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ON AW*-SUBALGEBRAS OF TYPE I AW*-ALGEBRAS

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1. Introduction.

The AW*-modules introduced by Kaplansky [8] have been considered as a substitute in the theory of AW*-algebras for Hilbert spaces in the theory of W*-algebras (=von Neumann algebras). For an AW*-module H over a commutative AW*-algebra Z the algebra $\mathrm{End}_Z(H)$ of all bounded module endomorphisms of H is a type I AW*-algebra with center Z, and conversely each type I AW*-algebra is an algebra of this form [7, 8]. The embedding of AW*-algebras as AW*-subalgebras of type I AW*-algebras may be regarded as a counterpart of the spatial theory of W*-algebras and has been developed by Yen [17], Goldman [3], Feldman [2], Widom [14], Saitô [11, 12], and others. Let A be an AW*-algebra and Z an AW*-subalgebra of its center. Then Widom [14, Theorem 3.1] proved that A is embedded as an AW*-subalgebra of a type I AW*-algebra B with center Z if and only if (*) there is a separating set $\{\varphi_\alpha\}$ of bounded positive module homomorphisms φ_α of A into Z which are completely additive on projections (i.e., $\varphi_\alpha(x)=0$ for all α with $x \in A^+$ implies $x=0$ and $\varphi_\alpha(\bigvee_\beta p_\beta)=\sum_\beta \varphi_\alpha(p_\beta)$ for each orthogonal family $\{p_\beta\}$ of projections in A, where $\bigvee_\beta p_\beta$ denotes the supremum of the p_β in the projection lattice of A and $\sum_\beta \varphi_\alpha(p_\beta)$ denotes the supremum of the finite sums in the boundedly complete lattice Z_{sa}). (Note that Widom assumes a condition slightly stronger than, but essentially equivalent to the above (*) (see [11, Lemma 1.1.2]).) In addition Saitô [12] proved that in this situation A coincides with its double commutant A'' in B. However we cannot a priori assume the condition (*). For example if A is of type II_1 and Z is the center of A, then (*) is equivalent to the existence of the center-valued trace on A ([17, Theorem 5.2], [3, Theorem 3]), which is still open. Moreover, if a non W*, AW*-factor A arises as an AW*-subalgebra of the type I AW*-algebra $\mathrm{End}_Z(H)$ (this is the case for Takenouchi's factor and Dyer's one [13]), then $\mathrm{End}_Z(H)$, hence Z also is non W* and A cannot contain the center Z of $\mathrm{End}_Z(H)$.

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In this paper we show that an AW*-algebra is an AW*-subalgebra of some type I AW*-algebra if and only if it is normal in the sense of Wright [16]. The necessity is clear, and to see the sufficiency we use two known facts: The construction by Paschke [10] of inner product modules over C*-algebras and the embedding of a C*-algebra into its Dedekind completion [15]. Each finite AW*-algebra is normal ([16, Theorem 4], [4, Lemma 3.12]) and so it is an AW*-subalgebra of some type I AW*-algebra. It is not known whether or not each AW*-algebra is normal.

We refer the reader to the book [1] of Berberian for basic fact on AW*-algebras.

