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ON THE UNIQUENESS OF SEQUENTIAL LIMITS OF QUASICONFORMAL MAPPINGS

MATTI VUORINEN

1. Introduction.

In this paper we shall complete some earlier results about the existence of angular limits of quasiconformal (qc) mappings [10], [11]. We shall discuss the following two uniqueness problems related to the boundary behavior of a qc mapping $f\colon \mathbf{R}^n_+ \to \mathbf{R}^n$.

PROBLEM 1. Suppose that f has an angular limit α at 0 and $b_k \in \mathbf{R}^n_+, b_k \to 0$, $f(b_k) \to \beta$. When is $\alpha = \beta$?

PROBLEM 2. Suppose that $a_k, b_k \in \mathbf{R}^n_+$, a_k, $b_k \to 0$ and $f(a_k) \to \alpha$, $f(b_k) \to \beta$. Under which conditions is $\alpha = \beta$?

Concerning Problem 1 we shall show that $\alpha = \beta$, if the sequence (b_k) approaches 0 in a way not "too" tangential in the sense of Corollary 4.5 and Remark 3.10 and if the sequence (b_k) satisfies an isolation condition. Concerning Problem 2 we shall show that $\alpha = \beta$, if $\sum \varrho(a_k, b_k)^{1-n} = \infty$, and if the geodesic segments (in the hyperbolic geometry of $\mathbf{R}^n_+$) joining the points a_k to the points b_k are uniformly isolated. Here ϱ is the hyperbolic metric of $\mathbf{R}^n_+$.

The proofs are based on the application of the modulus method and on a qc counterpart [10, 6.5] of a result due to Bagemihl and Seidel [1, Theorem 1] in the case of normal meromorphic functions. Some preliminary results are given in Section 2. In Section 3 the existence of some tangential sequences of points is proved. The main results are in Section 4. It is possible that the main results, applicable to the conformal mappings of the complex plane, are new in this particular case as well.

2. Preliminary results.

In this section we shall list some preliminary lemmas, which deal with the modulus of a path family or the hyperbolic geometry. Throughout the paper

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we shall employ the relatively standard notation and terminology in Väisälä's book [8].

2.1. For $x \in \mathbf{R}^n, n \geqq 2$, and $r>0$ let

$$B^n(x,r) = \{z \in \mathbf{R}^n : |z-x| < r\},$$

$$S^{n-1}(x,r) = \partial B^n(x,r), \quad B^n(r) = B^n(0,r),$$

$$S^{n-1}(r) = \partial B^n(r), \quad B^n = B^n(1), \quad \text{and} \quad S^{n-1} = \partial B^n .$$

The standard coordinate unit vectors are $e_1, \ldots, e_n$. If $x \in \mathbf{R}^n$ and $b>a>0$, we write

$$R(x,b,a) = B^n(x,b) \setminus \bar{B}^n(x,a) \quad \text{and} \quad R(b,a) = R(0,b,a) .$$

For the definition and basic properties of the modulus $M(\Gamma)$ of a curve family Γ, the reader is referred to [8]. If $E, F, G \subset \mathbf{R}^n$, then $\Delta(E,F;G)$ is the family of all non-constant paths $\gamma:[0,1] \to G$ with $\gamma(0) \in E$, $\gamma(1) \in F$, and $\gamma(t) \in G$, $t \in (0,1)$. If $\varphi \in (0,\pi/2)$,

$$C(\varphi) = \{x \in \mathbf{R}^n : (x \mid e_n) > |x| \cos\varphi\} .$$

The notation $c(t,u,v)$ means that $c(t,u,v)$ depends only on t, u, and v.

2.2. THE CONTINUUM CRITERION. Let $x \in \mathbf{R}^n$ and $C \subset \mathbf{R}^n$ be compact. Then [4] $M(x,C)<\infty$, if there exists a non-degenerate continuum $K \subset \{x\} \cup (\mathbf{R}^n \setminus C)$ with $x \in K$ and $M(\Delta(K,C,\mathbf{R}^n \setminus C))<\infty$. Otherwise $M(x,C)=\infty$. For $E \subset \mathbf{R}^n$, $b \in \mathbf{R}^n$, and $t>r>0$ let

$$M_t(E,r,b) = M(\Delta(S^{n-1}(b,t), \bar{B}^n(b,r) \cap E; \mathbf{R}^n)) ,$$

$$M(E,r,b) = M_{2r}(E,r,b) .$$

The lower and upper capacity densities of E at b are defined, respectively, by (cf. [6], [10])

$$\operatorname{cap} \underline{\operatorname{dens}} (E,b) = \liminf_{r \to 0} M(E,r,b) ,$$

$$\operatorname{cap} \overline{\operatorname{dens}} (E,b) = \limsup_{r \to 0} M(E,r,b) .$$

