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M-T TOPOLOGICALLY STABLE MAPPINGS ARE UNIFORMLY STABLE

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0. Introduction.

In [3] Mather proves that the topologically stable mappings from a compact source are open and dense in the space of mappings. This can be done by constructing a stratification of the jetspace, and showing that the mappings which are multitransverse to the stratification are topologically stable. In fact if f is multitransverse and g is sufficiently close to f, there exists a curve f_t, $t \in [0,1]$ such that $f_0 = f$, $f_1 = g$, and f_t are topologically equivalent to f. Inspecting Mather's proof more closely, one finds that the homeomorphisms conjugating f_t to f depends continuously on t.

A stronger continuity statement can be made; there exists a neighbourhood of f such that all mappings in this neighbourhood are topologically equivalent to f via a continuous mapping from the neighbourhood to the space of homeomorphisms in the source and the target. (The mapping spaces are given the Whitney C^∞ or C^0 topology.) This shows that the multitransverse (m-t) topologically stable mappings have the same continuity property as proper smoothly stable mappings (see [2, Theorem 2]). Quite likely it is known by the experts that m-t topologically stable mappings have this property, but I have never seen an explicit proof of this fact and the purpose of the article is to give such a proof.

1. Definitions and the Theorem.

This article deals with topologically stable mappings. A general reference for definitions and technical details is Mather's article [3].

Consider C^∞ mappings $f\colon N \to P$, where N and P are C^∞ manifolds of dimension n and p, respectively, and N is compact. Let $C^\infty(N,P)$ denote the space of such mappings. Let $\theta(N)$, $\theta(P)$, and $\theta(f)$ denote vector fields on N, P, and along f, respectively, and let $tf\colon \theta(N) \to \theta(f)$ be as described in [2]. Now let $f\colon (N,x) \to (P,f(x))$ be a germ of a C^∞ mapping, and let $\theta(N)_x$, $\theta(P)_{f(x)}$, and $\theta(f)_x$ denote the sets of the corresponding germs of vector fields. Define

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$$K(f,x) = \dim_{\mathbf{R}} \frac{\theta(f)_x}{tf(\theta(N)_x)+f^*[m_{f(x)}]\theta(f)_x}\,.$$

As explained in [3] whether $K(f,x) \leqq k$ depends only on the jet $j^{k+1}f(x)$.

In the jetspace $J^k(n,p)$, let Σ_k denote the set of jets z such that $K(f,0) \leqq k-1$ for one, hence for every, representative f: $(\mathbf{R}^n,0) \to (\mathbf{R}^p,0)$ of z.

In [3] Mather defines a canonical stratification of $J^k(n,p)-\Sigma_k$ called $S_1^{n,p,(k)}$. If N,P are C^∞ manifolds as above, we can construct the subbundle $\Sigma_k(N,P)$ and the corresponding stratification $S_1^{n,p,(k)}(N,P)$ of the space $J^k(N,P)-\Sigma_k(N,P)$.

Let $f \in C^\infty(N,P)$ and assume that for some k, $j^kf(N) \cap \Sigma_k(N,P) = \emptyset$ and that j^kf is multitransverse to $S_1^{n,p,(k)}(N,P)$. Then it is proven in [3] that f is topologically stable. We call mappings satisfying these two conditions m-t *topologically stable mappings.*

Definition. Let $f \in C^\infty(N,P)$ and assume that we can find a neighbourhood W of f in $C^\infty(N,P)$ in the Whitney C^∞ topology, and continuous mappings H: $W \to \mathrm{Hom}(N,N)$, K: $W \to \mathrm{Hom}(P,P)$ such that $H(f)=\mathrm{id}_N$, $K(f)=\mathrm{id}_P$ and

$$g = K(g)\circ f\circ H(g)\,.$$

We will call mappings satisfying this continuity condition uniformly topologically stable (here $\mathrm{Hom}(N,N)$ and $\mathrm{Hom}(P,P)$ are given the Whitney C^0 topology).

We have

Theorem. *Let $f \in C^\infty(N,P)$. If f is* m-t *topologically stable, then f is uniformly topologically stable.*

Remark. Since the property to be a m-t topologically stable mapping is generic (it is satisfied for a residual set of mappings in $C^\infty(N,P)$), Theorem 1 shows that there is a dense set in $C^\infty(N,P)$ consisting of topologically stable mappings with this continuity property.