2. Preliminaries.

Throughout the paper C*-algebras to be considered are always unital. For a C*-algebra A the self-adjoint part of A (respectively the set of all projections of A) is denoted by A_{sa} (respectively A_p) and the supremum in A_{sa} (respectively A_p) of a subset $\mathscr{F}$ of A_{sa} (respectively A_p), if it exists, is written as $\sup_A \mathscr{F}$ (respectively $\bigvee_A \mathscr{F}$) or simply as $\sup \mathscr{F}$ (respectively $\bigvee \mathscr{F}$). For a net $\{x_\alpha\}$ in A_{sa} we write $x_\alpha \nearrow x$ in A, if the net is increasing and has the supremum x in A_{sa}. The C*-algebra A is *monotone complete* (respectively an AW*-*algebra*) if each bounded increasing net in A_{sa} has a supremum in A_{sa} (respectively if each maximal abelian *-subalgebra of A is generated by projections and A_p is a complete lattice). If a C*-algebra A is a C*-subalgebra of another C*-algebra (respectively AW*-algebra) B, then A is *monotone closed* in B (respectively an AW*-*subalgebra* of B) if $x_\alpha \nearrow x$ in B with $\{x_\alpha\} \subset A_{sa}$ implies $x \in A_{sa}$, that is, $x_\alpha \nearrow x$ in A (respectively $\bigvee_A \mathscr{F} = \bigvee_B \mathscr{F}$ for each subset $\mathscr{F}$ of A_p, i.e., the suprema in A_p and in B_p coincide). An AW*-algebra A is *normal* in the sense of Wright [16] if $\sup p_\alpha$ exists and equals $\bigvee p_\alpha$ for each increasing net $\{p_\alpha\}$ in A_p. Any monotone complete C*-algebra is a normal AW*-algebra, an AW*-subalgebra of a monotone complete C*-algebra is normal [16], and an AW*-algebra A is normal if and only if it is an AW*-subalgebra of its regular monotone completion $\bar{A}$ [4, Remark 3.14]. Clearly a monotone closed C*-subalgebra of a monotone complete C*-algebra is also monotone complete.

A bounded positive linear map φ of a C*-algebra A into another B is *normal* if $x_\alpha \nearrow x$ in A implies $\varphi(x_\alpha) \nearrow \varphi(x)$ in B.

An AW*-*module* over a commutative AW*-algebra Z, [8], is a right Z-module H with a Z-valued inner product $(\cdot, \cdot)$: $H \times H \to Z$, that is, $(\xi, \xi) \geqq 0$ and $=0$ only if $\xi = 0$,

$$(\xi_1 \cdot a_1 + \xi_2 \cdot a_2, \eta) = (\xi_1, \eta)a_1 + (\xi_2, \eta)a_2$$

and $(\xi, \eta)^* = (\eta, \xi)$ for $\xi, \xi_1, \xi_2, \eta \in H$ and $a_1, a_2 \in Z$, which is self-dual in the sense that each bounded module homomorphism f of H into Z is of the form $f=(\cdot,\eta)$ for some $\eta \in H$. The type I AW*-algebra $\mathrm{End}_Z(H)$ of all module endomorphisms of H is monotone complete, the maps $\mathrm{End}_Z(H) \ni x \mapsto (x\xi,\xi) \in Z$, $\xi \in H$, are normal, and $x_\alpha \nearrow x$ in $\mathrm{End}_Z(H)$ if and only if $(x_\alpha\xi,\xi) \nearrow (x\xi,\xi)$ in Z for all $\xi \in H$ [14].

3. Theorem.

THEOREM. (i) *A* C*-*algebra is monotone complete if and only if it is a monotone closed* C*-*subalgebra of some type* I AW*-*algebra.*

(ii) *An* AW*-*algebra is normal if and only if it is an* AW*-*subalgebra of some type* I AW*-*algebra.*

The sufficiency is obvious from the facts stated above and the necessity follows from the following series of lemmas.

The next, known result is stated for the sake of convenience.

LEMMA 1. *Let H be an* AW*-*module over a commutative* AW*-*algebra Z, M a submodule of H such that the orthogonal complement $M^\perp$ in H is $\{0\}$ and $\{x_\alpha\}$ a bounded increasing net in* $\mathrm{End}_Z(H)$. *Then we have $x_\alpha \nearrow x$ in* $\mathrm{End}_Z(H)$ *if and only if $(x_\alpha\xi,\xi) \nearrow (x\xi,\xi)$ in Z for all $\xi \in M$.*

PROOF. We need only show the sufficiency. Since $\mathrm{End}_Z(H)$ is monotone complete, we have $x_\alpha \nearrow x'$ for some $x' \in \mathrm{End}_Z(H)_{\mathrm{sa}}$. Then $((x-x')\xi,\xi)=0$ for all $\xi \in M$, which together with the polarization identity and the fact that $M^\perp=\{0\}$ implies $x=x'$.