2.3. REMARKS. (1) Let $C \subset \mathbf{R}^n$ be compact. It follows from [6, 2.15] or [11, 3.8] that $\operatorname{cap} \overline{\operatorname{dens}} (C,0)>0$ implies $M(0,C)=\infty$.

(2) Let $E=\{r \in (0,1) : S^{n-1}(r) \cap C \neq \emptyset\}$. Then $\int_E dr/r=\infty$ implies $M(0,C)=\infty$ by [8, 10.9].

(3) From a result of Martio and Sarvas [6, p. 773] it follows that there are sets C with $M(0,C)=\infty$ and $\overline{\text{cap dens}}\,(C,0)=0$. The set

$$C = \bigcup_{j=3}^{\infty} \bar{B}^n(2^{-j}e_1, 2^{-j}/j) \cup \{0\}$$

satisfies these conditions if the dimension $n \geqq 3$. The proof follows, if we apply (2) and [8, 6.4, 7.5, 7.9] (cf. [6]).

2.4. HYPERBOLIC GEOMETRY. The hyperbolic metric ϱ in

$$\mathsf{R}^n_+ = \{x=(x_1,\ldots,x_n) \in \mathsf{R}^n : x_n>0\}$$

has the element of length $d\varrho=|dx|/x_n$. If $x \in \mathsf{R}^n_+$ and $M>0$, we write

$$D(x,M) = \{z \in \mathsf{R}^n_+ : \varrho(z,x) < M\}.$$

A basic fact is that the hyperbolic balls are euclidean ones, and for instance

$$D(te_n, M) = B^n((t\cosh M)e_n, t\sinh M) \tag{2.5}$$

for $t>0$ and $M>0$. Let $x,y \in \mathsf{R}^n_+$. As in the case $n=2$ [2, Theorem 6.3.1 (ii)] we have the formula

$$\cosh \varrho(x,y) = 1+\frac{|x-y|^2}{2x_ny_n}. \tag{2.6}$$

The geodesic curve joining $a \in \mathsf{R}^n_+$ to $b \in \mathsf{R}^n_+$ lies on an circular arc through a and b, perpendicular to ∂R^n_+. Making use of this fact one calculates

$$\varrho(e_n, (\sin\alpha)e_1+(\cos\alpha)e_n) = \log\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right) \tag{2.7}$$

for $\alpha \in (0,\pi/2)$. The closed geodesic segment joining a to b is denoted by $J[a,b]$.

2.8. LEMMA. *Let $a,b \in \mathsf{R}^n_+$, $J=J[a,b]$, $M>0$, and let $D(J,M)=\bigcup_{x\in J}D(x,M)$. Then there is a quasiconformal mapping $f\colon \mathsf{R}^n \to \mathsf{R}^n$ such that $fD(J,M)=B^n$ and*

$$K(f) \leqq c(n,M,\varrho(a,b)) \leqq d(n,M) < \infty.$$

PROOF. By performing an auxiliary Möbius transformation, if necessary, we may assume that $a=e_n, b=te_n, t=\exp\varrho(a,b)$. Let

$$B_u = B^n((u\cosh M)e_n, u\sinh M), \qquad u>0.$$

From (2.5) it follows that

$$D(J,M) = B_1 \cup (R(t,1) \cap C(\alpha_M)) \cup B_t,$$

where $\alpha_M = \overline{\text{arc}}\tan(\sinh M)$. Therefore

$$B_t \subset D(J, M) \subset B^n\big((t\cosh M)e_n, (t-1)\cosh M + \sinh M\big).$$

The proof follows now from [7, 2.4, 2.7].