2. Proof of the Theorem.

Let $f \in C^\infty(N,P)$ be a m-t topologically stable mapping. Since

$$j^kf(N) \cap \Sigma_k(N,P) = \emptyset \qquad \text{for some } k\,,$$

it follows that f is of finite singularity type, so that we can find a stable unfolding of f. Following [3] the unfolding can be chosen to be of the form

$$\begin{array}{ccc} N & \xrightarrow{i} & N' = N \times U \\ {\scriptstyle f}\downarrow & & {\scriptstyle F}\downarrow \\ P & \xrightarrow{j} & P' = P \times U \end{array}$$

where U is an open neighbourhood of the origin in $\mathbf{R}^l$ (where $l = \dim_{C^\infty(P)} \theta(f)/tf\theta(N)$), i and j are the canonical injections $N \to N \times \{0\}$, $P \to P \times \{0\}$, and F is of the form

$$F(x,t) = (f_1(x,t),t) \in P \times U \quad \text{for } (x,t) \in N \times U .$$

In the proof of the theorem we will need the following proposition:

PROPOSITION. *Let f: $N \to P$ be a mapping of finite singularity type. Let*

$$\begin{array}{ccc} N & \xrightarrow{i} & N' \\ {\scriptstyle f}\downarrow & & {\scriptstyle F}\downarrow \\ P & \xrightarrow{j} & P' \end{array}$$

be a stable unfolding of f. Then there is a neighbourhood W of f in $C^\infty(N,P)$ and continuous mappings H: $W \to C^\infty(N,N')$, K: $W \to C^\infty(P,P')$ with $H(f)=i$, $K(f)=j$, such that the diagram

$$\begin{array}{ccc} N & \xrightarrow{H(g)} & N' \\ {\scriptstyle g}\downarrow & & {\scriptstyle F}\downarrow \\ P & \xrightarrow{K(g)} & P' \end{array}$$

is a stable unfolding of g.

PROOF. Let k: $P' \to \mathbf{R}^n$ be a closed embedding into Euclidean space and let (U,π) be a tubular neighbourhood of $k(P')$. Using i we will consider N as a submanifold of N', and using j and k we will consider P as a submanifold of $\mathbf{R}^n$. Since $N \subset N'$ is compact, there are a finite number of charts (U_j,ψ_j), $j=1,\ldots,m$ in N' with $N \subset \bigcup_{j=1}^m U_j$ such that in the chart (U_j,ψ_j) we have coordinates (x,t), where $(x,0)$ are coordinates in $U_j \cap N$. Further we will choose $\bar{U}_j$ compact. Let V be an open set in N' with $\bar{V} \subset \bigcup_{j=1}^m U_j$, and $N \subset V$. Let $U_{m+1} = N' - \bar{V}$. Then $\{U_j\}_{j=1}^{m+1}$ is an open covering of N'. Let $\{\varphi_j\}_{j=1}^{m+1}$ be a partition of unity subordinate to $\{U_j\}$. Let $g \in C^\infty(N,P)$, and define $\tilde{G}_j$: $U_j \to \mathbf{R}^n$, $j=1,\ldots,m$ by

$$\tilde{G}_j(x,t) = \varphi_j(x,t)(g(x,0)-f(x,0)) .$$

Since $\operatorname{supp}\varphi_j \subset U_j$, we can extend $\tilde{G}_j(x,t)$ to N' setting $\tilde{G} \equiv 0$ outside U_j. Hence we get a mapping $g \to \tilde{G}_j$, $C^\infty(N,P) \to C^\infty(N',\mathbf{R}^n)$. Since the partial derivatives of φ_j are bounded on the compact set $\operatorname{supp}\varphi_j$ and vanish outside $\operatorname{supp}\varphi_j$, it is clear that this mapping is continuous in the Whitney C^∞ topology. Define

