Jónsson α-generated R-module primarily for the case where $\alpha \in \{\omega_0, \omega_1\}$, but several of our results in Section 1 are valid for an arbitrary infinite cardinal α. For example, Proposition 1.1 yields that if M is a faithful Jónsson α-generated R-module, then R is an integral domain, and Theorem 1.4 states that if M is a torsion-free Jónsson α-generated R-module, then M is isomorphic to the quotient field K of R. We prove (Corollary 1.2) that if an ideal I of R is a Jónsson α-generated R-module, then $I^2 = (0)$ and I is contained in each regular ideal of R. For each regular cardinal α, we confirm the existence of Jónsson α-generated modules and ideals. We also prove that if there exists a Jónsson α-generated module for α an irregular cardinal, then it must be a torsion module. In Theorem 1.6 we prove, using results from [2] and [8], that most rings normally encountered in commutative algebra do not admit Jónsson ω_1-generated modules. Theorem 1.8 shows that if R admits a Jónsson ω_1-generated module, then R does not satisfy the descending chain condition (d.c.c.) for prime ideals.

In Section 2 we study Jónsson ω_0-generated modules. We prove (Proposition 2.1) that a Jónsson ω_0-generated module is either torsion or

[1] This author received partial support from National Science Foundation Grant 7903123.
[2] This author received partial support from National Science Foundation Grant 8002201.
Received October 22, 1980; in revised form September 7, 1981.

torsion-free. If M is a faithful torsion Jónsson ω_0-generated module over an integral domain D, then Theorem 2.4 shows that there exists a maximal ideal P of D such that $P=\sqrt{\operatorname{Ann}(m)}$ for each nonzero $m \in M$. Moreover, $D/\operatorname{Ann}(m)$ is an Artinian ring and

$$(0) = \bigcap \{\operatorname{Ann}(m) \mid 0 \neq m \in M\} = \bigcap_{i=1}^{\infty} P^i .$$

We prove in Theorem 2.7 that a Noetherian ring R admits a torsion Jónsson ω_0-generated module if and only if $\dim R \neq 0$. Results of Armendariz in [1] completely determine the structure of torsion-free Jónsson ω_0-generated modules. An integral domain D admits a torsion-free Jónsson ω_0-generated module if and only if D is a 1-dimensional local domain such that the integral closure of D is a rank-one discrete valuation ring that is a finite D-module. In Theorem 2.9 we determine to within isomorphism the family of all torsion Jónsson ω_0-generated modules over a Prüfer domain.

The examples of Section 3 indicate certain restrictions on what can be said about the structure of a quasi-local domain D that admits a faithful Jónsson ω_0-generated torsion module. Such a domain D need not be Noetherian, for example, and even for a Noetherian domain D, no restriction can be placed on the (Krull) dimension of D. We conclude with an example that shows (in contrast with the situation for ω_0) that for each regular cardinal $\alpha > \omega_0$, there exists a domain D with infinitely many maximal ideals such that the quotient field K of D is a torsion-free Jónsson α-generated D-module.

All rings considered in this paper are assumed to be commutative and to contain an identity element; all modules considered are assumed to be unitary.

1. Jónsson α-generated Modules.

We begin with a proposition and a corollary that are valid for an arbitrary infinite cardinal α.

PROPOSITION 1.1. *Let M be a Jónsson α-generated module over the ring R, for α an infinite cardinal.*

(1) *If N is a proper submodule of M, then the quotient module M/N is again a Jónsson α-generated module over R.*

(2) *If N_1 and N_2 are submodules of M such that $M=N_1+N_2$, then either $M=N_1$ or $M=N_2$. In particular, M is indecomposable and has no maximal submodules.*

(3) *If $r \in R$, then either $rM=M$ or $rM=(0)$.*

(4) $\operatorname{Ann} M$ *is a prime ideal of* R.

PROOF. Statements (1) and (2) are clear from the structure of quotient modules and the fact that if α_1 and α_2 are cardinal numbers less than α, then $\alpha_1 + \alpha_2 < \alpha$. Thus, if N_1 and N_2 are proper submodules of M with generating sets of cardinality α_1 and α_2, then $N_1 + N_2$ has a generating set of cardinality $\alpha_1 + \alpha_2 < \alpha$ so that $N_1 + N_2 \neq M$.

For statement (3), consider the R-module homomorphism φ_r: $M \to rM$, $\varphi_r(m) = rm$. If rM is properly contained in M, then rM is not a Jónsson α-generated module. But by (1), we must then have $rM = (0)$, which proves (3).

It follows from (3) that if $x, y \in R - \operatorname{Ann} M$, then $M = xM = yM$, and hence $M = xyM$. Thus $xy \notin \operatorname{Ann} M$, and $\operatorname{Ann} M$ is prime in R, as asserted in (4).

COROLLARY 1.2. *Let I be a Jónsson α-generated ideal of the ring R, for α an infinite cardinal. Then $I^2 = (0)$ and I is contained in each regular ideal of R.*

PROOF. If $x \in R$ and $xI \neq (0)$, then by Proposition 1.1, $xI = I$ so that $I \subset (x)$. In particular, for $x, y \in I$ we must have $xy = 0$ since otherwise we would have $I = xI = (x)$, and I would be principal.

REMARK 1.3. We note that Proposition 1.1 implies that the only rings R that admit a faithful Jónsson α-generated module are domains. Moreover, since a vector space over a field is never a Jónsson α-generated module, we see from Proposition 1.1 that a 0-dimensional ring does not admit a Jónsson α-generated module.