2.9. LEMMA. *Let $\mu > 0, a, b \in \mathbf{R}^n_+$ with $\varrho(a,b) > 2\mu$, and let $\Gamma(a,b,\mu) = \Delta\big(D(a,\mu), D(b,\mu); \mathbf{R}^n_+\big)$. There exists a positive constant $c(n,\mu)$ such that*

$$M\big(\Gamma(a,b,\mu)\big) \geqq c(n,\mu)\varrho(a,b)^{1-n}.$$

If $\varrho = \varrho(a,b) \geqq 4\mu$, then

$$M\big(\Gamma(a,b,\mu)\big) \leqq \omega_{n-1}\left(\frac{\varrho}{2} + A\right)^{1-n}; \quad \exp A = \frac{2\sinh(\varrho/4)}{\sinh\mu}.$$

PROOF. By performing an auxiliary Möbius transformation, if necessary, we may assume that $a = e_n$ and $b = \big(\exp\varrho(a,b)\big)e_n$. From (2.5) it follows that

$$S_1 = S^{n-1}(|a|) \cap C(\alpha_\mu) = D(a,\mu),$$
$$S_2 = S^{n-1}(|b|) \cap C(\alpha_\mu) \subset D(b,\mu),$$

where $\alpha_\mu = \overline{\text{arc}}\tan(\sinh\mu)$. Hence by [8, 7.7]

$$M\big(\Gamma(a,b,\mu)\big) \geqq M\big(\Delta(S_1,S_2;\mathbf{R}^n_+)\big) \geqq m_{n-1}(S_1)\left(\log\frac{|b|}{|a|}\right)^{1-n}$$
$$= m_{n-1}(S_1)\varrho(a,b)^{1-n}.$$

Let

$$c(n,\mu) = m_{n-1}(S_1) = \omega_{n-2}\int_0^{\alpha_\mu}\sin^{n-2}\theta\,d\theta.$$

For the upper bound we note that by (2.5) and [8, 7.5], we get

$$M\big(\Gamma(a,b,\mu)\big) \leqq \omega_{n-1}\left(\log\frac{e^{\varrho(a,b)-\mu} - \cosh\mu}{\sinh\mu}\right)^{1-n}.$$

The desired estimate follows from this upper bound by straightforward estimation, since $\varrho(a,b) \geqq 4\mu$.

In the following result we consider the hyperbolic geometry in a 2-dimensional plane.

2.10. LEMMA. *Let $\alpha \in (0, \pi/2)$, $M > 0$, and define*

$$d = \min\{|x_1| : x \in \mathbf{R}^2_+, \varrho(x, e_2) \geqq M \text{ and } x_2 = 1 + x_1\tan\alpha\}.$$

Then

$$d = \left((\tan^2\alpha + 2A)^{1/2} - \tan\alpha\right)/A ,$$

where

$$A = (1+\tan^2\alpha)/(\cosh M - 1) .$$

PROOF. By (2.5) it suffices to find the common points of $x_2 = 1 + x_1 \tan\alpha$ and $S^1((\cosh M)e_2, \sinh M)$. The absolute value of the negative root of the corresponding equation is d.

3. On tangential sequences of points.

In Section 4 we shall consider some sequences in $\mathbf{R}^n_+$ which approach 0 in a tangential way and satisfy, in addition, an isolation condition and a particular divergence condition. In the present section we shall show that such sequences exist, and indicate how such sequences can be generated.

Throughout the paper $h: (0,c) \to (0,\infty)$, $c>0$, will be an increasing C^2-function with the properties

$$\text{(3.1)} \qquad \begin{cases} h'(t) > 0,\ h''(t) > 0 & \text{for } t \in (0,c) , \\ \lim\limits_{t\to 0+} h(t) = \lim\limits_{t\to 0+} h'(t) = 0 , \end{cases}$$

Let (b_k) be a sequence in $\mathbf{R}^n_+$ with

$$\text{(3.2)} \qquad \begin{cases} b_k = t_k u_k + h(t_k)e_n, & u_k \in S^{n-1} \cap \partial\mathbf{R}^n_+ , \\ 0 < t_{k+1} \leqq t_k < c, & \lim t_k = 0 . \end{cases}$$

We shall consider the following isolation condition

$$\text{(3.3)} \qquad \varrho(b_k, b_j) \geqq M > 0 \quad \text{for } \ j \neq k ,$$

and the divergence condition

$$\text{(3.4)} \qquad \sum_{k=1}^{\infty} (h(t_k)/t_k)^{n-1} = \infty .$$

3.5. A SUFFICIENT CONDITION FOR (3.3). Let (b_k) be a sequence as in (3.2). By (3.1) we have

$$\frac{|b_k - b_{k+1}|^2}{h(t_k)h(t_{k+1})} \geqq \left[\frac{t_k - t_{k+1}}{h(t_k)}\right]^2 .$$