$$\tilde{G}(g) = \sum_{j=1}^{m} \tilde{G}_j + F .$$

By Proposition 2 and Corollary 1, § 2 of [2], $g \to \tilde{G}(g)$ is a continuous mapping $C^\infty(N,P) \to C^\infty(N',\mathrm{R}^n)$. From the definition of $\tilde{G}(g)$ follows that $\tilde{G}(g) \mid N = g$. By continuity, $\tilde{G}(g)(N') \subset U$ for g sufficiently close to f. Hence in a neighbourhood W of f we can define $G(g) = \pi \circ \tilde{G}(g)$. By Proposition 2, § 2 of [2] this defines a continuous mapping $G\colon W \to C^\infty(N',P')$. Notice that $G(g) \mid N = g$ and that $G(f) = F$. From the continuity of the mapping follows that $G(g) \pitchfork j(P)$, if g is sufficiently close to f. Since F is stable, it also follows that $G(g)$ is smoothly equivalent to F. Then by [2, Theorem 2, § 3], it is possible to find $\tilde{H}(g) \in \operatorname{Diff} N'$, $\tilde{K}(g) \in \operatorname{Diff} P'$ depending continuously on g, such that

$$G(g) = \tilde{K}(g) \circ F \circ \tilde{H}(g) ,$$

where $\tilde{H}(f) = \mathrm{id}_{N'}$, $\tilde{K}(f) = \mathrm{id}_{P'}$. Hence in a sufficiently small neighbourhood W of f in $C^\infty(N,P)$, we can define

$$H(g) = \tilde{H}(g) \circ i \quad \text{and} \quad K(g) = \tilde{K}(g)^{-1} \circ j .$$

$H(g)$, $K(g)$ are continuously dependent on g by Proposition 1 and Proposition 5, § 2 of [2], and by construction $H(f) = i$, $K(f) = j$. Since

$$G(g) \circ i = j \circ g \quad \text{and} \quad G(g) = \tilde{K}(g) \circ F \circ \tilde{H}(g) ,$$

the diagram

$$\begin{array}{ccc} N & \xrightarrow{H(g)} & N' \\ {\scriptstyle g}\downarrow & & {\scriptstyle F}\downarrow \\ P & \xrightarrow{K(g)} & P' \end{array}$$

commutes. Since $G(g) \pitchfork j(P)$ and $F = \tilde{K}(g)^{-1} G(g) \tilde{H}(g)^{-1}$, $F \pitchfork \tilde{K}(g)^{-1} \circ j(P) = K(g)(P)$. Since i and j are closed embeddings $H(g)$ and $K(g)$ are also closed embeddings for W chosen sufficiently small.

At last since $F^{-1}(K(g)(P)) = (\tilde{K}(g) \circ F)^{-1}(j(P))$ and $\tilde{K}(g) \circ F$ is close to F, it follows that $F^{-1}(K(g)(P))$ and $F^{-1}(j(P)) = i(N)$ are diffeomorphic. Since these manifolds then are diffeomorphic with $H(g)(N)$ and $H(g)(N) \subseteq F^{-1}(K(g)(P))$, we have $F^{-1}(K(g)(P)) = H(g)(N)$. This completes the proof of the proposition.

Let us now return to the proof of the theorem. Let f be a m-t topologically stable mapping and let

$$\begin{array}{ccc} N & \xrightarrow{i} & N' = N \times U \\ {\scriptstyle f}\downarrow & & {\scriptstyle F}\downarrow \\ P & \longrightarrow & P' = P \times U \end{array}$$

be a stable unfolding as described above. Since $j^k f$ is multitransverse to the stratification $S_1^{n,p(k)}$ for some k, we get as explained in [3], Whitney stratifications $S_1^{n+l,p+l}(F)$, $S_3^{n+l,p+l}(F)$ of N' and $S_2^{n+l,p+l}(F)$ of P' such that $(S_3^{n+l,p+l}(F), S_2^{n+l,p+l}(F))$ will be a Thom stratification of F in the sense of [1]. As Mather explains in [3], i will be transverse to $S_1^{n+l,p+l}(F)$ and $S_3^{n+l,p+l}(F)$ and j will be transverse to $S_2^{n+l,p+l}(F)$.

Consider the two mappings

$$N \times U \xrightarrow{F} P \times U \xrightarrow{\text{proj.}} U .$$

From the fact that $j(P) = P \times \{0\}$ is transverse to $S_2^{n+l,p+l}(F)$, it follows that the restriction of the projection $P \times U \to U$ to the strata of $S_2^{n+l,p+l}(F)$ is a submersion in a neighbourhood of $P \times \{0\}$. Now give U the trivial stratification, and let N', P' be stratified by $S_3^{n+l,p+l}(F)$, $S_2^{n+l,p+l}(F)$. Shrinking U if necessary, the diagram

$$N \times U \xrightarrow{F} P \times U \xrightarrow{\text{proj.}} U$$

is a diagram of Thom stratified mappings in the sense of [1], hence the stratified mapping is trivial over U. (Note that when P is not compact, we have to modify the stratifications of N' and P' slightly the same way as done in [1, chapter IV (3.5)] to have control at infinity.)

