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LINEAR-TOPOLOGICAL CLASSIFICATION OF MATROID C*-ALGEBRAS

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Abstract.

We classify, up to a linear-topological isomorphism, all matroid C*-algebras (i.e. direct limits of a sequence of finite dimensional matrix algebras). There are two isomorphism classes: one is represented by $LC(l_2)$, the C*-algebra of all compact operators on the Hilbert space l_2, and the other – by the Fermion algebra $F = \otimes_{n=1}^{\infty} M_2$. In particular, any UHF algebra is isomorphic to F as a Banach space. We also show that $LC(l_2)$ is isometric to a 1-complemented subspace of F, but F is not isomorphic to a subspace of a quotient space of $LC(l_2)$.

1. Introduction.

Let M_n denote the C*-algebra of all complex $n \times n$-matrices with the usual algebraic operations and norms. A C*-algebra A is called a *matroid* C*-*algebra* (or, briefly, *a matroid*) if there exists a sequence $\{A_k\}_{k=1}^{\infty}$ of C*-subalgebras of A, possibly with different units, so that:

(i) $A_k \subsetneqq A_{k+1}$; $\quad k = 1,2,3,\ldots$

(ii) A_k is C*-isomorphic to $M_{n(k)}$ for some positive integer $n(k)$; $k = 1,2,3,\ldots$

(iii) $\bigcup_{k=1}^{\infty} A_k$ is dense in A in the norm topology.

If, moreover,

(iv) A has a unit e and $e \in A_k$, $\quad k = 1,2,\ldots$

then A is called a UHF *algebra* (i.e., uniformly hyper-finite algebra or, a Glim algebra, see [6, Chapter 6]). We call the sequence $\{A_k\}_{k=1}^{\infty}$ an *admissible sequence* for the matroid A.

The classification of matroids up to a C*-isomorphism is due to Glimm [2] (in the UHF case) and Dixmier [1] (in the general case). Glimm proved that if $\{A_k\}_{k=1}^{\infty}$ and $\{B_k\}_{k=1}^{\infty}$ are admissible sequences for the UHF algebras A and B

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respectively, with A_k C*-isomorphic to $M_{n(k)}$ and B_k C*-isomorphic to $M_{m(k)}$, then A is C*-isomorphic to B if and only if

$$\sup\{j;\ \exists\, k\,(p^j\,|\,n(k))\} = \sup\{j;\ \exists\, k\,(p^j\,|\,m(k))\}$$

for every prime number p. In particular, there exist uncountably many non-C*-isomorphic UHF algebras. Dixmier constructed a "dimension function" d_A on the set E_A of projections of a general matroid A, and showed that two matroids A and B are C*-isomorphic if and only if $d_A(E_A)=d_B(E_B)$. He also showed how to compute $d_A(E_A)$ in terms of the dimensions $\{n(k)\}_{k=1}^{\infty}$ of an arbitrary admissible sequence $\{A_k\}_{k=1}^{\infty}$ for A. Another (easy) remark of Dixmier is that a unital matroid is, in fact, a UHF algebra (see [1, 1.2]).

Using the fact that two C*-algebra are linearly isometric if and only if they are Jordan-*-isomorphic (see [3] and [5]) one obtains easily that the isometric classification of matroids coincides with the Glimm-Dixmier classification as C*-algebras.

We are interested here in the linear-topological classification of matroids, i.e., in the classification up to a Banach-space isomorphism. Our main result is the following theorem, which shows a completely different phenomenon.

THEOREM 1.1. (a) *Every matroid C*-algebra is isomorphic either to* $\mathrm{LC}(l_2)$, *the C*-algebra of all compact operators on* l_2, *or to the Fermion algebra* $F=\otimes_{n=1}^{\infty} M_2$;

(b) $\mathrm{LC}(l_2)$ *is isometric to a subspace of* F *which is the range of a contractive projection from* F.

(c) F *is not isomorphic to a subspace of a quotient space of* $\mathrm{LC}(l_2)$.

The representation of $\mathrm{LC}(l_2)$ as a matroid is quite obvious. Let $\{a_k\}_{k=1}^{\infty}$ be an increasing sequence of finite-rank projections on l_2 tending strongly to I, the identity operator. Let $A_k=a_k\cdot\mathrm{LC}(l_2)\cdot a_k$ and $n(k)=\mathrm{rank}\ (a_k)$. Then A_k is C*-isomorphic to $M_{n(k)}$, $A_k\subsetneqq A_{k+1}$, and $\bigcup_{k=1}^{\infty} A_k$ is norm-dense in $\mathrm{LC}(l_2)$.

Assuming the notion of infinite tensor product of C*-algebras (see [7, Section 1.23], [8], and section 2 below) the representation of $F=\otimes_{n=1}^{\infty} M_2$ as a UHF algebra is also obvious. For $k=1,2,\ldots$ let

$$A_k = \overbrace{M_2 \otimes M_2 \otimes \ldots \otimes M_2}^{k\text{-factors}}$$

then $\{A_k\}_{k=1}^{\infty}$ is (identified with) a strictly increasing sequence of unital C*-subalgebras of F, A_k is C*-isomorphic to M_{2^k}, and $\bigcup_{k=1}^{\infty} A_k$ is norm-dense in F.

Theorem 1.1 answers questions raised by A. Lazar, and may be helpful in the linear-topological classification of general AF-algebras. We thank Professor Lazar for valuable discussions.

Our methods are elementary and straightforward, and are independent of the delicate analysis of [1] and [2]. After replacing the above definition of matroids by the (equivalent) definition as a direct limit of matrix algebras, we analyze in a greater detail commutative diagrams of the form

$$\begin{array}{ccc} M_{n(1)} & \xrightarrow{f} & M_{n(2)} \\ \gamma_1 \downarrow & & \downarrow \gamma_2 \\ M_{m(1)} & \xrightarrow[g]{} & M_{m(2)} \end{array}$$

where f and g are C*-monomorphisms and γ_1 and γ_2 are linear isometries of a special kind. This analysis enables us to show that if A, B are matroids with $B \neq \mathrm{LC}(l_2)$, then A is isometric to a 1-complemented subspace of B. Then we show that every matroid A is isomorphic to $c_0(A)$. These two facts together easily imply parts (a) and (b) of Theorem 1.1. In order to prove part (c) we introduce the notion of the "diagonal" of a matroid which is always a 1-complemented, commutative C*-subalgebra, and show that the diagonal of F is $C(\Delta)$, the algebra of all continuous function on the cantor set Δ. A simple duality argument, together with the fact that $\mathrm{LC}(l_2)^* = C_1$ (=the trace class) is separable, imply (c).

A word of caution about our terminology is necessary. Throughout the entire work we shall stay in the category of Banach spaces; so by "operator", "isomorphism", "isometry", "projection", etc. we shall always mean linear, continuous maps with the specified properties. The prefix "C*" will switch us to the category of C*-algebras, so "C*-homomorphism" "C*-monomorphism", "C*-isomorphism", etc. will mean linear, multiplicative, *-preserving, continuous maps. We do not require, however, that a C*-homomorphism from one unital C*-algebra into another preserves the unit element (also, a C*-subalgebra B of a unital C*-algebra A need not have a unit, and if it does —the units of A and B need not be the same). Except for this—our notation and terminology are quite standard, and we refer to [4] for Banach space theory and to [6] and [7] for C*-algebras.