In view of (2.6) a sufficient condition for (3.3) is

$$A = \liminf_{k\to\infty} \frac{t_k - t_{k+1}}{h(t_k)} > 0\,. \tag{3.6}$$

3.7. EXAMPLE. (1) Let $h(t)=t^{1+\alpha}$, $\alpha>0$, $t>0$, and $t_k=k^{-1/\alpha}$, $k=1,2,\dots$. Now $\sum h(t_k)/t_k=\infty$ and

$$\frac{t_k-t_{k+1}}{h(t_k)} = \frac{k^{-1/\alpha}-(k+1)^{-1/\alpha}}{k^{-(1+\alpha)/\alpha}} = k(1-(k/(k+1))^{1/\alpha}) \to \frac{1}{\alpha}$$

as $k\to\infty$. Hence, by (3.6), the sequence $b_k=(t_k, h(t_k))$ satisfies (3.3) as well.

(2) Let $h(t)=t(\log 1/t)^{-1}$, $t\in(0,c)$, where $c>0$ is chosen so that (3.1) is fulfilled, and let $t_k=2^{-k\log k}<c$. Now $\sum h(t_k)/t_k=\infty$ and

$$\frac{t_k-t_{k+1}}{h(t_k)} = \log 2\,(\log k)[1-2^{-k\log((k+1)/k)-\log(k+1)}] \to \infty$$

and $t_{k+1}/t_k\to 0$ as $k\to\infty$. Thus both (3.3) and (3.4) are satisfied. The sequence $s_k=2^{-k}<c$ satisfies also conditions (3.3) and (3.4). Observe however that, $s_{k+1}/s_k \not\to 0$.

3.8. THE ITERATION $t_{k+1}=t_k-\lambda h(t_k)$. Fix h as in (3.1), $\lambda>0$, and $t_1\in(0,c)$ such that $t-\lambda h(t)>0$ for all $t\in(0,t_1]$. Let $t_{k+1}=t_k-\lambda h(t_k)\in(0,t_k)$ and $b_k=t_k e_1+h(t_k)e_n$, $k=1,2,\dots$. For this sequence (b_k) the number A in (3.6) has a positive value λ^{-2}, and hence (3.3) is satisfied. We next show that

$$\limsup \varrho(b_k, b_{k+1}) < \infty\,.$$

In fact, $h(t_{k+1})\geqq h(t_k)(1-\lambda h'(t_k))$ by (3.1) for $h'(t_k)<1/\lambda$ and further

$$\frac{|b_k-b_{k+1}|^2}{h(t_k)h(t_{k+1})} \leqq \frac{\lambda^2(1+h'(t_k)^2)}{1-h'(t_k)}\,.$$

Hence $\limsup \varrho(b_k,b_{k+1})\leqq \operatorname{ar\,cosh}(1+\lambda^2/2)$ by (2.6) and (3.1).

3.9. LEMMA. *If (t_k) is as in 3.8, then $\sum h(t_k)/t_k=\infty$.*

PROOF. Clearly

$$\frac{\lambda h(t_k)}{t_{k+1}} = \frac{t_k-t_{k+1}}{t_{k+1}} \geqq \int_{t_{k+1}}^{t_k}\frac{dt}{t}\,.$$

Since $t_k\to 0$ and $t_{k+1}/t_k\to 1$ by (3.1), the proof follows.

A set $E\subset \mathbf{R}^n_+$ with $0\in\bar{E}$ is said to be non-tangential at 0 if $E\subset C(\varphi)$ for some $\varphi\in(0,\pi/2)$. Otherwise E is tangential at 0.

3.10. REMARKS. (1) Let (a^k) be a non-tangential sequence at 0, $\lim a^k = 0$. Then there exists a constant $\mu > 0$ such that $a_n^k/|a^k| \geqq \mu > 0$ for all k, where a_n^k is the nth coordinate of a^k. In particular, $\sum a_n^k/|a^k| = \infty$. Due to (3.1)

$$h(t_k)/t_k = b_n^k/t_k \approx b_n^k/|b^k|$$

for large k, when (b^k) is a sequence as in (3.2). Hence we can consider the divergence condition (3.4) to be some sort of generalization of the non-tangentiality condition. More specifically, (3.4) requires that the sequence be thick enough for the given h, since not all sequences of the form (3.2) satisfy (3.4) as can be easily shown.