Let $\partial/\partial t_1, \ldots, \partial/\partial t_l$ be the coordinate vector fields on U. Since we have a diagram of stratified mappings we use the results of [1, chapter II] to lift these vectorfield to controlled vector fields $\xi_1, \ldots, \xi_l$ on $P \times U$, and these vector fields can be lifted further to controlled vector fields $\eta_1, \ldots, \eta_l$ on $N \times U$. Again using the results from [1], we can integrate these vector fields to get continuous flows θ_i of ξ_i, and ω_i of η_i, $i = 1, \ldots, l$.

After shrinking U if necessary, we use the proposition to find a neighbourhood W of f in $C^\infty(N,P)$ and continuous mappings H: $W \to C^\infty(N,N')$, K: $W \to C^\infty(P,P')$ such that

$$\begin{array}{ccccc} N & \xrightarrow{H(g)} & N' & \doteq & N \times U \\ {\scriptstyle g}\downarrow & & {\scriptstyle F}\downarrow & & \\ P & \xrightarrow{K(g)} & P' & = & P \times U \end{array}$$

is a stable unfolding of g.

Let us consider the two mappings $H(g)$, $K(g)$ and for $x \in N$, $y \in P$ write these mappings as

$$H(g)(x) = (h_g(x), \tilde{h}_g(x))$$

$$K(g)(y) = (k_g(y), \tilde{k}_g(y))$$

where $h_g(x) \in N$, $k_g(y) \in P$, and $\tilde{h}_g(x)$, $\tilde{k}_g(y) \in U$.

h_g, k_g will be close to the identity mappings id_N and id_P respectively so that, if

W is chosen small enough, we can assume that they are diffeomorphisms of N and P respectively.

Now let us define two other mappings.

$$\bar{H}(g) = H(g) \circ h_g^{-1} : N \to N' = N \times U$$
$$\bar{K}(g) = K(g) \circ k_g^{-1} : P \to P' = P \times U .$$

The mapping $\bar{H}(g)$ will associate to each point x in N the point in $\operatorname{im} H(g)$ having x as component in the N direction. (There is only one such point since h_g is a diffeomorphism.) $\bar{K}(g)$ has the similar property with respect to $K(g)$.

Note that $\bar{H}(g)$, $\bar{K}(g)$ are mappings continuously dependent on g by [2, Proposition 1 and 5, § 2], since h_g^{-1}, k_g^{-1} are diffeomorphisms and hence are proper mappings.

Define a mapping $\bar{g}$: $N \to P$ by

$$\bar{g} = \bar{K}(g)^{-1} \circ F \circ \bar{H}(g) .$$

This makes sense since $F(H(g)(N)) \subset K(g)(P)$, $\operatorname{im} \bar{H}(g) = \operatorname{im} H(g)$, and $\operatorname{im} \bar{K}(g) = \operatorname{im} K(g)$.

We can illustrate the mapping $\bar{g}$ by Figure 1. A direct computation shows that $g = k_g^{-1} \circ \bar{g} \circ h_g$.

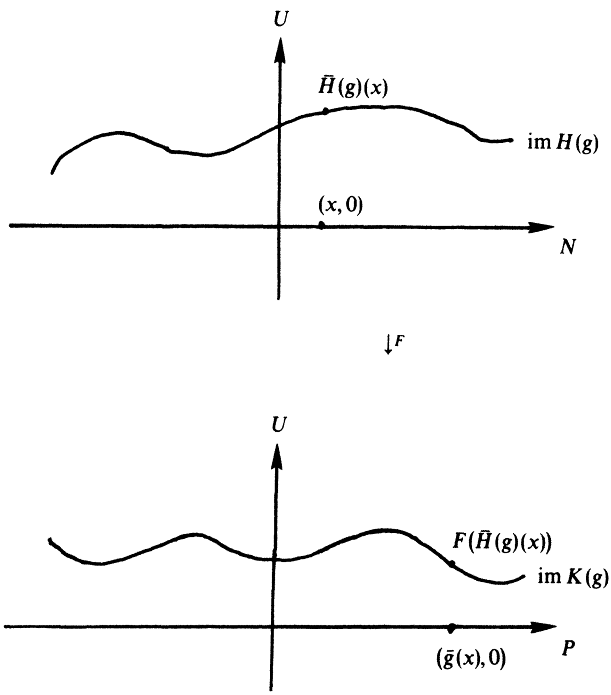

Figure 1.