2. Technical preparation

Let us start with some information on direct (or, inductive) limits of sequences of C*-algebras and infinite tensor products of matrix algebras. Our presentation is a variant of [7, Section 1.23] and [8]. Let $\{A_k\}_{k=1}^{\infty}$ be a sequence of C*-algebras so that for every k there exists a C*-monomorphism (i.e., an injective C*-homomorphism) $f_k: A_k \to A_{k+1}$. We call $\{A_k, f_k\}_{k=1}^{\infty}$ a *direct sequence.* Let $\tilde{A}$ be the *-subalgebra of $\prod_{k=1}^{\infty} A_k$ consisting of all

$a=(a_k)_{k=1}^{\infty}$ so that $a_{k+1}=f_k(a_k)$ for all $k \geqq k_a$, normed by $\|a\|=\sup_k \|a_k\|$, and let A be the completion of $\tilde{A}$. A is called the *direct limit* of the direct sequence$\{A_k, f_k\}_{k=1}^{\infty}$, in notation $A=\varinjlim\{A_k, f_k\}_{k=1}^{\infty}$. For every $n=1,2,\ldots$ the map $A_n \to \varinjlim\{A_k, f_k\}_{k=1}^{\infty}$ defined by

$$a \mapsto (\overbrace{0,0,\ldots,0}^{n-1 \text{ terms}}, a, f_n(a), f_{n+1}(f_n(a)),\ldots)$$

is a C*-monomorphism which identifies A_n with a C*-subalgebra of $\varinjlim\{A_k,f_k\}_{k=1}^{\infty}$. For simplicity, we shall regard A_n itself as being a C*-algebra of $\varinjlim\{A_k,f_k\}_{k=1}^{\infty}$.

Suppose now that $\{\nu(j)\}_{j=1}^{\infty}$ is some sequence of positive integers. Let

$$A_k = M_{\nu(1)} \otimes M_{\nu(2)} \otimes \ldots \otimes M_{\nu(k)},$$

with the norm of $B(l_2^{\nu(1)} \otimes l_2^{\nu(2)} \otimes \ldots \otimes l_2^{\nu(k)})$. Clearly, A_k is C*-isomorphic to $M_{n(k)}$, where $n(k)=\nu(1)\cdot\nu(2)\ldots\nu(k)$. Let $f_k\colon A_k \to A_{k+1}$ be defined by

$$f_k(x_1 \otimes x_2 \otimes \ldots \otimes x_k) = x_1 \otimes x_2 \otimes \ldots \otimes x_k \otimes I_{\nu(k+1)}$$

Then $\{A_k,f_k\}_{k=1}^{\infty}$ is direct sequence and its direct limit is called the *infinite tensor product* of $\{M_{\nu(j)}\}_{j=1}^{\infty}$, in notation

$$\bigotimes_{j=1}^{\infty} M_{\nu(j)} = \varinjlim \left\{ \bigotimes_{j=1}^{k} M_{\nu(j)}, f_k \right\}_{k=1}^{\infty}.$$

Next, let us show that the notion of a matroid coincides with the notion of a direct limit of matrix algebras. Let $\{A_k\}_{k=1}^{\infty}$ be an admissible sequence for a matroid A. Let $\varphi_k\colon A_k \to M_{n(k)}$ be a C*-isomorphism of A_k onto $M_{n(k)}$, and let $f_k: M_{n(k)} \to M_{n(k+1)}$ be defined by $f_k=\varphi_{k+1}\circ\varphi_k^{-1}$. Then $\{M_{n(k)},f_k\}_{k=1}^{\infty}$ is a direct sequence and A is C*-isomorphic to $\varinjlim\{M_{n(k)},f_k\}_{k=1}^{\infty}$. Using this identification one can easily prove the following.

PROPOSITION 2.1. *Let $\{M_{n(k)},f_k\}_{k=1}^{\infty}$ and $\{M_{m(k)},g_k\}_{k=1}^{\infty}$ be two direct sequences of matrix algebras. Suppose that for every k there exists an operator h_k: $M_{n(k)} \to M_{m(k)}$, so that $g_k\cdot h_k=h_{k+1}\cdot f_k$ for all k,* i.e., *the following diagram commutes*:

$$\begin{array}{ccc} M_{n(k)} & \xrightarrow{f_k} & M_{n(k+1)} \\ {\scriptstyle h_k}\downarrow & & \downarrow{\scriptstyle h_{k+1}} \\ M_{m(k)} & \xrightarrow{g_k} & M_{m(k+1)} \end{array}$$

Suppose also that $\sup_k \|h_k\| < \infty$. *Then there exists a unique operator*

$$h:\varinjlim\{M_{n(k)},f_k\}_{k=1}^{\infty} \to \varinjlim\{M_{m(k)},g_k\}_{k=1}^{\infty}$$

satisfying $h_{|M_{n(k)}}=h_k$ for all k and $\|h\|=\sup_k\|h_k\|$. Moreover, if all the h_k are *isometries (or, C*-homomorphisms) then h is an isometry (respectively, a C*-homomorphism).*

The operator h whose existence is ensured by Proposition 2.1 is called the *direct limit* of the sequence $\{h_k\}_{k=1}^{\infty}$, and is denoted by $h = \varinjlim h_k$. One can easily verify the following composition formula:

$$\varinjlim (h_k \circ h'_k) = (\varinjlim h_k) \circ (\varinjlim h'_k) .$$

As a consequence, we have the following.

PROPOSITION 2.2. *Let $\{M_{n(k)}, f_k\}_{k=1}^{\infty}$ and $\{M_{m(k)}, g_k\}_{k=1}^{\infty}$ be two direct sequences. Suppose that for every k there exists an isometry h_k of $M_{n(k)}$ into $M_{m(k)}$ and a contraction p_k from $M_{m(k)}$ into $M_{n(k)}$ so that*

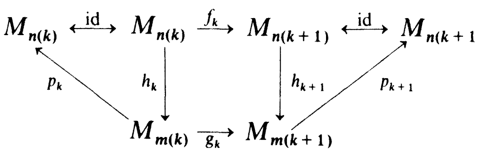
(i) $g_k \circ h_k = h_{k+1} \circ f_k$

(ii) $p_k \circ h_k = \mathrm{id}_{M_{n(k)}}$, *the identity operator on $M_{n(k)}$*

and

(iii) $f_k \circ p_k = p_{k+1} \circ g_k$

that is, the following diagram commutes



Then

$$A = \varinjlim \{M_{n(k)}, f_k\}_{k=1}^{\infty}$$

is isometric to a subspace X of

$$B = \varinjlim \{M_{m(k)}, g_k\}_{k=1}^{\infty} ,$$

and there exists a contractive projection from B onto X.

Indeed, $h = \varinjlim k_k$ is an isometry from A into B, $p = \varinjlim p_k$ is a contraction from B into A, and

$$p \circ h = \varinjlim (p_k \circ h_k) = \mathrm{id}_A ,$$

so $h \circ p$ is a contractive projection from B onto $X = h(A)$.

DEFINITION 2.3. Let n, m, r be positive integers with $rn \leqq m$. We define a map

$$\varphi_{n,m,r} : M_n \to M_m$$

by

$$(2.1) \qquad \varphi_{n,m,r}(a) = \left(\bigoplus_{j=1}^{r} a \right) \oplus 0_l = \begin{bmatrix} \overbrace{a \ \ddots \ a}^{r\text{-terms}} & 0 \\ 0 & 0_l \end{bmatrix}$$

where $l = m - nr$ and 0_l denotes the zero matrix of order $l \times l$.

Clearly, $\varphi_{n,m,r}$ is a C*-monomorphism of M_n into M_m.

PROPOSITION 2.4. *Let $n \leqq m$ and let $\varphi : M_n \to M_m$ be a C*-monomorphism. Then there exists a unitary matrix $v \in M_m$ so that*

$$\varphi(a) = v^* \cdot (\varphi_{n,m,r}(a)) \cdot v, \qquad a \in M_n$$

where $r = \operatorname{rank} \varphi(e_{1,1})$.

This is well-known (any C*-monomorphism maps elements with orthogonal ranges (or, orthogonal cokernels) into elements with the same properties. Now apply this to the system of matrix-units $\{e_{i,j}\}_{i,j=1}^{n}$ of M_n).