(2) Let $A > 0$, (b_k) as in (3.2), and let $a_k \in D(b_k, A)$, $a_k = s_k v_k + \sigma_k e_n$, $v_k \in S^{n-1} \cap \partial \mathbf{R}^n_+$. Then we get by (2.5) the following generous estimates

$$\frac{\sigma_k}{s_k} \geqq \frac{h(t_k)e^{-A}}{t_k + h(t_k)\sinh A} \geqq \frac{h(t_k)}{C t_k}; \quad \frac{\sigma_k}{s_k} \leqq C\frac{h(t_k)}{t_k},$$

where $C = 2e^A$. Hence (b_k) satisfies (3.4) if and only if (a_k) does. In conclusion, small changes of locations of the points b_k have no effect on the validity of (3.4).

We shall now show that there are sequences (b_k) satisfying (3.3) and (3.4), when $n = 3$. *If* $a \in \mathbf{R}$ write $T(a) = \{x \in \mathbf{R}^n : x_n = a\}$.

3.11. LEMMA. *Let h be as in* (3.1) *and let* $0 < t_{k+1} < t_k < c$, $\lim t_k = 0$, $\sum h(t_k)/t_k = \infty$ *and* $\varrho(b_k, b_j) \geqq a > 0$ *for* $k \neq j$, *where* $b_k = t_k e_1 + h(t_k)e_3 \in \mathbf{R}^3_+$. *Then there exists a sequence* (c_k) *in* $\bigcup T_j$, $T_j = T(h(t_j))$, *where* $c_k = s_k u_k + h(s_k)e_3$, $u_k \in S^2 \cap \partial \mathbf{R}^3_+$, *with* $c_k \to 0$ *and* $\varrho(c_k, c_j) \geqq a$ *for* $k \neq j$ *and such that*

$$\sum_{j=1}^{\infty} (h(s_j)/s_j)^2 = \infty .$$

PROOF. From (2.5) it follows that there exists a constant $d > 0$ and an integer j_0 such that for $j \geqq j_0$, there are at least $N_j \geqq d t_j/h(t_j)$ points b_j^i, $i = 1, \ldots, N_j$, in T_j with $|b_j^i|^2 = t_j^2 + h(t_j)^2$ and with $\varrho(b_j^i, b_j^k) \geqq a$ for $k \neq i$. Arrange the points (b_j^i) into a sequence (c_k) with $|c_{k+1}| \in (0, |c_k|]$, $c_k = s_k u_k + h(s_k)e_3$, $u_k \in S^2 \cap \partial \mathbf{R}^3_+$. For each $r \geqq j_0$ there are at least $d t_r/h(t_r)$ terms in the sequence $((h(s_j)/s_j)^2)$ having the value $(h(t_r)/t_r)^2$. Thus

$$\sum_{k=1}^{\infty} (h(s_k)/s_k)^2 \geqq d \sum_{r \geqq j_0} h(t_r)/t_r = \infty .$$

In view of 3.8 the following lemma is seen to be a generalization of Lemma 3.9.

3.12. LEMMA. *Let* $a_k = (s_k, h(s_k)) \in \mathbf{R}^2_+$ *be a sequence as in* (3.1) *and* (3.2) *such that* $\varrho(a_k, a_{k+1}) < 2C$ for all $k = 1, 2, \ldots$. *Then* $\sum h(s_k)/s_k = \infty$.

