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WHITNEY C^∞-TOPOLOGIES AND THE BAIRE PROPERTY

JENS GRAVESEN

A short examination of the proof that a complete metric space (X,d) is a Baire space, as presented e.g. in ([4, 5.6]), reveals that the proof only uses that the set of balls in (X,d) is a base for the topology with the following

Intersection property (For complete metric spaces).
Let

$$\{B(x_n,r_n) \mid x_n \in X,\ r_n \in \mathbf{R}_+\}$$

be a sequence of balls in (X,d) such that $r_n \searrow 0$ and $\overline{B(x_{n+1},r_{n+1}} \subseteq B(x_n,r_n)$. Then $\bigcap_{n=1}^{\infty} B(x_n,r_n) \neq \emptyset$.

As usual we define the ball with center $x \in X$ and radius $r \in \mathbf{R}_+$ in a metric space (X,d) by

$$B(x,r) = \{y \in X \mid d(x,y) < r\} .$$

We shall indicate how to adapt this proof to show that Whitney C^∞-topologies have the Baire property.

Let X and Y be finite dimensional, paracompact smooth manifolds. Denote by $J^k(X,Y)$ the space of k-jets of X into Y. It is well known that X, Y and $J^k(X,Y)$ are completely metrizable spaces, see e.g. ([3, Proposition 5.11]). Choose complete metrics d_X on X, d_Y on Y, and d_k on $J^k(X,Y)$, such that $d_0 = d_X \times d_Y$, and such that the projections $\pi_{l,k}\colon J^k(X,Y) \to J^l(X,Y)$ are contracting for $l \leqq k$, i.e.

$$d_l(\pi_{l,k}(j^k f(x)), \pi_{l,k}(j^k g(y))) \leqq d_k(j^k f(x), j^k g(y))$$

for arbitrary k-jets $j^k f(x), j^k g(y) \in J^k(X,Y)$. Such a choice of metrics is possible, since we can always put $d_0 = d_X \times d_Y$, and if $\{\tilde{d}_k \mid k \in \mathbf{N}\}$ is an arbitrary family of complete metrics, then we can put

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$$d_k = d_0 \circ (\pi_{0,k} \times \pi_{0,k}) + \sum_{l=1}^{k} \tilde{d}_l \circ (\pi_{l,k} \times \pi_{l,k}),$$

and thereby obtain a family of complete metrics $\{d_k \mid k \in \mathbb{N}_0\}$ with the properties required.

For $f \in C^\infty(X, Y)$, $k \in \mathbb{N}$ and $\delta \in C(X, \mathbb{R}_+)$, we define the

Whitney ball

$$B^k_\delta(f) = \{g \in C^\infty(X, Y) \mid \forall x \in X : d_k(j^k f(x), j^k g(x)) < \delta(x)\}.$$

The Whitney balls $B^k_\delta(f)$ shall play the role played by the balls in the case of metric spaces. The Whitney balls $B^k_\delta(f)$ do indeed form a base for the Whitney C^∞-topology, see e.g. ([3, p. 43]), and we have the following

Intersection property (For Whitney C^∞-topologies).

Let

$$\{B^{k_n}_{\delta_n}(f_n) \mid f_n \in C^\infty(X, Y),\ k_n \in \mathbb{N},\ \delta_n \in C(X, \mathbb{R}_+)\}$$

be a sequence of Whitney balls in $C^\infty(X, Y)$ with the Whitney C^∞-topology, such that $k_n \nearrow \infty$, $\delta_n \searrow 0$ uniformly on X, and

$$B^{k_{n+1}}_{\delta_{n+1}}(f_{n+1}) \subseteqq B^{k_n}_{\delta_n}(f_n).$$

Then $\bigcap_{n=1}^{\infty} B^{k_n}_{\delta_n}(f_n) \neq \varnothing$.

PROOF. Let $x \in X$ be given. We have that

$$d_y(f_n(x), f_{n+p}(x)) \leqq d_{k_n}(j^{k_n} f_n(x), j^{k_n} f_{n+p}(x)) < \delta_n(x).$$

Hence $(f_n(x))_{n \in \mathbb{N}}$ is a Cauchy sequence in Y and therefore convergent. If we denote the limit point by $f(x)$, then $x \to f(x)$ is a map $f\colon X \to Y$. We will show that f is smooth. Let $k \in \mathbb{N}$ be given. For sufficiently large n, we have $k_n > k$, and therefore

$$d_k(j^k f_n(x), j^k f_{n+p}(x)) \leqq d_{k_n}(j^{k_n} f_n(x), j^{k_n} f_{n+p}(x)) < \delta_n(x).$$

Hence the sequnce of k-jets converges uniformly. In a chart all the derivatives up to order k converges uniformly on compact sets in the standard metric, because uniform convergence on compact sets gives the compact open topology on the continuous functions, see ([2, Theorem 4.2.17]). It is now easy to show that f is smooth and belongs to $\bigcap_{n=1}^{\infty} B^{k_n}_{\delta_n}(f_n)$. Thus

$$\bigcap_{n=1}^{\infty} B_{\delta_n}^{k_n}(f_n) \neq \emptyset$$

as should be proved.

If we in the proof ([4, 5.6]) replace the balls with the Whitney balls, then we easily get that $C^\infty(X, Y)$ with the Whitney C^∞-topology is a Baire space. We can in fact prove that it is strongly α-favorable, see ([1, pp. 115–120]).

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MATEMATISK INSTITUT
KØBENHAVNS UNIVERSITET
UNIVERSITETSPARKEN 5
2100 KØBENHAVN Ø
DENMARK