According to the terminology of [3, Example 17, p. 245], the infinite cardinal α is said to be *regular* if $\alpha \neq \sum_{i \in I} \alpha_i$ for each nonempty family $\{\alpha_i\}_{i \in I}$ of cardinals with $|I| < \alpha$ and $\alpha_i < \alpha$ for each i. As noted by Simis in [15], this condition is equivalent to the statement that there is no cofinal set of cardinality less than α in the set of ordinals preceding the first ordinal of cardinality α.

We remark that for each regular cardinal α, Jónsson α-generated ideals exist. In fact, if ϱ is the first ordinal of cardinality α, then there exists a valuation ring V whose set of nonzero prime ideals, ordered under reverse inclusion, is order-isomorphic to the ordered set of ordinals preceding ϱ, and a standard argument shows that if K is the quotient field of V, then K and K/V are Jónsson α-generated modules over V [9, p. 797]. For $M = K$ or K/V, one then obtains by passage to the idealization of V and the V-module M a Jónsson α-generated ideal of a ring R [14, p. 2]. We note that this in fact gives for α a regular cardinal a Jónsson α-generated module or ideal for which the submodules are linearly ordered under inclusion — that is, a chained module.

For α an irregular infinite cardinal, it is easy to see that there does not exist a chained module M that is a Jónsson α-generated module. For M would have a generating set A with $|A|=\alpha$ and $A=\bigcup\{A_i \mid i \in I\}$ where $|A_i|=\alpha_i<\alpha$ and $|I|<\alpha$. Hence A_i would generate a proper submodule M_i of M. If $x_i \in M-M_i$, then $M_i \subseteq Rx_i$, so $\{x_i\}_{i\in I}$ is a generating set for M of cardinality $|I|<\alpha$. From our next result we conclude, moreover, that a Jónsson α-generated module for α an irregular infinite cardinal (if such exists) must of necessity be a torsion module.

THEOREM 1.4. *Let D be an integral domain with quotient field K. If M is a torsion-free Jónsson α-generated D-module, then $M \cong K$.*

PROOF. By part (3) of (1.1), M is divisible and hence a vector space over K. Part (2) of Proposition 1.1 shows that M is indecomposable as a D-module, and hence M is also indecomposable as a K-module, and this implies that $M \cong K$.

We are indebted to David Lantz for the elegant proof of Theorem 1.4 given above.

REMARK 1.5. If D is an integral domain with quotient field K and if for some infinite cardinal α, K is a Jónsson α-generated D-module, then for each overring V of D properly contained in K, it is easy to see that K is also a Jónsson α-generated V-module. By taking V to be a valuation overring of D properly contained in K, we realize K as a chained V-module that is also a Jónsson α-generated V-module. It follows that α is a regular cardinal. Thus, from Theorem 1.4 and Proposition 1.1 we conclude that if α is an irregular cardinal, then the only possible Jónsson α-generated module M is a torsion module. For we may assume that M is a faithful module and then of necessity it is a module over an integral domain D; and if M is not torsion and N is the torsion submodule of M, then M/N is a torsion-free Jónsson α-generated D-module so that $M/N \cong K$, the quotient field of D. But as we just observed, this implies that α is a regular cardinal.

We consider next the question of what rings R admit a Jonsson ω_1-generated module. Results of [2] and [8] are useful in this connection. Following the terminology of [2], we call a module M a (**)-*module* if M cannot be expressed as the union of a countable strictly ascending sequence $M_1 < M_2 < \dots < M_n < \dots$ of submodules; we denote by $\mathscr{F}$ the class of rings R such that each (**)-module over R is finitely generated (clearly a finitely generated module is a (**)-module for any R). Theorems 4.2, 4.7, and 4.10 of

[2] show that $\mathcal{F}$ contains the subclasses of Noetherian rings, finite-dimensional chained rings and W*-rings; Theorem 6.1 of [8] shows that $\mathcal{F}$ also contains each ring R such that (1) R has Noetherian spectrum, (2) d.c.c. for prime ideals is satisfied in R, and either (3) each ideal of R is countably generated or (4) each ideal of R contains a power of its radical.

If α is an infinite cardinal, we say that α is *countably inaccessible from below* if $\alpha \neq \sum_{i \in I} \alpha_i$ for each nonempty countable family $\{\alpha_i\}_{i \in I}$ of cardinals $\alpha_i < \alpha$. According to this terminology, ω_0 is countably accessible from below, while each infinite cardinal with an immediate predecessor (in particular, ω_1) is countably inaccessible from below.

THEOREM 1.6. *If the cardinal α is countably inaccessible from below, then any Jónsson α-generated R-module M is a non-finitely generated* (**)*-module. Hence if R is in the class $\mathcal{F}$, then R admits no Jónsson α-generated module.*

PROOF. If $M_1 < M_2 < \dots$ is a strictly ascending sequence of submodules of M, then each M_i has a generating set of cardinality $\alpha_i < \alpha$. Thus, $N = \sum_{i=1}^{\infty} M_i$ has a generating set of cardinality $\sum_{i=1}^{\infty} \alpha_i < \alpha$, so that $N \neq M$. It follows that M is a non-finitely generated (**)-module.

REMARK 1.7. If we let $\mathcal{G}$ denote the class of rings R such that R admits no Jónsson ω_1-generated module, then by Theorem 1.6, $\mathcal{F} \subseteqq \mathcal{G}$. We remark that this inclusion is, in fact, proper. For if R is an infinite product of copies of a field, then by [2, Example 2.4], $R \notin \mathcal{F}$. However, R is 0-dimensional, so by Remark 1.3, $R \in \mathcal{G}$.