PROOF. Fix $\lambda > 0$ and $t_1 \in (0, s_1)$ such that the sequence $t_{k+1} = t_k - \lambda h(t_k)$, $b_k = (t_k, h(t_k))$, satisfies $\varrho(b_k, b_{k+1}) < C$ (cf. 3.8). Choose a subsequence (s_{k_j}) of (s_k) such that

$$2C \leqq \varrho(a_{k_j}, a_{k_{j+1}}) \leqq 4C$$

for all $j = 1, 2, \ldots$. For each a_{k_j} there is b_{i_j} such that

$$\varrho(a_{k_j}, b_{i_j}) < C .$$

It follows that $b_{i_k} \neq b_{i_j}$ for $j \neq k$. We estimate

$$\infty = \sum_{k=1}^{\infty} h(t_k)/t_k = \sum_{j=1}^{\infty} \sum_{k=i_j}^{i_{j+1}-1} h(t_k)/t_k$$

$$\leqq \sum_{j=1}^{\infty} (i_{j+1} - i_j) h(t_{i_j})/t_{i_j} .$$

For the last inequality note that $h(t)/t$ is increasing by (3.1). From (2.5) it follows that $i_{j+1} - i_j \leqq N = N(C)$ for $j \geqq j_0$, where j_0 is chosen so that $h'(t)$ is small enough for $t \in (0, t_{i_{j_0}})$ (cf. Lemma 2.10). From the above estimates it follows, in view of 3.10(2), that

$$\sum h(s_k)/s_k \geqq \sum_{j \geqq j_0} h(s_{k_j})/s_{k_j} \geqq \text{const} \sum_{j \geqq j_0} h(t_{i_j})/t_{i_j} = \infty$$

as desired.

3.13. COROLLARY. *Let* $a_k = (s_k, h(s_k)) \in \mathbf{R}^2_+$ *be a sequence as in* (3.1) *and* (3.2) *such that* $\varrho(a_k, a_{k+1}) < 2C$. *Then there exists a subsequence* (s_{p_k}) *of* (s_k) *such that* $\varrho(a_{p_k}, a_{p_{k+1}}) \to \infty$ *and* $\sum h(s_{p_k})/s_{p_k} = \infty$.

4. The main results.

For the basic properties of quasiconformal mappings the reader is referred to [8]. The next lemma is well-known ([8, 17.13], [11, 4.6]).

4.1. LEMMA. *Let* $f: \mathbf{R}^n_+ \to \mathbf{R}^n$ *be a* qc *mapping,* $E_j \subset \mathbf{R}^n_+$, $0 \in \bar{E}_j$, *and suppose that the limits*

$$\lim_{x \to 0,\, x \in E_j} f(x) = \alpha_j, \quad j = 1,2$$

exist. Then $\alpha_1 = \alpha_2$ *if* $M(\Gamma(r)) = \infty$ *for all* $r > 0$, *where*

$$\Gamma(r) = \Delta(E_1 \cap B^n(r), E_2 \cap B^n(r); \mathsf{R}^n_+) .$$

The following lemma indicates a connection between the continuum criterion and the isolation and divergence conditions in Section 3.

4.2. LEMMA. *Let* (b_k) *be a sequence in* R^n_+ *satisfying* (3.1)–(3.4). *Then* $M(\Delta(E, C(\varphi); \mathsf{R}^n_+)) = \infty$ for all $\varphi \in (0, \pi/2)$ and $M(0, \bar{E}) = \infty$, where $E = \bigcup D(b_k, 1)$.

PROOF. From (3.1) it follows that for each $\varphi \in (0, \pi/2)$ there exists an integer k_φ such that

$$\overline{C(\varphi)} \cap (\bar{E}_k \setminus \{0\}) = \emptyset \qquad \text{for } k \geqq k_\varphi ,$$

$E_k = \bigcup_{j \leqq k} D(b_j, 1)$. To prove $M(0, \bar{E}) = \infty$ it suffices to show that

$$M(\Delta(E_k, C(\varphi); \mathsf{R}^n)) = \infty \qquad \text{for } k \geqq k_\varphi .$$

This sufficient condition for $M(0, \bar{E}) = \infty$ follows from the proof of [9, 8.7].

Fix $\varphi \in (0, \pi/2)$ and $k_0 \geqq k_\varphi$ such that $h'(t) < 1/2$ for $t \in (0, t_{k_0}]$. Such a choice is possible by (3.1). From (2.5) it follows that

$$B^n(b_k, (1 - 1/e)h(t_k)) \subset D(b_k, 1) .$$

Applying Lemma 2.10 with $\tan \alpha = 1/2$ we see that there exists a number $\lambda = \lambda(M) < 1 - 1/e$, depending only on the number M in (3.3), such that the projections of the balls $B^n(b_k, \lambda h(t_k))$ on the $(n-1)$-dimensional plane $x_n = 0$, call them A_k, are pairwise disjoint for $k \geqq k_0$. Let Γ be the family of all segments parallel to the x_n-axis joining $\bigcup_{k \geqq k_0} A_k$ to $C(\varphi)$. We obtain by virtue of [8, 7.2, 6.4, 6.7] and (3.4)

$$\begin{aligned} M(\Gamma) &\geqq c_1(\varphi, n) \sum_{k \geqq k_0} t_k^{1-n} m_{n-1}(A_k) \\ &= c_2(\varphi, n, M) \sum_{k \geqq k_0} (h(t_k)/t_k)^{n-1} = \infty . \end{aligned}$$