Now we will construct homeomorphisms $h_{\bar{g}}$ and $k_{\bar{g}}$ of N and P respectively such that

$$k_{\bar{g}} \circ \bar{g} = f \circ h_{\bar{g}} .$$

To construct these homeomorphisms look at the mapping $\bar{h}_g \circ h_g^{-1}$: $N \to U$ which is the U component of $\bar{H}(g)$. Let the different components of this mapping be $h_g^1, \ldots, h_g^l$. Given a point x in N associate the point

$$\omega(x) = \omega_1(-h_g^1(x), \omega_2(\ldots, \omega_{l-1}(-h_g^{l-1}(x), \omega_l(-h_g^l(x), \bar{H}_g(x)) \ldots) .$$

Note that since the mapping F is of the form $F(x,t) = (f_1(x,t), t)$ and the ω_i's are lifts of the coordinate flows t_i's in U, ω_i will be the standard linear flow in the t_i direction. Since $\bar{H}(g)(x)$ has U component $(h_g^1(x), \ldots, h_g^l(x))$, it follows that $\omega(x)$ is a point in $N \times \{0\}$.

Now define $h_{\bar{g}}$ by $h_{\bar{g}}(x) = i^{-1} \circ \omega(x)$.

If $g = f$ we will have $h_{\bar{g}}(x) = x$. Inspecting the formula for $h_{\bar{g}}$, it follows from Proposition 2, § 2 of [2] that $h_{\bar{g}}$ is continuously dependent on g. Hence if the neighbourhood W is chosen small enough, $h_{\bar{g}}$ will be close to id_N, and we can assume that $h_{\bar{g}}$ is a homeomorphism of N.

We define $k_{\bar{g}}$ in exactly the same way in terms of $\bar{K}(g)$, the θ_i's and the embedding j.

Since F commutes with the flows it follows that $k_{\bar{g}} \circ g = f \circ h_{\bar{g}}$.

To prove this: first suppose $l (= \dim U) = 1$.

From the fact that F commutes with the flow it follows that

$$F(\omega_1(s, (x,t))) = \theta_1(s, F(x,t)) .$$

Substituting $(x,t) = \bar{H}(g)(x)$, $s = -h_g^1(x)$, and identifying $\omega(x)$ with $h_{\bar{g}}(x)$ and $F \mid N \times \{0\}$ with f, we get

$$\theta_1(-h_g^1(x), F(\bar{H}(g)(x))) = f \circ h_{\bar{g}}(x) .$$

Now identifying $k_{\bar{g}}$ with $j \circ k_{\bar{g}}$, we get

$$k_{\bar{g}} \circ \bar{g}(x) = \theta_1(-k_g^1(\bar{g}(x)), \bar{K}(g)(\bar{g}(x))) .$$

Put $\bar{g} = \bar{K}(g)^{-1} \circ F \circ \bar{H}(g)$. Since F is U-level-preserving, the U component of $F(\bar{H}(g)(x))$ is $h_g^1(x)$. Since k_g^1 is the U component of $\bar{K}(g)$ it follows that.

$$-k_g^1(\bar{g}(x)) = -h_g^1(x) .$$

Hence we get

$$k_{\bar{g}} \circ \bar{g}(x) = \theta_1(-h_g^1(x), F \circ \bar{H}(g)(x)) = f \circ h_{\bar{g}}(x) .$$

When $l > 1$ the computations are similar.

Hence

$$g = k_g^{-1} \circ \bar{g} \circ h_g = k_g^{-1} \circ k_{\bar{g}}^{-1} \circ f \circ h_{\bar{g}} \circ h_g \, .$$

Since k_g^{-1}, $k_{\bar{g}}^{-1}$, $h_{\bar{g}}$, h_g are continuously dependent on g and are proper mappings (they are either diffeomorphisms or homeomorphisms), it follows that the homeomorphisms conjugating f to g depend continuously on g. This proves that f is a uniformly topologically stable mapping, completing the proof of the theorem.

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