DEFINITION 2.5. Let $\varphi : M_n \to M_m$ $(n \leqq m)$ be a C*-monomorphism. We put

$$r(\varphi) = \operatorname{rank} \varphi(e_{1,1}) .$$

It is clear that $r(\varphi) = \operatorname{rank} \varphi(e)$ for every rank-one projection $e \in M_n$. The functional "r" is multiplicative: if $M_n \xrightarrow{\varphi} M_m$ and $M_m \xrightarrow{\psi} M_k$ are C*-monomorphisms, then $r(\psi \circ \varphi) = r(\psi) \cdot r(\varphi)$.

We state without proof the following elementary proposition.

PROPOSITION 2.6. *Let $\{A_k, f_k\}_{k=1}^{\infty}$ be a direct sequence and let $\{k_j\}_{j=1}^{\infty}$ be any increasing sequence of positive integers. Let for $j = 1, 2, \ldots$*

$$g_j = f_{k_{j+1}-1} \circ \ldots \circ f_{k_j+1} \circ f_{k_j} : A_{k_j} \to A_{k_{j+1}} .$$

Then $\varinjlim \{A_k, f_k\}_{k=1}^{\infty}$ is C-isomorphic to $\varinjlim \{A_{k_j}, g_j\}_{j=1}^{\infty}$.*

In particular, if $\{M_{n(k)}, f_k\}_{k=1}^{\infty}$ is any direct sequence of matrix algebras with $r(f_k) = 1$ for $k \geqq k_0$, then $\varinjlim \{M_{n(k)}, f_k\}_{k=1}^{\infty}$ is C-isomorphic to*

$$\mathrm{LC}(l_2) = \varinjlim \{M_k, \varphi_{k,k+1,1}\}_{k=1}^{\infty} .$$

DEFINITION 2.7. Let $n \leqq m$. We define $p_{m,n}: M_m \to M_n$ by $(p_{m,n}(a))(i,j) = a(i,j)$, $1 \leqq i,j \leqq n$, $a \in M_m$.

Clearly, $p_{m,n}$ is a contraction and $p_{m,n} \circ \varphi_{n,m,r} = \mathrm{id}_{M_n}$ for all positive integers n, m, r with $nr \leqq m$.

DEFINITION 2.8. Two maps $f, g: M_n \to M_m$ are said to be *equivalent* if there exist unitary matrices $u_1, u_2 \in M_n$ and $v_1, V_2 \in M_m$ so that

$$f(a) = v_2(g(u_2 a u_1))v_1, \qquad a \in M_n .$$

DEFINITION 2.9. For positive integers $n \leqq m$ let $\Gamma_{n,m}$ be the set of all linear maps $\gamma: M_n \to M_m$ that are equivalent to a map $\tilde{\gamma}: M_n \to M_m$ of the form

$$\tilde{\gamma}(a) = a \oplus p_{n,n_1}(a) \oplus p_{n,n_2}(a) \oplus \ldots \oplus p_{n,n_s}(a) \oplus 0_l \tag{2.2}$$

where $1 \leqq n_j < n$, $1 \leqq l$, $0 \leqq s$, and $n + \sum_{j=1}^{s} n_j + l = m$.

Notice that, up to a permutation, the sequence $\{n_j\}_{j=1}^{s}$ depends only on γ (not on $\tilde{\gamma}$). Also, $\tilde{\gamma}(a^*) = \tilde{\gamma}(a)^*$ for all $a \in M_n$, and $\tilde{\gamma}$ is multiplicative if and only if $s = 0$, i.e., $\tilde{\gamma}(a) = a \oplus 0_l$.

PROPOSITION 2.10. *Let $n \leqq m$. Then every $\gamma \in \Gamma_{n,m}$ is an isometry and there is a contractive* (i.e., *norm-one*) *projection from M_m onto $\gamma(M_n)$.*

PROOF. It is clearly enough to prove this in the case where $\gamma = \tilde{\gamma}$ is given by (2.2). Now, for any matrices a, b

$$\|a \oplus b\| = \max\{\|a\|, \|b\|\} .$$

So, using the fact that p_{n,n_j} are contractions, we get

$$\|\gamma(a)\| = \max\{\|a\|, \|p_{n,n_1}(a)\|, \ldots, \|p_{n,n_s}(a)\|\} = \|a\| .$$

Since $p_{m,n} \circ \gamma = \mathrm{id}_{M_n}$, we get that $\gamma \circ p_{m,n}$ is a contractive projection from M_m onto $\gamma(M_n)$.

The following Lemma is the heart of the proof of Theorem 1.1.

LEMMA 2.11. *Let $f: M_{n(1)} \to M_{n(2)}$ and $g: M_{m(1)} \to M_{m(2)}$ be C*-monomorphism. Let $\nu = r(f)$, $\mu = r(g)$ and $\sigma = n(2) - n(1)\nu$, and suppose that $\mu = \prod_{j=1}^{\sigma+1} \mu(j)$, where $\mu(j)$ are positive integers satisfying $\mu(1) \geqq \nu$ and $\mu(j) \geqq 3$ for all j. Assume also that $m(1) \geqq n(1) + 1$ and let $\gamma_1 \in \Gamma_{n(1),m(1)}$. Then there exists a $\gamma_2 \in \Gamma_{n(2),m(2)}$ so that $\gamma_2 \circ f = g \circ \gamma_1$,* i.e., the following diagram commutes

$$\begin{array}{ccc} M_{n(1)} & \xrightarrow{f} & M_{n(2)} \\ \gamma_1 \downarrow & & \downarrow \gamma_2 \\ M_{m(1)} & \xrightarrow{g} & M_{m(2)} \end{array}$$

PROOF. By Proposition 2.4, there is no loss of generality in assuming that

$$f = \varphi_{n(1), n(2), \nu}, \qquad g = \varphi_{m(1), m(2), \mu}$$

and that for all $a \in M_{n(1)}$,

$$\gamma_1(a) = a \oplus p_{n(1), k(1)}(a) \oplus \ldots \oplus p_{n(1), k(s)}(a) \oplus 0_l, \tag{2.3}$$

where $1 \leqq k(j) < n(1)$, $0 \leqq s$, and $1 \leqq l$. Next, let us factor f as $f = f_\sigma \circ \ldots \circ f_1 \circ f_0$, where

$$f_0 = \varphi_{n(1), n(1)\nu, \nu}$$

and

$$f_j = \varphi_{n(1)\nu + j - 1, n(1)\nu + j, 1}, \qquad 1 \leqq j \leqq \sigma .$$

By our assumption on $\mu = r(g)$ there is also a factorization $g = g_{\sigma+1} \circ \ldots \circ g_1 \circ g_0$, where for $0 \leqq j \leqq \sigma$,

$$g_j = \varphi_{m(1)\mu(j)!,\ m(1)\mu(j+1)!,\ \mu(j+1)}$$

(here $\mu(j)! = \prod_{i=1}^{j} \mu(i)$, with the understanding that $\mu(0)! = 1$) and

$$g_{\sigma+1} = \varphi_{m(1)\mu, m(2), 1} \cdot$$

It is therefore enough to prove the existence of maps

$$\gamma^{(j)} \in \Gamma_{n(1)\nu + j,\ m(1)\mu(j+1)!}, \qquad j = 0, 1, \ldots, \sigma$$

satisfying

$$g_j \circ \gamma^{(j-1)} = \gamma^{(j)} \circ f_j, \qquad j = 0, 1, \ldots, \sigma ,$$

(where $\gamma_1 = \gamma^{(-1)}$). Indeed, using these $\gamma^{(j)}$ we define $\gamma_2 = g_{\sigma+1} \circ \gamma^{(\sigma)}$. It is clear that $\gamma_2 \in \Gamma_{n(2), m(2)}$ and that $\gamma_2 \circ f = g \circ \gamma_1$.