The proof is complete, since $M(\Delta(E_{k_0}, C(\varphi); \mathsf{R}^n)) \geqq M(\Gamma)$ by [8, 6.4].

4.3. LEMMA. *Let* $f\colon \mathsf{R}^n_+ \to \mathsf{R}^n$ *be a* qc *mapping, let* (a_k) *be a sequence in* R^n_+ *with* $a_k \to 0$, $f(a_k) \to \alpha$ *and let* $M \in (0, \infty)$. *Then* $f(x) \to \alpha$ *as* $x \to 0$, $x \in E$, $E = \bigcup D(a_k, M)$.

Lemma 4.3 is a qc counterpart of a result due to Bagemihl and Seidel [1, Theorem 1] and it was proved in [10, 6.5].

A mapping $f\colon \mathbf{R}^n_+ \to \mathbf{R}^n$ is said to have an angular limit α at 0, if $f(x) \to \alpha$, when $\varphi \in (0, \pi/2)$ and $x \to 0$, $x \in C(\varphi)$.

4.4. THEOREM. *Let $f\colon \mathbf{R}^n_+ \to \mathbf{R}^n$ be a* qc *mapping, $E \subset \mathbf{R}^n_+$ with* cap $\underline{\text{dens}}$ $(E, 0) > 0$ *and* $\lim_{x\to 0, x\in E} f(x) = \alpha$ *and let (b_k) be a sequence in $\mathbf{R}^n_+$ satisfying* (3.1)–(3.4) *and $f(b_k) \to \beta$ as $k \to \infty$. Then $\alpha = \beta$.*

PROOF. By [10, 4.4] f has an angular limit α at 0. The proof follows from Lemmas 4.1, 4.2, and 4.3.

In the special case (cf. Remark 2.3) cap $\overline{\text{dens}}$ $(F, 0) > 0$, $F = \bigcup D(b_k, 1)$, Theorem 4.4 follows from [11, 4.7].

4.5. COROLLARY. *Let $f\colon \mathbf{R}^n_+ \to \mathbf{R}^n$ be a* qc *mapping having an angular limit α at 0 and let (b_k) be a sequence in $\mathbf{R}^n_+$ satisfying* (3.1)–(3.4) *and with $f(b_k) \to \beta$ as $k \to \infty$. Then $\alpha = \beta$.*

4.6. REMARKS. (1) For sequences of the form $b_k = t_k e_1 + h(t_k) e_n$ it suffices to require $\sum h(t_k)/t_k = \infty$ in 4.2, 4.4, and 4.5 instead of the stronger condition (3.4) $\sum (h(t_k)/t_k)^{n-1} = \infty$. This state of things is due to the fact that for a fixed $\varphi \in (0, \pi/2)$, there is a number $\mu(M) < 1 - 1/e$ (cf. (3.3) and (2.5)) such that the curve families

$$\Gamma_k = \Delta\big(C(\varphi), B^n(b_k, r_k); R(|b_k| + r_k, |b_k| - r_k)\big),$$

where $r_k = \mu(M) h(t_k)$, are separate for large k (cf. (3.1)), and $M(\Gamma_k) \geqq \text{const}\, (h(t_k)/t_k)$ [8, 10.2].

(2) It should be observed that the set E in 4.2 may be, in view of 4.6(1), so small that cap $\overline{\text{dens}}$ $(E, 0) = 0$. In fact, let

$$h(t) = t\left(\log\frac{1}{t}\right)^{-1}$$

(cf. 3.7(2)) and

$$b_k = 2^{-k} e_1 + h(2^{-k}) e_n, \quad n \geqq 3.$$

Then (b_k) is of the form 4.6(1) and the conditions (3.1)–(3.3), $\sum 2^k h(2^{-k}) = \infty$ are satisfied. Let $E = \{0\} \cup (\bigcup D(b_k, 1))$. It follows from 2.3(3) that cap $\overline{\text{dens}}$ $(E, 0) = 0$.