The next result shows that, in the notation of Remark 1.7, the class $\mathcal{G}$ contains each ring satisfying d.c.c. on prime ideals.

THEOREM 1.8. *Assume that D is an integral domain and α is an infinite cardinal that is countably inaccessible from below. If D admits a Jónsson α-generated module M, then D does not satisfy* d.c.c. *on prime ideals.*

PROOF. We assume, to the contrary, that D satisfies d.c.c. on primes. Without loss of generality, we also assume that M is faithful. We consider first the case where M is a torsion module. Then if $S = D - \{0\}$, we have $M_S = (0)$. On the other hand, $M \neq (0)$ implies $M_Q \neq (0)$ for some maximal ideal Q of D. We choose a prime ideal P minimal among primes T such that $M_T \neq (0)$. We consider M_P as a D_P-module. Take $x \in P$, $x \neq 0$, and take $m \in M_P$, $m \neq 0$. By choice of P, Ann (m) is contained in no prime of D_P properly contained in PD_P.

Hence Ann (m) is primary for PD_P and $x \in \sqrt{\text{Ann}(m)}$. Thus, if $H(x^k)$ is the submodule of M_P annihilated by x^k, then

$$(0) \subsetneqq H(x) \subsetneqq H(x^2) \subsetneqq \dots \quad \text{and} \quad M_P = \bigcup_{k=1}^{\infty} H(x^k) .$$

Moreover, $x^kM = M$ implies $x^kM_P = M_P$, so that $M_P > H(x^k)$ for each k. Let H_k be the inverse image in M of $H(x^k)$ for each k. Then $M > H_k$ for each k and $M = \bigcup_{k=1}^{\infty} H_k$. For each k we choose a generating set S_k for H_k of cardinality $\alpha_k \leqq \alpha$. Since $\bigcup_{k=1}^{\infty} S_k$ generates M, we have

$$\alpha = \sum_{k=1}^{\infty} \alpha_k ,$$

contrary to the assumption that α is countably inaccessible from below. Thus, M a torsion module is impossible.

If M is not a torsion module, then the torsion submodule N of M is a proper submodule of M and M/N is a torsion-free Jónsson α-generated module. By Theorem 1.4, $M/N \cong K$, the quotient field of D. Hence K/D is a faithful torsion Jónsson α-generated module, contrary to what was proved in the preceding paragraph. We conclude that D does not satisfy d.c.c. for primes, as asserted.

In relation to the proof of Theorem 1.8, we remark that, conversely, if K/D is a Jónsson α-generated D-module, then K is also such a module. For a proof, note that it is clear that K has a generating set of cardinality α, but no generating set of smaller cardinality. Let N be a nonzero proper submodule of K and let n be a nonzero element of N. The mapping $x \to n^{-1}x$ is a D-module automorphism of K. Since $n^{-1}N \supseteqq D$, then $n^{-1}N/D$, and hence $n^{-1}N$, has a generating set of cardinality less than α. Thus, N also has such a generating set, and this completes the proof.

2. Jónsson ω_0-generated modules.

Theorems 1.6 and 1.8 show that the rings normally encountered in commutative algebra admit no Jónsson ω_1-generated modules. The situation, of course, is quite different for ω_0; for example, the p-quasicyclic group $\mathbf{Z}(p^\infty)$ is a Jónsson ω_0-generated $\mathbf{Z}$-module. (Theorem 2.9 shows that the p-quasicyclic groups are, in fact, the only torsion Jónsson ω_0-generated modules over $\mathbf{Z}$; cf. [6, Example 4, p. 105].) In this section we examine Jónsson ω_0-generated modules.

Proposition 2.1. *If M is a faithful Jonsson ω_0-generated module over R, then either M is a torsion module or M is torsion-free.*

PROOF. Since M is faithful, Proposition 1.1 shows that R is an integral domain. Let N be the torsion submodule of M and assume that $N \neq M$. Then N is finitely generated, and hence there exists a nonzero element $r \in R$ such that $rN = (0)$. Since M is faithful, $rM = M$. This implies that $N = (0)$, for if $n \in N$ and if $m \in M$ is such that $n = rm$, then m is also a torsion element of M. Consequently, $m \in N$ and $rm = n = 0$. This completes the proof of Proposition 2.1.

REMARK 2.2. For α an arbitrary infinite cardinal, if M is a faithful Jónsson α-generated R-module and N is the torsion submodule of M, then $xN = N$ for each nonzero $x \in R$. It would be interesting to know if in general $N = M$ or $N = (0)$—that is, to answer the following question.

Must a faithful Jónsson α-generated module be either torsion or torsion-free?

If M is a Jónsson ω_0-generated module over R, then replacing R by $R/\text{Ann}\, M$, there is no loss of generality in assuming that M is faithful, and Proposition 1.1 shows that $R/\text{Ann}\, M$ is an integral domain. Thus we turn to a consideration of faithful torsion Jónsson ω_0-generated modules over an integral domain.