The following commutative diagram describes the factorizations of f and g and the maps $\gamma^{(j)}$ (the broken lines describes the maps to be constructed):

$$\begin{array}{ccccccccccccc} M_{n(1)} & \xrightarrow{f_0} & M_{n(1)\nu} & \xrightarrow{f_1} & M_{n(1)\nu+1} & \xrightarrow{f_2} & M_{n(1)\nu+2} & \xrightarrow{f_3} & \ldots & \xrightarrow{f_\sigma} & M_{n(2)} & & \\ \gamma_1 \downarrow & & \vdots\ \gamma^{(0)} & & \vdots\ \gamma^{(1)} & & \vdots\ \gamma^{(2)} & & & & \vdots\ \gamma^{(\sigma)} & \ddots\ \gamma_2 & \\ M_{m(1)} & \xrightarrow{g_0} & M_{m(1)\mu(1)} & \xrightarrow{g_1} & M_{m(1)\mu(2)!} & \xrightarrow{g_2} & M_{m(1)\mu(3)!} & \xrightarrow{g_3} & \ldots & \xrightarrow{g_\sigma} & M_{m(1)\mu} & \xrightarrow{g_{\sigma+1}} & M_{m(2)} \end{array}$$

Thus, it is enough to prove the lemma in the following special cases:

CASE 1. $n(2)=n(1)\nu, \quad m(2)=m(1)\mu, \quad \nu \leqq \mu$, and

$$f = \varphi_{n(1),n(1)\nu,\nu}, \qquad g=\varphi_{m(1),m(1)\mu,\mu} .$$

CASE 2. $n(2)=n(1)+1, \quad m(2)=m(1)\mu, \quad \mu \geqq 3$, and

$$f = \varphi_{n(1),n(1)+1,1}, \qquad g = \varphi_{m(1),m(1)\mu,\mu} .$$

PROOF OF THE LEMMA IN CASE 1. Write (2.3) as

$$\gamma_1(a) = a \oplus p(a) \oplus 0_l = \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0_l \end{bmatrix}, \quad a \in M_{n(1)} . \tag{2.4}$$

By our assumption,

$$f(a) = \begin{bmatrix} a & & 0 \\ & a & \\ & & \ddots & \\ 0 & & & a \end{bmatrix} = \overbrace{a \oplus a \oplus \ldots \oplus a}^{\nu\text{-terms}}, \quad a \in M_{n(1)}$$

and

$$g(b) = \begin{bmatrix} b & & 0 \\ & b & \\ & & \ddots & \\ 0 & & & b \end{bmatrix} = \overbrace{b \oplus b \oplus \ldots \oplus b}^{\mu\text{-terms}}, \quad b \in M_{m(1)} . \tag{2.5}$$

Define $\gamma_2 : M_{n(2)} \to M_{m(2)}$ by

$$\gamma_2 \begin{bmatrix} a_{1,1} & \ldots & a_{1,\nu} \\ \vdots & & \vdots \\ a_{\nu,1} & \ldots & a_{\nu,\nu} \end{bmatrix} =$$

$$
\left[\begin{array}{c|c|c|c|c|c|c}
\begin{matrix} a_{1,1} & 0 & 0 \\ 0 & p(a_{1,1}) & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} a_{1,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \cdots & \begin{matrix} a_{1,\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 & \cdots & 0 \\ \hline
\begin{matrix} a_{2,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} a_{2,2} & 0 & 0 \\ 0 & p(a_{1,1}) & 0 \\ 0 & 0 & 0 \end{matrix} & \cdots & \begin{matrix} a_{2,\nu} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & 0 & \cdots & 0 \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hline
\begin{matrix} a_{\nu,1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} a_{\nu,2} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \cdots & \begin{matrix} a_{\nu,\nu} & 0 & 0 \\ 0 & p(a_{1,1}) & 0 \\ 0 & 0 & 0 \end{matrix} & 0 & \cdots & 0 \\ \hline
0 & 0 & \cdots & 0 & \begin{matrix} a_{1,1} & 0 & 0 \\ 0 & p(a_{1,1}) & 0 \\ 0 & 0 & 0 \end{matrix} & \cdots & 0 \\ \hline
\vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \hline
0 & 0 & \cdots & 0 & 0 & \cdots & \begin{matrix} a_{1,1} & 0 & 0 \\ 0 & p(a_{1,1}) & 0 \\ 0 & 0 & 0 \end{matrix}
\end{array}\right]
$$

where the large blocks belong to $M_{m(1)}$, $a_{i,j} \in M_{n(1)}$ and $p(a_{1,1}) \in M_{m(1)-n(1)-l}$.

It is clear that γ_2 is unitarily equivalent to the map $\tilde{\gamma}: M_{n(2)} \to M_{m(2)}$ defined by

$$
\gamma_2 \begin{bmatrix} a_{1,1} & \cdots & a_{1,\nu} \\ \vdots & & \vdots \\ a_{\nu,1} & \cdots & a_{\nu,\nu} \end{bmatrix} = \begin{bmatrix} a_{1,1} & \cdots & a_{1,\nu} \\ \vdots & & \vdots \\ a_{\nu,1} & \cdots & a_{\nu,\nu} \end{bmatrix}
$$

$$
\oplus \Big(\overbrace{a_{1,1} \oplus \cdots \oplus a_{1,1}}^{\mu\text{-terms}}\Big) \oplus \Big(\overbrace{p(a_{1,1}) \oplus \cdots \oplus p(a_{1,1})}^{(\mu-\nu)\text{-terms}}\Big)
$$

So $\gamma_2 \in \Gamma_{n(2),m(2)}$. Also, for a $\in M_{n(1)}$

$$\gamma_2(f(a)) = \gamma_2 \begin{bmatrix} a & & 0 \\ & a & \\ & & \ddots \\ 0 & & a \end{bmatrix}$$

$$= \overbrace{\begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \dots \oplus \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{\mu\text{-terms}}$$

$$= g(\gamma_1(a)) .$$

This completes the proof of the lemma in Case 1.

PROOF OF THE LEMMA IN CASE 2. In this case $f\colon M_{n(1)} \to M_{n(1)+1} = M_{n(2)}$ is given by

$$f \begin{bmatrix} a_{1,1} & \dots & a_{1,n(1)} \\ \vdots & & \vdots \\ a_{n(1),1} & \dots & a_{n(1),n(1)} \end{bmatrix} = \begin{bmatrix} a_{1,1} & \dots & a_{1,n(1)} & 0 \\ \vdots & & \vdots & \vdots \\ a_{n(1),1} & \dots & a_{n(1),n(1)} & 0 \\ 0 & \dots 0 & & 0 \end{bmatrix}$$

while γ_1 and g are given by (2.4) and (2.5) respectively. Let every $x = (x(i,j))_{i,j=1}^{n(2)} \in M_{n(2)}$ be written as

$$x = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where

$$a = \begin{bmatrix} x(1,1) & \dots & x(1,n(1)) \\ \vdots & & \vdots \\ x(n(1),1) & \dots & x(n(1),n(1)) \end{bmatrix} = P_{n(2),n(1)}(x) \in M_{n(1)}$$

$$b = \begin{bmatrix} x(1,n(2)) \\ \vdots \\ x(n(1),n(2)) \end{bmatrix} \in M_{n(1),1}$$

$$c = \quad (x(n(2),1), \dots, x(n(2),n(1))) \in M_{1,n(1)}$$

and

$$d = x(n(2),n(2)) \in M_{1,1}$$

Using the fact that $\mu \geqq 3$ we define a map $\gamma_2 : M_{n(2)} \to M_{m(2)}$ by

$$\gamma_2(x) = \gamma_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} =$$

$$\left[\begin{array}{c|c|c|c|c|c}
\begin{matrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{matrix} & \mathbf{0} & \begin{matrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline
\begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ c & 0 & 0 \end{matrix} & \begin{matrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{matrix} & \begin{matrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & d \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline
\mathbf{0} & \mathbf{0} & \begin{matrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{matrix} & \mathbf{0} & \cdots & \mathbf{0} \\ \hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \begin{matrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{matrix} & \cdots & \mathbf{0} \\ \hline
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \hline
\mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \cdots & \begin{matrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{matrix}
\end{array}\right]$$

($\mu \times \mu$ block matrices from $M_{m(1)}$). Clearly, $\gamma_2 \in \Gamma_{n(2),m(2)}$. Also, for $a \in M_{n(1)}$ we have

$$\gamma_2(f(a)) = \gamma_2\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \oplus \cdots \oplus \begin{bmatrix} a & 0 & 0 \\ 0 & p(a) & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (\mu \text{ blocks})$$

$$= \gamma_1(a) \oplus \cdots \oplus \gamma_1(a) \qquad (\mu \text{ blocks})$$

$$= g(\gamma_1(a))$$

This completes the proof of the Lemma in Case 2.