Let A be a C*-algebra, Z a commutative AW*-algebra and φ a bounded positive linear map of A into Z. Then φ is completely positive and it induces a *-representation of A on an inner product module over Z as follows [10]. Let $A \odot Z$ be the algebraic tensor product of A and Z and regard this as a right Z-module by the action $(\sum a_j \otimes z_j)\cdot z = \sum a_j \otimes z_j z$, $a_j \in A$ and z_j, $z \in Z$. Let

$$\langle \sum a_j \otimes z_j, \sum b_k \otimes w_k \rangle_\varphi = \sum_{j,k} w_k^* \varphi(b_k^* a_j) z_j ,$$

$$\sum a_j \otimes z_j, \sum b_k \otimes w_k \in A \odot Z .$$

Then $\langle \cdot, \cdot \rangle_\varphi$ is a Z-valued pre-inner product and so putting

$$\mathfrak{N}_\varphi = \{u \in A \odot Z : \langle u,u \rangle_\varphi = 0\}, \qquad \mathfrak{M}_\varphi = A \odot Z/\mathfrak{N}_\varphi ,$$

$$(u_\varphi, v_\varphi)_\varphi = \langle u, v \rangle_\varphi, \qquad u, v \in A \odot Z, \text{ where } u_\varphi = u + \mathfrak{N}_\varphi \in \mathfrak{M}_\varphi ,$$

$$\pi_\varphi^0(x)(\sum a_j \otimes z_j)_\varphi = (\sum x a_j \otimes z_j)_\varphi, \qquad x \in A, \sum a_j \otimes z_j \in A \odot Z ,$$

we obtain a Z-valued inner product $(\cdot,\cdot)_\varphi$ on $\mathfrak{M}_\varphi$ and a *-representation π_φ^0 of

A as bounded module endomorphisms of $\mathfrak{M}_\varphi$. Let H_φ be the AW*-completion of $\mathfrak{M}_\varphi$ in the sense of Widom [14], i.e., the unique AW*-module containing $\mathfrak{M}_\varphi$ as a submodule so that $\mathfrak{M}_\varphi^\perp = \{0\}$. Then the bounded module endomorphism $\pi_\varphi^0(x)\colon \mathfrak{M}_\varphi \to \mathfrak{M}_\varphi$ extends to a unique one $\pi_\varphi(x)\colon H_\varphi \to H_\varphi$ and so π_φ is a *-homomorphism of A into $\mathrm{End}_Z(H_\varphi)$. (The AW*-module H_φ is regarded as the "bidual" of $\mathfrak{M}_\varphi$ and so we may take $\pi_\varphi(x)$ as the bitranspose of $\pi_\varphi^0(x)$.)

LEMMA 2. *Let A, Z and φ be as above.*

(i) *If φ is normal, then so is π_φ.*

(ii) *If A is an AW*-algebra and φ is completely additive on projections (see the introduction), then $\pi_\varphi(A)$ is an AW*-subalgebra of $\mathrm{End}_Z(H_\varphi)$.*

PROOF. (i) If $x_\alpha \nearrow x$ in A, then

$$(\pi_\varphi(x_\alpha)(\textstyle\sum a_j \otimes z_j)_\varphi, (\sum a_j \otimes z_j)_\varphi)_\varphi = \displaystyle\sum_{j,k} z_k^* \varphi(a_k^* x_\alpha a_j) z_j \nearrow$$

$$\sum_{j,k} z_k^* \varphi(a_k^* x a_j) z_j = (\pi_\varphi(x)(\textstyle\sum a_j \otimes z_j)_\varphi, (\sum a_j \otimes z_j)_\varphi)_\varphi$$

in Z for all $\sum a_j \otimes z_j \in A \odot Z$ (see [6, Lemma 2.1] or [5, Lemma 1.2]); hence Lemma 1 implies that π_φ is normal.