PROPOSITION 2.3. *Let M be a faithful torsion Jónsson ω_0-generated module over the integral domain D. Let m be a nonzero element of M and choose $x \neq 0$ in D such that $xm = 0$. Since $xM = M$, there exist elements $m = m_1, m_2, m_3, \ldots$ of M such that $m_i = xm_{i+1}$ for each i. Then*

$$Dm_1 < Dm_2 < \ldots \quad \text{and} \quad M = \bigcup_{i=1}^{\infty} Dm_i .$$

Moreover, if $M(x^i)$ denotes the submodule of M annihilated by x^i, then $M(x^i)$ is Noetherian,

$$M(x) < M(x^2) < \ldots, \quad \text{and} \quad M = \bigcup_{i=1}^{\infty} M(x^i) .$$

PROOF. Since $x^i m_i = 0$ while $x^{i-1} m_i = m \neq 0$, we have $Dm_1 < Dm_2 < \ldots$. Hence $\bigcup_{i=1}^{\infty} Dm_i$ is a submodule of M that is not finitely generated, so that $M = \bigcup_{i=1}^{\infty} Dm_i$. Moreover, $m_i \in M(x^i) - M(x^{i-1})$ implies that $M(x^1) < M(x^2) < \ldots$, and hence that

$$M = \bigcup_{i=1}^{\infty} M(x^i) .$$

THEOREM 2.4. *Assume that M is a faithful torsion Jónsson ω_0-generated module over the domain D. There exists a maximal ideal P of D such that $P = \sqrt{\operatorname{Ann}(m)}$ for each nonzero m of M. Moreover, each $D/\operatorname{Ann}(m)$ is an Artinian ring and*

$$(0) = \bigcap \{\operatorname{Ann}(m) \mid 0 \neq m \in M\} = \bigcap_{i=1}^{\infty} P^i .$$

If H_i is the submodule of M annihilated by P^i, then the sequence $\{H_i\}_{i=1}^{\infty}$ properly ascends and $M = \bigcup_{i=1}^{\infty} H_i$.

PROOF. For m, n nonzero elements of M, Proposition 2.3 implies that $\operatorname{Ann}(m) \subseteq \sqrt{\operatorname{Ann}(n)}$, and by symmetry we conclude that $\sqrt{\operatorname{Ann}(m)} = \sqrt{\operatorname{Ann}(n)}$. Let P be the ideal $\sqrt{\operatorname{Ann}(m)}$; we show that P is a maximal ideal of D. Choose $y \in D - P$, $x \in P - \{0\}$, and let N be the submodule of M annihilated by x. Since $x \in P$ and since M is faithful, $(0) < N < M$. Hence N is finitely generated. We prove that $yN = N$. By Proposition 1.1, $yM = M$, so if $t \in N$, then there exists $s \in M$ such that $t = ys$. Then $0 = xt = xys$, and $y \notin P$ implies that $xs = 0$ so that $s \in N$. Therefore $yN = N$, and since N is finitely generated, there exists $d \in D$ such that $(1 - dy)N = (0)$ [12, Theorem 76]. Then $N \neq (0)$ implies that $1 - dy \in P$, so $D = P + Dy$ and P is maximal in D as we wished to prove.

The assertions in the second sentence of Theorem 2.4 follow easily from the proceeding paragraph. Thus, $D/\operatorname{Ann}(m)$ is Noetherian, since $D/\operatorname{Ann}(m)$ and Dm are isomorphic D-modules, and $D/\operatorname{Ann}(m)$ is zero-dimensional since $P = \sqrt{\operatorname{Ann}(m)}$ is maximal in D. Whence $D/\operatorname{Ann}(m)$ is Artinian. The equality

$$(0) = \bigcap \{\operatorname{Ann}(m) \mid 0 \neq m \in M\}$$

holds since M is faithful, and $(0) = \bigcap_{i=1}^{\infty} P^i$, since the ideal $P/\operatorname{Ann}(m)$ is nilpotent for each nonzero $m \in M$.

The inclusion $H_i \subseteq H_{i+1}$ is clear, and Proposition 2.3 shows that $M = \bigcup_{i=1}^{\infty} H_i$. Moreover, H_i is properly contained in M since $M = P^i M \neq P^i H_i = (0)$. Finally, the definition of H_i is such that an equality $H_i = H_{i+1}$ would imply $H_i = H_{i+k}$, and hence

$$H_i = \bigcup_{k=1}^{\infty} H_{i+k} = M ,$$

a contradiction. Consequently, $H_i < H_{i+1}$, and this completes the proof of Theorem 2.4.

Let the notation and hypothesis be as in the statement of Theorem 2.4. We remark that it is possible to extend the scalar multiplication between D and M to a scalar multiplication between D_P and M in such a way that M is a D_P-

module. To wit, for $m \in M$ and $d/s \in D_P$, we define the product $(d/s)\cdot m$ to be dm_1, where $sm_1 = m$. The product is well-defined since multiplication by s defines a D-automorphism of M. And M as a D_P-module has precisely the same submodules as does M as a D-module; the proof amounts to showing that if $m \in M - \{0\}$ and if $s \in D - P$, then $m \in Dsm$, and this statement is true since (s) is comaximal with Ann (m).

We note the following general result concerning the structure of Jónsson ω_0-generated modules. The proof is straightforward and will be omitted.

PROPOSITION 2.5. *Let M be an R-module that can be expressed as the union of a countably infinite strictly ascending sequence $\{M_i\}_{i=1}^{\infty}$ of Noetherian submodules. The following conditions are equivalent.*

(1) *M is a Jónsson ω_0-generated module.*
(2) *Each proper submodule of M is contained in some M_i.*
(3) *If $x_i \in M - M_i$, then $\{x_i\}_{i=1}^{\infty}$ generates M.*

Armendariz in [1] has proved some interesting results about what we term Jónsson ω_0-generated modules. In [1], a module M is said to be *almost Noetherian* if each proper submodule of M is Noetherian. Thus, an almost Noetherian module that is not Noetherian is the same as a Jónsson ω_0-generated module in our terminology. From [1, Theorems 2.1 and 2.2] we have the following definitive result concerning the structure of torsion-free Jónsson ω_0-generated modules.