4.7. REMARK. Gaier and Pommerenke [3] have proved that if $\{z_p\}$ is any sequence in B^2 with $z_p \to 1 = (1, 0)$ and $\arg(z_p - 1) \to \pi/2$, then there exists a bounded conformal mapping $f\colon B^2 \to \mathbf{R}^2$ having a radial limit at 1 but such that $\{f(z_p)\}$ diverges. Hence we see that the assumption concerning the existence of $\lim f(b_k)$ does not follows from the other assumptions in 4.4 or 4.5.

Let (a_k) and (b_k) be sequences in R^n_+ with $a_k \to 0$, $b_k \to 0$ and such that for some $M > 0$

$$\varrho(J_k, J_h) \geqq 2M \quad \text{for } k \neq h\,, \tag{4.8}$$

where $J_k = J[a_k, b_k]$.

4.9. THEOREM. *Let* (a_k) *and* (b_k) *satisfy the condition* (4.8) *and let* $f\colon \mathsf{R}^n_+ \to \mathsf{R}^n$ *be a* qc *mapping with* $f(a_k) \to \alpha$, $f(b_k) \to \beta$ *as* $k \to \infty$. *Then*

$$\sum_{k=1}^{\infty} \varrho(a_k, b_k)^{1-n} = \infty$$

implies $\alpha = \beta$.

PROOF. By Lemma 4.3, $\liminf \varrho(a_k, b_k) < \infty$ implies $\alpha = \beta$. Thus we may assume $\varrho(a_k, b_k) \to \infty$ as $k \to \infty$ and, after relabeling if necessary, that $\varrho(a_k, b_k) > M$ for all k. By performing an auxiliary Möbius transformation if necessary we may assume that $\alpha, \beta \neq \infty$. By Lemma 4.3 there exists an integer k_0 such that

$$fD(b_k, M/2) \subset B^n(\beta, |\alpha-\beta|/3)$$

and

$$fD(a_k, M/2) \subset B^n(\alpha, |\alpha-\beta|/3) \quad \text{for } k \geqq k_0\,.$$

Let

$$\Gamma_k = \Delta\big(D(a_k, M/2), D(b_k, M/2); D_k\big)\,,$$

where $D_k = \bigcup\{D(x, M) : x \in J[a_k, b_k]\}$. By virtue of (4.8) the families Γ_k are separate and thus [8, 6.7]

$$M(\Gamma) \geqq M\Bigg(\bigcup_{k \geqq k_0} \Gamma_k\Bigg) \geqq \sum_{k \geqq k_0} M(\Gamma_k)\,,$$

where

$$\Gamma = \Delta\Bigg(\bigcup_{k \geqq k_0} D(a_k, M/2),\ \bigcup_{k \geqq k_0} D(b_k, M/2); \mathsf{R}^n_+\Bigg)\,.$$

From a symmetry property of the modulus [12, 2.21] and Lemmas 2.8 and 2.9 (or directly from the proof of 2.9) it follows that

$$M(\Gamma_k) \geqq c\varrho(a_k, b_k)^{1-n}$$

for $k \geqq k_0$, where $c = c(n, M)$. Since $M(f\Gamma) < \infty$ [8, 7.5] this last lower bound leads to a contradiction with the quasi-invariance of the modulus [8, 13.1].

4.10. REMARK. Let $b_k = (2^{-k}, 2^{-2k}) \in \mathsf{R}^2_+$. Suppose that $f\colon \mathsf{R}^2_+ \to \mathsf{R}^2_+$ is a qc mapping having an angular limit α at 0 and such that $f(b_k) \to \beta$. Then $f(a_k) \to \alpha$, $a_k = |b_k| e_2$ and (4.8) holds. It follows from (2.7) that $\sum \varrho(a_k, b_k)^{-1} = \infty$ and hence we conclude by Theorem 4.9 that $\alpha = \beta$. Observe that the sequence (b_k) satisfies the conditions (3.1)–(3.3) but fails to satisfy (3.4) and hence the assumptions in Corollary 4.5 are not satisfied.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HELSINKI
HALLITUSKATU 15
SF-00100 HELSINKI 10
FINLAND