(ii) If $\{p_\alpha\}$ is an orthogonal family in A_p, then $\pi_\varphi(\bigvee p_\alpha) = \bigvee \pi_\varphi(p_\alpha)$. In fact, a slight modification of the argument in [2, Lemma 3] shows that

$$\varphi(a^*(\bigvee p_\alpha)a) = \sum \varphi(a^* p_\alpha a)$$

for all $a \in A$, hence that

$$\varphi(b^*(\bigvee p_\alpha)a) = \sum \varphi(b^* p_\alpha a)$$

for all $a, b \in A$. (Note that since Z is commutative, the Schwarz inequality $\varphi(y^*x)^*\varphi(y^*x) \leqq \varphi(x^*x)\varphi(y^*y)$, $x, y \in A$, holds.) The desired equality follows as in (i).

The preceding equality shows that the ideal $\mathrm{Ker}\,\pi_\varphi$ is a direct summand of A, hence that $\pi_\varphi|_{hA}\colon hA \to \pi_\varphi(A)$ is a *-isomorphism for some central projection h of A. If $\{q_\alpha\}$ is an orthogonal family of projections in $\pi_\varphi(A)$, then $\pi_\varphi(p_\alpha) = q_\alpha$ for some orthogonal family $\{p_\alpha\}$ of projections in hA and again the preceding equality implies that $\bigvee q_\alpha$ (the supremum in $\pi_\varphi(A)_p) = \pi_\varphi(\bigvee p_\alpha) = \bigvee \pi_\varphi(p_\alpha)$ $= \bigvee q_\alpha$ (the supremum in $\mathrm{End}_Z(H)_p$).

LEMMA 3. *Let A be a C*-algebra and Z a commutative AW*-algebra.*

(i) *There is an AW*-module H over Z so that A is (*-isomorphic to) a C*-*

subalgebra of $\mathrm{End}_Z(H)$ *and the embedding* $A \hookrightarrow \mathrm{End}_Z(H)$ *is normal if and only if there is a separating set* $\{\varphi_\alpha\}$ *of bounded normal positive linear maps* φ_α *of* A *into* Z.

(ii) *There is an* AW*-*module* H *over* Z *so that* A *is an* AW*-*subalgebra of* $\mathrm{End}_Z(H)$ *if and only if* A *is an* AW*-*algebra with a separting set* $\{\varphi_\alpha\}$ *of bounded positive linear maps* φ_α *of* A *into* Z *which are completely additive on projections.*

PROOF. The necessity (of (i) and (ii)) is clear from the fact that the maps

$$\mathrm{End}_Z(H) \ni x \mapsto (x\xi, \xi) \in Z, \qquad \xi \in H,$$

are normal and completely additive on projections. To see the sufficiency take the direct sum of the *-representations $\{\pi_{\varphi_\alpha}, H_{\varphi_\alpha}\}$ constructed above and apply Lemma 2.

Let A be a C*-algebra, $X=\overline{P(A)}$ the pure state space of A (i.e., the weak* closure of the set $P(A)$ of all pure states of A), $C(X)$ (respectively $\mathscr{B}(X)$) the C*-algebra of all complex-valued continuous (respectively bounded Borel) functions on X and $\mathscr{M}(X)$ the ideal of $\mathscr{B}(X)$ consisting of $a \in \mathscr{B}(X)$ such that $\{f \in X: a(f) \neq 0\}$ is meager (i.e., of first category) in X. Then $\mathscr{D}(A) = \mathscr{B}(X)/\mathscr{M}(X)$ is a commutative AW*-algebra and we obtain the canonical embedding:

$$\varkappa\colon A \xrightarrow{\varkappa_1} C(X) \hookrightarrow \mathscr{B}(X) \xrightarrow{\varkappa_2} \mathscr{B}(X)/\mathscr{M}(X) = \mathscr{D}(A)$$

$\varkappa_1(x)(f)=f(x)$, $\varkappa_2(a)=a+\mathscr{M}(X)$, $x \in A$, $f \in X$, $a \in \mathscr{B}(X)$. Clearly the restriction

$$\varkappa|_{A_{\mathrm{sa}}} : A_{\mathrm{sa}} \to \mathscr{D}(A)_{\mathrm{sa}}$$

is a positive linear isometry.