(2.6) If D is an integral domain with quotient field K and if D admits a torsion Jónsson ω_0-generated module, then $M \cong K$, D is a 1-dimensional local domain, and the integral closure of D is a rank-one discrete valuation ring that is a finite D-module. Conversely, if D is a 1-dimensional local domain such that the integral closure of D is a rank-one discrete valuation ring that is a finite D-module, then K is a Jónsson ω_0-generated D-module.

In view of Armendariz's result (2.6), it is natural to ask what rings R admit a torsion Jónsson ω_0-generated module. As noted in Remark 1.3, such an R must have positive dimension, and Theorem 2.4 shows that R must have some Noetherian-type qualities such as the existence of a non-idempotent maximal ideal. In Section 3 we give examples of non-Noetherian rings R that admit faithful Jónsson ω_0-generated modules. We prove next that torsion Jónsson ω_0-generated modules exist over any Noetherian ring of positive dimension.

THEOREM 2.7. *The Noetherian ring R admits a torsion Jónsson ω_0-generated module if and only if* $\dim R \neq 0$.

PROOF. As noted above, it suffices to show that R admits a Jónsson ω_0-generated module if $\dim R \neq 0$. Let P be a maximal ideal of R of positive height. By passing from R to $R/Q = D$, where $Q < P$ is a prime ideal such that there are no primes properly between Q and P, we may assume that $R = D$ is an integral domain and that P is a maximal ideal of D of height one. Let K be the quotient field of D and let V be a valuation overring of D with center P on D. We claim that K/V is a Jónsson ω_0-generated D-module. To verify the claim, let D^* be the integral closure of D. If P^* is the center of V in D^*, then P^* is of height one. It follows that $D^*_{P^*} = V$, so V is rank-one discrete, and $D^*/P^* = k_V$, the residue field of V, is a finite algebraic extension of D/P [14, (33.10)]. Let v denote the valuation on K associated with V and having value group Z. Choose $\theta_1, \ldots, \theta_n \in D^*$ such that the residues of $\theta_1, \ldots, \theta_n$ in k_v generated k_v over D/P, and a $y \in D^*$ such that $v(y) = 1$. Let $D' = D[\theta_1, \ldots, \theta_n, y]$, and let C be the conductor of D in D'. Note that $C \neq (0)$ since D' is a finite D-module. Let P' denote the center of V on D'. If $P'_1, \ldots, P'_t$ are the minimal primes of C in D' other than P', we can multiply y by an element of $(P'_1 \cap \ldots \cap P'_t) - P'$ so that some power of the product is in C—that is, we may assume without loss of generality that $y \in D'$ is such that $v(y) = 1$ and a power of y, say y^i, is in C. For any $x \in K - V$ with $v(x) = -m$, we establish the following statement.

(2.8) The D-submodule of K generated by V and x contains all elements of K of v-value $\geqq -m + i$.

To prove (2.8), consider any $z \in K - V$ with $v(z) \geqq -m + i$. Then $v(z/x) = s \geqq i$. Let $u = z/xy^s$, so that $v(u) = 0$, and consider the residue of u in k_v. Since $D'/P' = k_v$, there exists $\theta \in D'$ so that $u - \theta$ has residue 0 in k_v. Hence $v((z/xy^s) - \theta) > 0$ and $v(z/x - \theta y^s) > s$; moreover the fact that $y^s\theta \in C$ implies that $z \in V + Dx$ if and only if $z_1 = z - xy^s\theta \in V + Dx$. Since

$$v(z_1) = v(x) + v(z/x - y^s\theta) > -m + s = v(z)$$

and since each element of K of nonnegative v-value is in $V + Dx$, a proof by induction establishes (2.8).

It follows immediately from (2.8) that $\{(V + Dy^{-n})/V\}_{n=1}^{\infty}$ is a strictly ascending sequence of Noetherian D-submodules of K/V such that

$$K/V = \bigcup_{n=1}^{\infty} (V + Dy^{-n})/V$$

and each proper submodule of K/V is contained in some $(V + Dy^{-n})/V$.

Consequently, K/V is a Jónsson ω_0-generated D-module by Proposition 2.5. This completes the proof of Theorem 2.7.

Using Theorem 2.4 and the results of Armendariz in (2.6), it is possible to determine to within isomorphism the class of all Jónsson ω_0-generated modules over a Prüfer domain D.

THEOREM 2.9. *Assume that D is a Prüfer domain and let* $\{P_i\}_{i\in I}$ *be the family of maximal ideals of D such that the powers of* P_i *properly descend. For each i, let* $Q_i = \bigcap_{k=1}^{\infty} P_i^k$, *let* $V_i = D_{P_i}/Q_iD_{P_i}$, *and let* K_i *be the quotient field of* V_i. *Let* $\{P_j\}_{j\in J}$ *denote the subset of* $\{P_i\}_{i\in I}$ *consisting of those* P_i *for which* P_i *is the unique maximal ideal of D containing* Q_i. *Then, to within isomorphism,*

$$\{K_i/V_i\}_{i\in I} \cup \{K_j\}_{j\in J}$$

is the family of Jónsson ω_0*-generated modules over D.*

PROOF. Using (2.4), the paragraph following the proof of (2.4), and (2.6), it is routine to verify that each K_i/V_i and each K_j is, in fact, a Jónsson ω_0-generated D-module. We remark that Q_i is the annihilator of K_i/V_i and Q_j is the annihilator of K_j.