LEMMZ 2.12. *Let* $n \leqq m$ *and let* $\gamma = \tilde{\gamma}$ *be given by* (2.2). *Let* $\beta : M_m \to M_n$ *be a contraction satisfying* $\beta \circ \gamma = \mathrm{id}_{M_n}$. *Then* $\beta = p_{m,n}$.

PROOF. Let $\{e_{i,j}\}_{i,j=1}^{m}$ denote the matrix units of M_m, and let $k = \max_{1 \leqq j \leqq s} n_j$, where s and n_j appear in (2.2). Let us write for short

$$\gamma(a) = a \oplus p(a), \qquad a \in M_n$$

instead of (2.2). Then for (i,j) with $k < \max\{i,j\} \leqq n$ we have $p(e_{i,j}) = 0$ and so

$$e_{i,j} = \beta(\gamma(e_{i,j})) = \beta(e_{i,j}) .$$

If $\max\{i,j\} \leqq k$ and $a = \beta(e_{i,j})$ then for any $(i_1,j_1) \neq (i,j)$ with $\max\{i_1,j_1\} \leqq n$ we have

$$\|e_{i_1,j_1} + \lambda e_{i,j}\| \leqq (1+|\lambda|^2)^{\frac{1}{2}}, \qquad |\lambda| \leqq 1$$

So, for all $|\lambda| \leqq 1$,

$$\begin{aligned} |1+\lambda a(i_1,j_1)| &\leqq \|e_{i_1,j_1} + \lambda a\| \\ &= \|\beta(\gamma(e_{i_1,j_1}) + \lambda e_{i,j})\| \\ &\leqq \|\gamma(e_{i_1,j_1}) + \lambda e_{i,j}\| \\ &= \|e_{i_1,j_1} + \lambda e_{i,j}\| \leqq (1+|\lambda|^2)^{\frac{1}{2}} \end{aligned}$$

and thus $a(i_1, j_1) = 0$. It follows that $\beta(e_{i,j}) = \lambda_{i,j} e_{i,j}$. Now

$$\|\beta(e_{i,n} \oplus p(e_{i,j}))\| \leqq \|e_{i,n} \oplus p(e_{i,j})\| = 1 .$$

But also

$$\|\beta(e_{i,n} \oplus p(e_{i,j}))\| = \|e_{i,n} + (1-\lambda_{i,j})e_{i,j}\| = (1+|1-\lambda_{i,j}|^2)^{\frac{1}{2}} .$$

So $\lambda_{i,j} = 1$ and $\beta(e_{i,j}) = e_{i,j}$.

Finally, let $\max\{i,j\} > n$, let $a = \beta(e_{i,j})$ and let $\max\{i_1,j_1\} \leqq n$. Then $\|e_{i_1,j_1} + \lambda e_{i,j}\| \leqq (1+|\lambda|^2)^{\frac{1}{2}}$, but

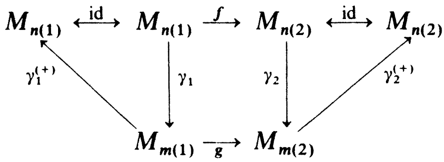
$$\|\beta(e_{i_1,j_1} + \lambda e_{i,j})\| = \|e_{i_1,j_1} + \lambda a\| \geqq |1+\lambda a(i_1,j_1)|$$

for all λ. This implies that $a(i_1,j_1) = 0$ for all $i_1, j_1 \leqq n$. So $a = 0$. This proves that $\beta(e_{i,j}) = e_{i,j}$ if $\max\{i,j\} \leqq n$ and $\beta(e_{i,j}) = 0$ if $\max\{i,j\} > n$. So $\beta = p_{m,n}$.

COROLLARY 2.13. *Any* $\gamma \in \Gamma_{n,m}$ $(n \leqq m)$ *has a unique contractive left inverse, denoted* $\gamma^{(+)}$. *So*

$$\gamma^{(+)} : M_m \to M_n, \quad \|\gamma^{(+)}\| = 1, \quad \gamma^{(+)} \circ \gamma = \mathrm{id}_{M_n}.$$

COROLLARY 2.14. *Under the assumptions of Lemma* 2.11 *we have* $f \circ \gamma_1^{(+)} = \gamma_2^{(+)} \circ g$, i.e., *the following diagram commutes*:



PROOF. We present the proof in Case 1 of the Proof of Lemma 2.11; the proof in Case 2 is essentially the same, but the (obvious) formula for $\gamma_2^{(+)}$ happens to be more complicated. We have

$$\gamma_1^{(+)}(b) = p_{m(1),n(1)}(b), \qquad b \in M_{m(1)},$$

and for $b = (b_{i,j})_{i,j=1}^{\mu} \in M_{m(2)}$ with $b_{i,j} \in M_{m(1)}$,

$$\gamma_2^{(+)}(b) = \gamma_2^{(+)}\left((b_{i,j})_{i,j=1}^{\mu}\right) = (a_{i,j})_{i,j=1}^{\nu} \in M_{n(2)},$$

where

$$a_{i,j} = p_{m(1),n(1)}(b_{i,j}) \in M_{n(1)}.$$

So, if $b \in M_{m(1)}$ and $a = p_{m(1),n(1)}(b)$, then

$$\gamma_2^{(+)}(g(b)) = \gamma_2^{(+)} \begin{bmatrix} b & & 0 \\ & b & \\ & & \ddots \\ 0 & & b \end{bmatrix} \qquad (\mu \text{ blocks})$$

$$= \begin{bmatrix} a & & 0 \\ & a & \\ & & \ddots \\ 0 & & a \end{bmatrix} \qquad (\nu \text{ blocks})$$

$$= f(a) = f(\gamma_1^{(+)}(b)).$$

3. The main results.

Let us start with the following

LEMMA 3.1. *Let* $\{M_{m(k)}, g_k\}_{k=1}^{\infty}$ *be a direct sequence, and let the positive integers* $r(g_k)$, $k = 1, 2, \ldots$, *the defined by Definition* 2.5. *Assume that* $\limsup_{k\to\infty} r(g_k) \geqq 2$, *and let* $\{M_{n(k)}, f_k\}_{k=1}^{\infty}$ *be any other direct sequence. Then* $A = \varinjlim \{M_{n(k)}, f_k\}_{k=1}^{\infty}$ *is isometric to a* 1-*complemented subspace of* $B = \varinjlim \{M_{m(k)}, g_k\}_{k=1}^{\infty}$.