LEMMA 4. *With the above notation the map* $\varkappa$ *is a faithful normal positive linear map. Hence if in addition* A *is a normal* AW*-*algebra, then* $\varkappa$ *is completely additive on projections.*

PROOF. The pair $(\mathscr{D}(A)_{\mathrm{sa}}, \varkappa|_{A_{\mathrm{sa}}})$ is the injective envelope (or the maximal essential extension) of A_{sa} as a real Banach space (cf. [9, p. 89, Theorem 4]). In fact, $(C(X)_{\mathrm{sa}}, \varkappa_1|_{A_{\mathrm{sa}}})$ is an essential extension of A_{sa}, since for each proper closed subset F of X, there is an $x \in A_{\mathrm{sa}}$ with

$$\sup\{|f(x)| : f \in F\} < \|x\|$$

[9, p. 91, Theorem 5] and $(\mathscr{D}(A)_{\mathrm{sa}}, \varkappa_2|_{C(X)_{\mathrm{sa}}})$ is the injective envelope of $C(X)_{\mathrm{sa}}$.

Hence $(\mathscr{D}(A)_{sa}, \varkappa|_{A_{sa}})$ is also the Dedekind completion of A_{sa} [4, Proposition 2.6], and so the map $\varkappa|_{A_{sa}}$ preserves the suprema which exists in A_{sa}, from which the conclusion follows.

Lemmas 3 and 4 complete the proof of the theorem.

4. Remarks.

(i) Let B be a type I AW*-algebra and A a C*-subalgebra of B and consider the following conditions:

(a) A is an AW*-subalgebra of B;
(b) A is a monotone closed C*-subalgebra of B;
(c) A is a weakly closed C*-subalgebra of B [14] (i.e., if B is written as $B = \mathrm{End}_Z(H)$ with H an AW*-module over a commutative AW*-algebra Z, then $(x_\alpha \xi, \eta) \to (x\xi, \eta)$ (order convergence in Z) for all $\xi, \eta \in H$ with $x_\alpha \in A$ and $x \in B$ implies $x \in A$).

Obviously (c) $\Rightarrow$ (b) $\Rightarrow$ (a). Is it true that (a) $\Leftrightarrow$ (b) $\Leftrightarrow$ (c)? Saitô's double commutation theorem stated in the introduction shows that this is the case, if A contains the center of B. If (a) $\Rightarrow$ (b) always holds, then any normal AW*-algebra would be monotone complete.

(ii) For a C*-algebra A, let $\mathscr{D}(A)$ and $\varkappa$ be as above and let $\pi_\varkappa$ be the faithful normal *-representation of A on the AW*-module $H_\varkappa$ over $\mathscr{D}(A)$ induced by $\varkappa$. Then the monotone closure of $\pi_\varkappa(A)$ in $\mathrm{End}_{\mathscr{D}(A)}(H_\varkappa)$ (the smallest monotone closed C*-subalgebra of $\mathrm{End}_{\mathscr{D}(A)}(H_\varkappa)$ containing $\pi_\varkappa(A)$) is the regular monotone completion $\bar{A}$ of A [4]. To see this let $\bar{\varkappa}: \bar{A} \to \mathscr{D}(\bar{A})$ be the embedding $\varkappa$ defined for $\bar{A}$ instead of A. Then the uniqueness of the Dedekind completion and the construction of $\bar{A}$ in [4, Theorem 3.1] give the following commutative diagram:

$$\begin{array}{ccc} \bar{A} & \xrightarrow{\bar{\varkappa}} & \mathscr{D}(\bar{A}) \\ \cup\!\uparrow & & \| \\ A & \xrightarrow{\varkappa} & \mathscr{D}(A) \end{array}$$