Conversely, let M be a Jónsson ω_0-generated module over D and let $Q = \mathrm{Ann}\,(M)$. Then M is a faithful Jónsson ω_0-generated module over $D^* = D/Q$. If M is torsion-free as a D^*-module, then (2.6) shows that D/Q is a rank-one discrete valuation ring and M is isomorphic to the quotient field of D/Q. Hence $Q = Q_j$ for some $j \in J$ and $M \cong K_j$ in this case. On the other hand, M a torsion D^*-module implies, by Theorem 2.4, that there exists $i \in I$ such that $\mathrm{Ann}\,(x)$ is P_i-primary for each $x \in M \setminus \{0\}$. Since D is a Prüfer domain, then $Q = Q_i$ and there exists no prime ideal of D properly between Q_i and P_i [7, Chapter 23]. The paragraph following the proof of (2.4) shows that M is a faithful Jónsson ω_0-generated module over $(D/Q_i)_{(P_i/Q_i)} \cong V_i$. We show that $M \cong K_i/V_i$ in this case. Thus, let x be a generator for the maximal ideal of V_i. Choose elements $m = m_1, m_2, \ldots$ in M as in Proposition 2.3; that is, $m \neq 0$, $xm = 0$, and $m_i = xm_{i+1}$ for each $i \geq 1$. Then

$$M = \bigcup_{j=1}^{\infty} V_i m_j$$

and it is straightforward to show that the mapping $m_j \to x^{-j} + V_i$ admits a unique extension to a V_i-module isomorphism of M onto K_i/V_i. This shows that $M \cong K_i/V_i$, and hence the proof of Theorem 2.9 is complete.

We remark that in Theorem 2.9, $K_j = D_{Q_j}/Q_jD_{Q_j}$, and

$$K_i/V_i \cong (D_{Q_i}/Q_iD_{Q_i})/(D_{P_i}/Q_iD_{P_i}) \cong D_{Q_i}/D_{P_i}.$$

Thus if D is a Prüfer domain, then each Jónsson ω_0-generated D-module has the form L/N, where N and L are D-submodules of the quotient field of D. It would be interesting to know whether this is still true for D an arbitrary integral domain. The examples we give in Section 3 of Jónsson ω_0-generated modules all have this form. It can be seen that for D a rank-two valuation ring with principal maximal ideal P, height-one prime $Q=\bigcap_{n=1}^{\infty} P^n$, and quotient field K, the Jónsson ω_0-generated D-module D_Q/D is not isomorphic to K/N for any D-submodule N of K. Thus it is necessary, even in describing the Jónsson ω_0-generated modules over a Prüfer domain D, to allow L to be a proper D-submodule of K.

3. Some Examples.

The examples in this section are intended to serve two purposes. First, they indicate certain limitations on what can be said about the structure of a quasi-local domain (D, P) such that D admits a faithful torsion Jónsson ω_0-generated module. In particular, Example 3.1 shows that D need to be Noetherian, and the class of examples included in Example 3.3 is large enough to show that even for D Noetherian, no restriction on the dimension of D is possible. The second purpose served by the examples is to indicate some methods for constructing Jónsson ω_0-generated modules other than those already encountered in the paper.

EXAMPLES 3.1. Assume that D_1 and D_2 are quasi-local domains with quotient field K. Let M_i be the maximal ideal of D_i and assume that there exists a subfield k of K such that $D_i=k+M_i$ for each i. Assume that D_2 is a rank-one discrete valuation ring, that $D_1 \nsubseteq D_2$, and let $D=k+(M_1 \cap M_2)$. Then D_1/D is a faithful Jónsson ω_0-generated D-module.

PROOF. Pick $x \in D_1-D_2$, let v be a valuation on K associated with D_2, and assume that $v(x)=t<0$. To prove that D_1/D is a Jónsson ω_0-generated D-module, we show that the hypothesis and condition (2) of Proposition 2.5 are satisfied for the sequence $\{(D+Dx^i)/D\}_{i=1}^{\infty}$ of submodules of D_1/D. As a first step in this process, we show that $D+Dx^i<D+Dx^{i+1}$ for each i and that $D_1 =\bigcup_{i=1}^{\infty} (D+Dx^i)$. Toward this end, we prove the following assertion.

(3.2) If $r \in D_1$, if $s \in D_1-D_2$, and if $v(s)<v(r)$, then $r \in D+Ds$.

To prove (3.2), consider first the case where s is a unit and r is a nonunit of D_1. Then $r/s \in M_1$, and since $v(r/s)>0$, $r/s \in M_2$ as well. Hence $r \in Ds$ in this case. On the other hand, if s is a nonunit of D_1, then we can replace s by the unit $s_1=s+1$ without affecting the hypothesis or the conclusion since $s_1 \in D_1-D_2$, $v(s)=v(s_1)$ and $D+Ds=D+Ds_1$. Similarly, if r is a unit of D_1, then $r_1=r-u \in M_1$ for some nonzero element u of k, and replacing r by r_1 yields the desired conclusion. This establishes (3.2).