PROOF. By the multiplicativity of the functiona "r", we get

$$r(g_{k_{j+1}-1}\circ\dots\circ g_{k_j+1}\circ g_{k_j}) = \prod_{k=k_j}^{k_{j+1}-1} r(g_k),$$

$j=1,2,\dots$ for any increasing sequence $\{k_j\}$ of positive integers. Using Proposition 2.6 we can assume without loss of generality that $m(k)$ and $r(g_k)$ are arbitrarily large. Precisely, if $\nu_k=r(f_k)$, $\mu_k=r(g_k)$, $\sigma_k=n(k+1)-n(k)\nu_k$, then we assume that

$$\mu_k = \prod_{j=1}^{q_k+1} \mu_k(j),$$

where $\mu_k(j)$ are positive integers satisfying $\mu_k(1)\geqq\nu_k$ and $\mu_k(j)\geqq 3$ for all j. We also assume that $m(1)>n(1)$.

We define $\gamma_1\in\Gamma_{n(1),m(1)}$ by $\gamma_1=\varphi_{n(1),m(1),1}$. Using Lemma 2.11 we construct inductively a sequence $\gamma_k\in\Gamma_{n(k),m(k)}$ so that $g_k\circ\gamma_k=\gamma_{k+1}\circ f_k$, $k=1,2,3,\dots$. Let

$$\gamma_k^{(+)}: M_{m(k)} \to M_{n(k)}$$

be the (unique) contractive left inverse of γ_k (see Lemma 2.12 and Corollary 2.13), i.e., $\|\gamma_k^{(+)}\|=1$ and

$$\gamma_k^{(+)}\circ\gamma_k = \mathrm{id}_{M_{n(k)}}, \qquad k=1,2,\dots.$$

By Corollary 2.13 we have also $f_k\circ\gamma_k^{(+)}=\gamma_{k+1}^{(+)}\circ g_k$ for all k. Let

$$\gamma = \varinjlim\gamma_k : A \to B \quad\text{and}\quad \gamma^{(+)} = \varinjlim\gamma_k^{(+)} : B \to A.$$

By Proposition 2.2 γ is an isometry of A onto a subspace, say X, of B, $\|\gamma^{(+)}\|=1$ and $\gamma^{(+)}\circ\gamma=\mathrm{id}_A$. So $p=\gamma\circ\gamma^{(+)}$ is a contractive projection from B onto $X=\gamma(A)$.

Let us concentrate now on the Fermion algebra

$$F = \bigotimes_{n-1}^{\infty} M_2^{(n)} = \varinjlim\{M_{2^k}, \varphi_{2^k,2^{k+1},2}\}_{k=1}^{\infty},$$

where $M_2^{(n)}$ denotes the nth factor M_2 (for basic information see [5]). The canonical, normalized trace of F is given by

$$\tau = \varinjlim\{2^{-k}\cdot\mathrm{trace}|_{M^k}\}_{k=1}^{\infty}$$

(see Proposition 2.1 and the discussion preceding it). The action of τ on an elementary tensor is

$$\tau\left(\bigotimes_{n=1}^{m} x_n\right) = \prod_{n=1}^{m} 2^{-1}\cdot(\mathrm{trace}\, x_n), \qquad x_n\in M_2^{(n)}.$$

Let also $\{e_{i,j}^{(n)}\}_{i,j=0}^{1}$ denote the standard matrix units of $M_2^{(n)}$.

PROPOSITION 3.2. *Let $v_m = \otimes_{n=1}^{m} e_{1,1}^{(n)}$. Then for all $x \in F$*

$$\delta_{(1,1)}(x) = \lim_{m\to\infty} \tau(2^m v_m x)$$

exists. $\delta_{(1,1)}$ a norm-one linear functional on F.

PROOF. Let x be an elementary tensor from $\otimes_{n=1}^{k} M_2^{(n)}$,

$$x = \bigotimes_{n=1}^{k} x_n, \qquad x_n \in M_2^{(n)} .$$

Then for $m \geqq k$,

$$\begin{aligned} \tau(2^m v_m x) &= 2^m \prod_{n=1}^{k} 2^{-1}(\operatorname{trace} e_{1,1}^{(n)} x_n) \cdot \prod_{n=k+1}^{m} 2^{-1} \cdot \operatorname{trace} e_{1,1}^{(n)} \\ &= 2^k \prod_{n=1}^{k} 2^{-1} \operatorname{trace}(e_{1,1}^{(n)} x_n) = \tau(2^k v_k x) . \end{aligned}$$

This clearly implies that $\delta_{(1,1)}(x) = \lim_{m\to\infty} \tau(2^m v_m x)$ exists for every x in the dense *-subalgebra $\bigcup_{k=1}^{\infty} \otimes_{n=1}^{k} M_2^{(n)}$ of F. Since each functional $x \mapsto \tau(2^m v_m x)$ has norm one,

$$\delta_{(1,1)}(x) = \lim_{m\to\infty} \tau(2^m v_m x)$$

exists for every $x \in F$, and $\|\delta_{(1,1)}\| \leqq 1$. Finally, if 1 denotes the unit of F, then

$$\delta_{(1,1)}(1) = \lim_{m\to\infty} \tau(2^m v_m 1) = \lim_{m\to\infty} \tau(2^m v_m) = 1 .$$

So $\|\delta_{(1,)}\| = 1$.

REMARK. $\delta_{(1,1)}$ correspond to "point-evaluation at $(1,1)$". If $0 \leqq s,\ t \leqq 1$ are given by

$$s = \sum_{i=1}^{\infty} s_i 2^{-i} \quad \text{and} \quad t = \sum_{i=1}^{\infty} t_i 2^{-i}$$

(where $s_i, t_i \in \{0,1\}$ and $\sum_{i=1}^{\infty} s_i = \sum_{i=1}^{\infty} t_i = \infty$), we define

$$\delta_{(s,t)}^{(m)}(x) = \tau\left[2^m \left(\bigotimes_{i=1}^{m} e_{s_i,t_i}^{(i)}\right) x\right].$$

Then

$$\delta_{(s,t)}(x) = \lim_{m\to\infty} \delta_{(s,t)}^{(m)}(x)$$

exists for all $x \in F$. $\delta_{(s,t)}$ is a norm-one functional which corresponds to "point-evaluation at (s,t)". This exhibits F as a space of functions on the unit square $[0,1]\times[0,1]$ (which, however, is very different from the classical function spaces).

For any Banach space X we denote by $c_0(X)$ the space of all sequences $x=(x_1,x_2,\ldots)$ with $x_j \in X$ and $\|x_j\| \to 0$, normed by $\|x\|=\sup\|x_j\|$. If X is a C*-algebra, then $c_0(X)$ is also C*-algebra.

LEMMA 3.3 *The Fermion algebra* $F=\otimes_{n=1}^{\infty} M_2^{(n)}$ *has a* C*-*subalgebra* A *which is* C*-*isomorphic to* $c_0(F)$, *and there is a projection* P *from* F *onto* A *with* $\|P\| \leqq 2$.

PROOF. Let $\delta_{(1,1)}$ and v_m have the same meaning as in Proposition 3.2. Let $F_0 = \ker\delta_{(1,1)}$ and let $Q: F \to F_0$ be given by $Qx = x - \delta_{(1,1)}(x)1$. Then Q is a projection of norm 2. Define for $j=1,2,\ldots$

$$a_j = \left(\bigotimes_{i=1}^{j-1} e_{1,1}^{(i)}\right)\otimes e_{0,0}^{(j)} .$$

Then $\{a_j\}$ are mutually orthogonal projections. Also

$$a_jFa_j = a_jF_0a_j = a_j\otimes\left(\bigotimes_{n=j+1}^{\infty} M_2^{(n)}\right), \qquad j=1,2.\ldots .$$

So a_jFa_j is C*-isomorphic in the natural way to F. Let $A=\overline{\operatorname{span}}\{a_jFa_j\}_{j=1}^{\infty}$. Then A is a C*-subalgebra of F which is C*-isomorphic to $c_0(F)$. We now claim that $A\subset F_0$ and that

$$\tilde{P}(x) = \sum_{j=1}^{\infty} a_jxa_j$$

defines a contractive projection from F_0 onto A. Proving this, we complete the proof of the lemma by letting $P=\tilde{P}Q$.