Moreover we see that $H_{\bar{\varkappa}} = H_\varkappa$ and $\pi_{\bar{\varkappa}}|_A = \pi_\varkappa$. In fact, we have canonically

$$\mathfrak{M}_\varkappa = A \odot \mathscr{D}(A)/\mathfrak{N}_\varkappa \subset \bar{A} \odot \mathscr{D}(\bar{A})/\mathfrak{N}_{\bar{\varkappa}} = \mathfrak{M}_{\bar{\varkappa}} \subset H_{\bar{\varkappa}}$$

as right $\mathscr{D}(A)$-modules. From the normality of the maps

$$\bar{A} \ni x \mapsto (\pi_{\bar{\varkappa}}(x)\xi, \xi)_{\bar{\varkappa}} \in \mathscr{D}(A), \qquad \xi \in H_{\bar{\varkappa}}$$

and the fact that $\bar{A}$ is the monotone closure of A, it follows that

$$(\eta, \sum (a_j \otimes z_j)_{\bar{\varkappa}})_{\bar{\varkappa}} = (\eta, \sum \pi_{\bar{\varkappa}}(a_j)(1 \otimes z_j)_{\bar{\varkappa}})_{\bar{\varkappa}} = 0$$

for all $\sum a_j \otimes z_j \in A \odot \mathscr{D}(A)$ implies

$$(\eta, \sum (x_j \otimes z_j)_{\bar{\varkappa}})_{\bar{\varkappa}} = (\eta, \sum \pi_{\bar{\varkappa}}(x_j)(1 \otimes z_j)_{\bar{\varkappa}})_{\bar{\varkappa}} = 0$$

for all $\sum x_j \otimes z_j \in \bar{A} \odot \mathscr{D}(A)$, hence that

$$(\mathfrak{M}_{\varkappa})^{\perp} = (\mathfrak{M}_{\bar{\varkappa}})^{\perp} \quad \text{and} \quad H_{\bar{\varkappa}} = (\mathfrak{M}_{\bar{\varkappa}})^{\perp\perp} = (\mathfrak{M}_{\varkappa})^{\perp\perp} = H_{\varkappa}$$

by the uniqueness of the AW*-completion. It is also clear that $\pi_{\bar{\varkappa}}|_A = \pi_{\varkappa}$. Since $\pi_{\bar{\varkappa}}$ is faithful and normal, $\pi_{\bar{\varkappa}}(\bar{A}) \cong \bar{A}$ is a monotone closed C*-subalgebra of $\mathrm{End}_{\mathscr{D}(\bar{A})}(H_{\bar{\varkappa}}) = \mathrm{End}_{\mathscr{D}(A)}(H_{\varkappa})$, which is the monotone closure of $\pi_{\bar{\varkappa}}(A) = \pi_{\varkappa}(A)$.

(iii) We will show in a succeeding paper that Lemma 3(i) holds with Z replaced by any monotone complete C*-algebra, i.e., that for a C*-algebra A and a monotone complete C*-algebra Z, there is a self-dual inner product module H over Z in the sense of Paschke [10], so that A is a C*-subalgebra of $\mathrm{End}_Z(H)$ and the embedding $A \hookrightarrow \mathrm{End}_Z(H)$ is normal, if and only if there is a separating set $\{\varphi_\alpha\}$ of normal completely positive maps φ_α of A into Z. In fact, Paschke's construction of $\mathfrak{M}_{\varphi_\alpha}$ and $\pi^0_{\varphi_\alpha}$ applies in this situation. Hence we need only show that the inner product module $\mathfrak{M}_{\varphi_\alpha}$ over the monotone complete C*-algebra Z is "completed" to a self-dual inner product module over Z.

(iv) An alternate proof of Theorem (ii) follows directly from Theorem (i) and the fact that a normal AW*-algebra is an AW*-subalgebra of its regular monotone completion. This proof does not use Lemma 2(ii) and Lemma 3(ii). But is seems worth to write down Lemma 3(ii) as a slight extension of Widom's result stated in the introduction.

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