It follows from (3.2) that

$$D_1 = \bigcup_{i=1}^{\infty} (D+Dx^i)$$

and that $D+Dx^i \subseteqq D+Dx^{i+1}$. The minimum of the v-values of elements of $D+Dx^i$ is $v(x^i)=it$, and hence $x^{i+1} \notin D+Dx^i$. Thus, the inclusion $D+Dx^i \subseteqq D+Dx^{i+1}$ is proper. Statement (3.2) also implies that if N is a proper D-submodule of D_1 containing D, then the set of v-values of elements of N is bounded below, and hence $N \subseteqq D+Dx^i$ for some i. Thus, to complete the proof that D_1/D is a Jónsson ω_0-generated module, we need only show that $(D+Dx^i)/D$ is Noetherian for each i. It is clear that $M_1 \cap M_2^{-it}$ is contained in the annihilator of $(D+Dx^i)/D$, and we show that $(D+Dx^i)/D$ is Noetherian by showing that $D/(M_1 \cap M_2^n)$ is a Noetherian ring for each positive integer n. Since $D/(M_1 \cap M_2^n)$ is zero-dimensional, it suffices to show that $(M_1 \cap M_2)/(M_1 \cap M_2^n)$ is finitely generated. Assume that $r_1<r_2<\ldots<r_h$ are the values less than n that are realized as the v-value of an element of $M_1 \cap M_2$, and choose $y_1, y_2, \ldots y_h \in M_1 \cap M_2$ such that $v(y_i)=r_i$ for each i. We show that

$$M_1 \cap M_2 = (M_1 \cap M_2^n)+\sum_{i=1}^{h} Dy_i .$$

Clearly $(M_1 \cap M_2^n)$ contains each element of $M_1 \cap M_2$ of v-value greater than r_h. Assume that $(M_1 \cap M_2)^n+\sum_{i=1}^{h} Dy_i$ contains each element of $M_1 \cap M_2$ of v-value greater than r_i, and pick $y \in M_1 \cap M_2$ such that $v(y)=r_i$. Then $y/y_i=a+m_2$ for some nonzero element a of k and some $m_2 \in M_2$. Therefore

$$y-ay_i = m_2y_i \in M_1 \cap M_2 \quad \text{and} \quad v(m_2y_i) > v(y_i) = r_i .$$

It follows that $y-ay_i \in (M_1 \cap M_2^n)+\sum_1^h Dy_j$, and consequently, y belongs to this set as well. By induction, we conclude that

$$M_1 \cap M_2 = (M_1 \cap M_2^n)+\sum_1^h Dy_j ,$$

and this completes the proof that D_1/D is a Jónsson ω_0-generated module. To see that D_1/D is faithful, take $d \in D-\{0\}$. For i sufficiently large, $dx^i \notin D_2$, and hence $dx^i \notin D$. Therefore D_1/D is faithful, which establishes Example 3.1.

If k is a field and $\{X_i\}_{i=1}^{\infty}$ is a set of indeterminates over k, then the field $K = k(\{X_i\}_1^{\infty})$ admits independent valuations v, w such that v is rank-one discrete, the valuation ring D_2 of v is of the form $k+M_2$, and the valuation ring D_1 of w is of the form $k+M_1$, where M_i is the maximal ideal of D_i. If $D=k+(M_1 \cap M_2)$, then Example 3.1 shows that D_1/D is a Jónsson ω_0-generated D-module. We note that $\dim D = \dim D_1$, and $\dim D_1$ can be any positive integer or it can be infinite. Moreover, if D_1 is chosen so that M_1 is unbranched (that is, so that M_1 is the only M_1-primary ideal) [7, p. 189], then no principal ideal of D is primary for $M_1 \cap M_2$. Thus, the assumption that a quasi-local domain admits a faithful Jónsson ω_0-generated module does not imply that the domain is Noetherian, and it imposes no restriction on its dimension. In the case where D_1 is rank-one non-discrete, if $B \neq M_1$ is any M_1-primary ideal, then the domain $D_3 = k+B$ is quasi-local with quotient field K and in the domain $D^* = k+(B \cap M_2)$, the residue class ring $D^*/(B \cap M_2)^n$ is non-Noetherian for each $n>1$. The approximation theorem for independent valuations can be used to show that $(B \cap M_2)/(B \cap M_2)^n \cong B/B^n$, which is not finitely generated.

The next example will be used to show that even in the case of a Noetherian domain D, existence of a faithful non-finitely generated Jónsson ω_0-generated module over D imposes no restriction on the dimension of D.

EXAMPLES 3.3. Assume that D is an integral domain with quotient field K, that (W, M) is a rank-one discrete valuation ring on K containing D, and that $W/M = D/P$, where P is the center of W on D. Then K/W is a Jónsson ω_0-generated module over D.

PROOF. Let w be a valuation associated with W, and assume without loss of generality that $\mathbf{Z}$ is the value group of w. Choose $y \in K$ of w-value 1, and express y as r/s with $r, s \in D$. Then $w(r)-w(s)=1$, so the additive subsemigroup of $\mathbf{Z}^+$ generated by $w(r)$ and $w(s)$ contains all integers $\geqq c$ for some positive integer c [13, Theorem 1.4.1]. We proceed to establish the following statement (3.4) that is analogous to (3.2) in the preceding proof.

(3.4) If $q \in K-\{0\}$ and if $w(q)=b$, then $W+Dq$ contains each element of K of w-value $\geqq b+c$.

For a proof, we note that $W+Dq$ contains each element of K of w-value $\geqq 0$. Thus, assume that $s \geqq b+c$ and that $W+Dq$ contains each element of K of w-value $>s$. Pick $t \in K$ of w-value s. By hypothesis, D contains an element d of w-value $s-b$. Thus, $w(t/dq)=0$, so $t/dq = e+m$ for some $e \in D$, $m \in M$ since $W/M = D/P$. Then $t-deq = dqm$ has w-value greater than s, and, is therefore in $W+Dq$. Consequently, $t \in W+Dq$, and this establishes (3.4).