Indeed, for all j and m

$$v_ma_j = a_jv_m = \begin{cases} 0 & ; \quad j\leqq m \\ a_j & ; \quad j>m \end{cases} . \tag{3.1}$$

This implies that for all $x \in F$,

$$\delta_{(1,1)}(a_jxa_j) = \lim_{m\to\infty} \tau(2^m v_ma_jxa_j) = 0$$

and so $A \subset F_0 = \ker\delta_{(1,1)}$. Next, let us define

$$P_m(x) = \sum_{j=1}^{m} a_j x a_j + \tau(2^m v_m x) v_m, \qquad x \in F,\ m = 1, 2, \ldots .$$

By (3.1), $P_m^2 = P_m$ and $\|P_m\| = 1$. If $x \in \otimes_{n=1}^{k} M_2^{(n)}$, then for all $m \geqq k$,

$$a_m x a_m = \tau(2^k v_k x) a_m, \qquad \tau(2^m v_m x) = \tau(2^k v_k x)$$

and also

$$\sum_{j=k+1}^{m} a_j + v_m = v_k$$

(the last formula follows by an easy induction on $m \geqq k$). So

$$\begin{aligned} P_m(x) &= \sum_{j-1}^{m} a_j x a_j + \tau(2^m v_m x) v_m \\ &= \sum_{j=1}^{k} a_j x a_j + \sum_{j=k+1}^{m} \tau(2^k v_k x) a_j + \tau(2^k v_k x) v_m \\ &= \sum_{j=1}^{k} a_j x a_j + \tau(2^k v_k x) v_k \\ &= P_k(x) . \end{aligned}$$

This clearly implies that $\tilde{P}(x) = \lim_{m \to \infty} P_m(x)$ exists for all $x \in F$, and that $\tilde{P}$ is a contractive projection. If $x \in F_0$ then $\tau(2^m v_m x) \to \delta_{(1,1)}(x) = 0$, and so

$$\tilde{P}(x) = \lim_{m \to \infty} \sum_{j=1}^{m} a_j x a_j = \sum_{j=1}^{\infty} a_j x a_j, \qquad x \in F_0 .$$

So $\tilde{P}(F_0) \subset A$. Finally, $P_m(a_j x a_j) = a_j x a_j$ for $m \geqq j$, so $\tilde{P}|_A = \mathrm{id}_A$.

For Banach spaces X, Y let $X \cong Y$ (respectively, $X \hookrightarrow Y$) denotes that X is isomorphic to Y (respectively, to a complemented subspace of Y).

LEMMA 3.4. *Let A be any matroid* C*-*algebra. Then $A \cong c_0(A)$.*

PROOF. It is enough to prove that $c_0(A)$ is isometric to a complemented subspace of A. Indeed, proving this, we get for some Banach space X that

$$\begin{aligned} A &\cong c_0(A) \oplus X \cong c_0(A) \oplus c_0(A) \oplus X \\ &\cong c_0(A) \oplus A \cong c_0(A) . \end{aligned}$$

If $A = \mathrm{LC}(l_2)$, let $\{a_j\}_{j=1}^{\infty}$ be a sequence of mutually orthogonal infinite-rank projections on l_2. Then $Px = \sum_{j=1}^{\infty} a_j x a_j$ defines a contractive projection in A and $P(A)$ is isometric to $c_0(A)$, since $a_j A a_j$ is C*-isomorphic to A.

If $A \neq \mathrm{LC}(l_2)$ and $A = \varinjlim \{M_{m(k)}, g_k\}_{k=1}^{\infty}$ is some representation of A as a direct limit of matrix algebras, then by Proposition 2.6, $\limsup_{k\to\infty} r(g_k) \geqq 2$. So, by Lemma 3.1, $F \hookrightarrow A$. It follows by Lemma 3.3 that

$$c_0(A) \hookrightarrow c_0(F) \hookrightarrow F \hookrightarrow A ,$$

where the isomorphisms are actually isometries. It follows that $c_0(A)$ is isometric to a 2-complemented subspace of A.

PROOF OF PART (a) OF THEOREM 1.1. Let A be any matroid C*-algebra, and let $A = \varinjlim \{M_{n(k)}, f_k\}_{k=1}^{\infty}$ be any representation of A as a direct limit of matrix algebras. If $\limsup_{k\to\infty} r(f_k) = 1$, then by Proposition 2.6, A is C*-isomorphic (and therefore linearly isometric) to $\mathrm{LC}(l_2)$. If $\limsup_{k\to\infty} r(f_k) \geqq 2$, then by Lemma 3.1, $A \cong F \oplus X$ and $F \cong A \oplus Y$ for some Banach spaces X and Y. Also, by Lemma 3.4, $A \cong A \oplus A$ and $F \cong F \oplus F$. Thus

$$A \cong F \oplus X \cong F \oplus F \oplus X \cong F \oplus A$$

and similarly $F \cong A \oplus F$. So $A \cong F$.

This proof shows, in fact, that the isomorphism type of a matroid C*-algebra can be decided by the asymptotic behavior of the numbers $\{r(f_k)\}_{k=1}^{\infty}$ (see Definition 2.5) in the representations $A = \varinjlim \{M_{n(k)}, f_k\}_{k=1}^{\infty}$. Precisely, we have the following:

COROLLARY 3.5. *Let A be a matroid C*-algebra,*

(i) *If* $\limsup_{k\to\infty} r(f_k) = 1$ *for some representation* $A = \varinjlim \{M_{n(k)}, f_k\}_{k=1}^{\infty}$,

then this is the case for all other representations;

(ii) $A \cong \mathrm{LC}(l_2)$ *if and only if* $\limsup_{k\to\infty} r(f_k) = 1$;

(iii) $A \cong F$ *if and only if* $\limsup_{k\to\infty} r(f_k) \geqq 2$.

PROOF OF PART (b) OF THEOREM 1.1. Let us apply Lemma 3.1 in the special case where $m(k) = 2^k$, $n(k) = k$, $g_k = \varphi_{2^k, 2^{k+1}, 2}$, and $f_k = \varphi_{k, k+1, 1}$. We have

$$A = \varinjlim \{M_k, \varphi_{k,k+1,1}\} = \mathrm{LC}(l_2)$$

and

$$B = \varinjlim \{M_{2^k}, \varphi_{2^k, 2^{k+1}, 2}\}_{k=1}^{\infty} = F .$$

So $\mathrm{LC}(l_2)$ is isometric to a 1-complemented subspace of F.

For the proof of part (c) of Theorem 1.1 we need the following Lemma. Here $\Delta=\{0,1\}^{\aleph_0}$ is the Cantor set and $C(\Delta)$ denotes the C*-algebra of all complex-valued continuous functions on Δ with the supremum norm.