To verify the assertions of Example 3.3, we choose $y \in P-\{0\}$ and let $x = y^{-1}$. Then

$$W \subsetneqq W+Dx \subsetneqq W+Dx^2 \subsetneqq \dots ,$$

and we show that the sequence $\{(W+Dx^i)/W\}_{i=1}^{\infty}$ satisfies the hypothesis and condition (2) of Proposition 2.5. It follows from (3.4) that

$$K = \bigcup_{i=1}^{\infty} (W+Dx^i) ,$$

and the inclusion $W+Dx^i \subsetneqq W+Dx^{i+1}$ is proper since $iw(x)$ is the minimum of the w-values of elements of $W+Dx^i$. Also, (3.4) shows that each proper submodule of K/W is contained in some $(W+Dx^i)/W$. To show that $(W+Dx^i)/W$ is Noetherian, we note that $D \cap M^n$ is contained in the annihilator of $(W+Dx^i)/W$ for n sufficiently large, and hence it suffices to show that $D/(D \cap M^n)$ is a Noetherian ring; equivalently, we show that $P/(D \cap M^n)$ is finitely generated. To this end, let $r_1 < r_2 < \dots < r_h$ be the integers less than n that are assumed as w-values of elements of P, and assume that $y_1, y_2, \dots, y_h \in P$ are such that $w(y_i) = r_i$ for each i. We show that

$$P \subseteq (D \cap M^n) + \sum_1^h Dy_j .$$

Each element of P of w-value greater than r_h is in $(D \cap M^n) + \sum_1^h Dy_j$. Assume that this is the case for each element of P of w-value greater than r_i, and take $p \in P$ such that $W(p) = r_i$. Then $p/y_i = d + m$ for some $d \in D$, $m \in M$. Hence $p - dy_i = my_i$ is an element of P of w-value greater than r_i, so $p - dy_i$, and hence p, belongs to $(D \cap M^n) + \sum_1^h Dy_j$. This completes the proof that K/W is a Jónsson ω_0-generated module.

To obtain an example of the type alluded to before Example 3.3, let k be a field, let n be a positive integer, and choose elements $x_1, \dots, x_n \in Yk[[Y]]$ such that $\{x_i\}_{i=1}^n$ is algebraically independent over k. Then

$$D = k[x_1, \dots, x_n]_{(x_1, \dots, x_n)}$$

is an n-dimensional regular local ring and

$$W = k[[Y]] \cap k(x_1, \dots, x_n)$$

is a rank-one discrete valuation overring of D such that D and W have residue field k. By Example 3.3, $k(x_1, \dots, x_n)/W$ is a faithful Jónsson ω_0-generated module over D.

Let D be an integral domain with quotient field K. In contrast with the situation for ω_0 described in (2.6), we remark that for each regular cardinal

$\alpha > \omega_0$, there exists a domain D with infinitely many maximal ideals such that K is a torsion-free Jónsson α-generated D-module. To obtain such an example, let F be a field of cardinality less than α, let t be an indeterminate over F, and let $k = F(t)$. It X is a set of indeterminates over k of cardinality α, then as in the paragraph preceding Theorem 1.4, we can construct a valuation ring V on $K = k(X)$ such that the set of nonzero prime ideals of V, ordered under reverse inclusion, is order-isomorphic to the set of ordinals preceding the first ordinal of cardinality α [9, p. 797]. Moreover, this can be done in such a way that $V = k + M$, where M is the maximal ideal of V. As previously noted, K is a Jónsson α-generated V-module in this case. Let $D = F[t] + M$. The cardinality of the set of maximal ideals of D is $\sup\{\omega_0, |F|\}$ and we claim that K is a Jónsson α-generated D-module. For a proof, we first observe that V is generated as a module over D by $F(t)$, a set of cardinality less than α. Hence, each proper V-*submodule* of K can be generated, *as a D-module*, by a set of cardinality less than α. By the same token, K can be generated as a D-module by a set of cardinality α, but by no set of smaller cardinality. Now let H be an arbitrary proper D-submodule of K. We show that the assumption that $VH = K$ leads to a contradiction. Thus, choose $m \in M$, $m \neq 0$. Then

$$K = mK = mVH \subseteqq MH \subseteqq DH = H ,$$

contrary to hypothesis. Hence $VH < K$, so VH has a generating set of cardinality less than α as a V-module. Since MH is a V-module, the proof above shows that MH also has a generating set of cardinality less than α as a D-module. Moreover, $H/MH \subseteqq VH/MH$, and since $V/M \cong F(t)$ has cardinality less than α, the vector space VH/MH over V/M also has cardinality less than α. Hence H/MH, *as a set*, has cardinality less than α so that H as a D-module is generated by fewer than α elements. This completes justification of our claim that K is a Jónsson α-generated D-module.

By considering the torsion Jónsson α-generated D-module K/D in the example above, we see that, also in contrast with the situation for ω_0 in Theorem 2.4, the set of zero divisors on such a module need not form a prime ideal for $\alpha > \omega_0$.

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FLORIDA STATE UNIVERSITY
TALLAHASSEE, FLORIDA 32304, U.S.A.

AND

PURDUE UNIVERSITY
W. LAFAYETTE, INDIANA 47907, U.S.A.