LEMMA 3.6. *$C(\Delta)$ is C*-isomorphic to a 1-complemented C*-subalgebra of the Fermion algebra F.*

PROOF. Write $F=\varinjlim\{M_{2^k}, f_k\}_{k=1}^{\infty}$ with $f_k=\varphi_{2^k,2^{k+1},2}$. Let D_k denote the diagonal projection in M_{2^k} (i.e., $(D_k a)(i,j)=\delta_{i,j}a(i,i)$). Then $f_k\circ D_k=D_{k+1}\circ f_k$. So $D=\varinjlim D_k$ exists and is a contractive projection from F onto its C*-subalgebra $A=\varinjlim\{D_k M_{2^k}, f_k\}_{k=1}^{\infty}$. Let

$$\psi_k:\ C(\{0,1\}^k)\to C(\{0,1\}^{k+1})$$

be the natural map:

$$(\psi_k g)(t_1,\ldots,t_{k+1}) = g(t_1,\ldots,t_k);\qquad t_j\in\{0,1\}\ .$$

Then there exist C*-isomorphisms h_k from $C(\{0,1\}^k)$ onto $D_k M_{2^k}$, $k=1,2,\ldots$, so that $f_k\circ h_k=h_{k+1}\circ\psi_k$ for all k. It follows by Proposition 2.2 that $h=\varinjlim h_k$ is a C*-isomorphism from

$$B = \varinjlim\left\{C(\{0,1\}^k),\psi_k\right\}_{k=1}^{\infty}$$

onto A. Finally, B is C*-isomorphic to $C(\Delta)$. Indeed, if $u=(u_k)$, $u_k\in C(\{0,1\}^k)$, is so that $\psi_k(u_k)=u_{k+1}$ for $k>k_u$, let $\varphi(u)=v$ be defined on Δ by

$$v(t_1,\ldots,t_j,\ldots) = u_k(t_1,\ldots,t_k);\qquad t_j\in\{0,1\}\ ,$$

where $k\geqq k_u$. Clearly, v is well defined and φ extends to a unital C*-isomorphism of B onto $C(\Delta)$.

PROOF OF PART (c) OF THEOREM 1.1. Assume the converse, i.e., that F is isomorphic to a subquotient (that is, a subspace of a quotient space) of $\mathrm{LC}(l_2)$. By Lemma 3.6, $C(\Delta)$ is also isomorphic to a subquotient of $\mathrm{LC}(l_2)$. By standard arguments, this implies that $C(\Delta)^*$ is isomorphic to a subquotient of $\mathrm{LC}(l_2)^*$ $=C_1$ (=the trace class), which is separable. This contradicts the well-known fact that $C(\Delta)^*$ is not separable.

The construction in Lemma 3.6 can be generalized to an arbitrary matroid C*-algebra A, as follows. Let

$$A = \varinjlim\{M_{n(j)}, f_j\}_{j=1}^{\infty},\qquad f_j = \varphi_{n(j),n(j+1),r(j)},\ (r(j)=r(f_j))\ ,$$

be any representation of A as a direct limit of matrix algebras. Then

$$B = \varinjlim \{D_{n(j)}M_{n(j)}, f_j|_{D_{n(j)}M_{n(j)}}\}_{j=1}^{\infty}$$

is a commutative C*-subalgebra of A, and $D = \varinjlim D_{n(j)}$ is a contractive projection from A onto B. We call B "*the*" *diagonal of* A and denote it by DA.

We now establish the properties of DA, and in particular its independence of the particular representation $A = \varinjlim \{M_{n(j)}, f_j\}_{j=1}^{\infty}$. Let $K_j = \{1, 2, \ldots, n_{(j)}\}$ be regarded as a discrete topological space and let

$$\alpha_j : C(K_j) \to D_{n(j)}M_{n(j)}$$

be defined by

$$\alpha_j(u) = \operatorname{diag}\big(u(1), u(2), \ldots, u(n(j))\big), \qquad u \in C(K_j) .$$

Then DA is C*-isomorphic to $\varinjlim \{C(K_j), g_j\}_{j=1}^{\infty}$, where $g_j = \alpha_{j+1}^{-1} \circ f_j \circ \alpha_j$. In the UHF (i.e., unital) case there exist quotient maps $g_j^* : K_{j+1} \to K_j$ so that

$$u\big(g_j^*(i)\big) = \big(g_j(u)\big)(i), \qquad u \in C(K_j), \qquad i \in K_{j+1} .$$

The inverse limit of the sequence

$$\ldots \leftarrow K_j \xleftarrow{g_j^*} K_{j+1} \leftarrow \ldots ,$$

namely

$$K = \varprojlim \{K_j, g_j^*\}_{j=1}^{\infty}$$

$$= \left\{ x = (x(j)) \in \prod_{j=1}^{\infty} K_j; \quad x(j) = g_j^*(x(j+1)), \quad j = 1, 2, \ldots \right\}$$

is homeomorphic to $\prod_{j=1}^{\infty} K_{j+1}/K_j$, and thus to Δ. So $\varinjlim \{C(K_j), g_j\}_{j=1}^{\infty}$ is C*-isomorphic to $C(K)$ and thus to $C(\Delta)$.

In the non-unital case we let $\tilde{K} = K_j \cup \{0\} = \{0, 1, 2, \ldots, n(j)\}$, and we identify $C(K_j)$ with

$$C^{(0)}(\tilde{K}_j) = \{u \in C(\tilde{K}_j); \quad u(0) = 0\} .$$

The inverse system of quotient maps is now

$$\ldots \leftarrow \tilde{K}_j \xleftarrow{g_j^*} \tilde{K}_{j+1} \leftarrow \ldots ,$$

where $g_j^*(0) = 0$ and $u(g_j^*(i)) = (g_j(u))(i)$ for all $i \in \tilde{K}_{j+1}$ and $u \in C^{(0)}(\tilde{K}_j)$. Let

$$\vec{0} = (0, 0, \ldots) \in \prod_{j=1}^{\infty} \tilde{K}_j \quad \text{and} \quad \tilde{K} = \varprojlim \{K_j, g_j\}_{j=1}^{\infty} .$$

Then

$$\varinjlim \{C(K_j), g_j\}_{j=1}^{\infty} = \varinjlim \{C^{(0)}(\tilde{K}_j), g_j\}_{j=1}^{\infty}$$

$$= C^{(\vec{0})}(\tilde{K}) = \{u \in C(\tilde{K});\ u(\vec{0})=0\} .$$

If $A \cong \mathrm{LC}(l_2)$, i.e., $\limsup_{k\to\infty} r(f_k)=1$, then $\tilde{K}$ is homeomorphic to a sequence converging to $\vec{0}$. So DA is C*-isomorphic to c_0, the space of all numerical sequences converging to zero with the "sup" norm. If $\limsup_{k\to\infty} r(f_k) \geqq 2$ then $\tilde{K}$ does not have isolated points. Thus, being zero-dimensional, compact and metrizable (as a closed subspace of $\prod_{j=1}^{\infty} \tilde{K}_j$), $\tilde{K}$ is homeomorphic to Δ. Thus, DA is C*-isomorphic to

$$C^{(\vec{0})}(\Delta) = \{u \in C(\Delta);\ u(\vec{0})=0\} = C_0(\Delta \setminus \{\vec{0}\}) .$$

It follows that DA is (linearly) isomorphic to $C(\Delta)$.

Let us summarize this discussion formally:

THEOREM 3.7. *Let A be a matroid* C*-*algebra.*

(1) *Up to a* C*-*isomorphism, the above definition of the diagonal DA of A is independent of the particular representation*

$$A = \varinjlim \{M_{n(j)}, f_j\}_{j=1}^{\infty} ;$$

(ii) *If A is a* UHF *algebra, then DA is* C*-*isomorphic to $C(\Delta)$;*

(iii) *If A is non-unital and $A \ncong \mathrm{LC}(l_2)$, that is $\limsup_{j\to\infty} r(f_j) \geqq 2$, then DA is* C*-*isomorphic to $C_0(A \setminus \{\vec{0}\})$, and thus linearly isomorphic to $C(\Delta)$;*

(iv) *If $\limsup_{j\to\infty} r(f_j)=1$, that is $A \cong \mathrm{LC}(l_2)$, then DA is* C*-*isomorphic to c_0;*

(v) *In all cases, DA is a 1-complemented* C*-*subalgebra of A.*

We conclude the paper by suggesting the following problem:

PROBLEM. Characterize, up to a linear-topological isomorphism, all AF-algebras (i.e., direct limits of finite dimensional C*-algebras).

Our methods and results might be helpful in studying this general problem, since every finite-dimensional C*-algebra is the finite direct sum of matrix algebras.

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