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REMARKS ON NAMIOKA SPACES AND R. E. JOHNSON'S THEOREM ON THE NORM SEPARABILITY OF THE RANGE OF CERTAIN MAPPINGS

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Abstract.

We give miscellaneous remarks on the class of Hausdorff spaces (and metrizable spaces) for which Namioka's theorem on separately continuous functions holds. We also show that this can be used to give a simplified and vastly generalized proof of a remarkable theorem of R. E. Johnson on the norm separability of the range of certain weakly continuous mappings. The measure in Johnsons theorem is replaced by a countable chain condition on the open sets and the class of domain spaces, for which the theorem holds, is vastly extended.

KEY WORDS AND PHRASES. Norm separability off range. Countable chain condition.

We shall call a Hausdorff topological space X a Namioka space, if for any separately continuous function $f\colon X \times Y \to Z$, where Y is a compact Hausdorff space and Z is a metrizable space, we have that there exists a dense G_δ subset $A \subseteq X$, such that f is jointly continuous at each point of $A \times Y$.

It is indeed possible to replace an arbitrary metric space Z with the interval $[-1,1]$ (usual metric) using a trick discussed in [2].

It is not known what metric spaces are Namioka spaces. Below we shall make some preliminary observations, which may eventually lead to a solution of this problem.

First let us recall the definitions of σ-well and τ-well α-favorable spaces in [2]. Let us consider the following topological "game" on the Hausdorff space X between the players α and β.

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The player β starts the game by choosing an open non empty subset U_1 of X. Then the player α chooses an open subset V_1 of U_1 and a point $x_1 \in V_1$ (his "move" is the pair (V_1, x_1) with $x_1 \in V_1$ and $V_1 \subseteqq U_1$). Then β chooses an open non empty subset U_2 of V_1 (he may choose as he wishes but is expected to try to escape x_1). Next α chooses an open subset V_2 of U_2 and a point $x_2 \in V_2 \ldots$

We shall fix the rule that α wins in the σ-game if any subsequence x_{n_p} of the sequence x_n accumulates to at least one point of $I = \bigcap_{n=1}^{\infty} U_n = \bigcap_{n=1}^{\infty} V_n$. In the τ-game we demand (for α to win) that any subnet x_{n_d} $(d \in D)$ accumulates to a point in I.

We shall consider only strategies for α depending on the previous moves by β. We shall call X a σ-well or τ-well α-favorable space if there exists a winning strategy for α in the σ-game or the τ-game respectively.

Any σ-well α-favorable space is a Namioka space. The more restricted class of τ-well α-favorable spaces has the property of being closed under arbitrary topological products (see [2]).

It is a remarkable observation, that any metrizable space X which is α-favorable in the original Choquet sense (see [1], it seems to be known that for metrizable spaces this is equivalent with X containing a dense G_δ subset of its completion) is automatically τ-well α-favorable! Hence an arbitrary product of such spaces is a Namioka space!

Let us sketch the proof of this statement. Any strategy σ for α in the original Choquet game which is contained in a winning strategy is itself winning. If therefore ψ is a winning strategy for α in the Choquet game, we can choose a winning strategy $\sigma \subseteqq \psi$ such that

$$\operatorname{diam}(\sigma(U)) \leqq \tfrac{1}{2}\operatorname{diam}(U) \wedge 1$$

for all non empty open sets. U. If we choose an arbitrary point in $\sigma(U)$ we get a winning strategy for α in the τ-game.

The fact that α-favorable implies τ-well α-favorable is not true in general as the real line with the Sorgenfrey topology shows.

The attention of the author was drawn to the above mentioned facts abour metrizable spaces and the Sorgenfrey topology by F. Topsøe and J. Hoffmann-Jørgensen (oral discussion).

Although the real line with the Sorgenfrey topology is α-favorable but not σ-well α-favorable, it is still a Namioka space. To see this one need only the fact that a non empty Sorgenfrey open subset of the real line has non empty interior in the usual topology, and a slight modification of the proof of Theorem 1 in [2] shows that (R. Sorgenfrey) is a Namioka space.

It is not known whether there exists a non Baire metrizable Namioka space. The theorem below might support the conjecture that any metrizable Namioka space is Baire.

THEOREM 1. *The rational numbers* Q *with the usual topology is* not *a Namioka space.*

PROOF. The proof uses a slight modification of J. Hoffmann-Jørgensen's example discussed in [2]. Since Q is homeomorphic with the space of rationals in $[-1,1]$, it is no restriction to let Q in the proof denote this space. Let Z be the set of mappings from $\mathbf{Q}^2$ into $[-1,1]$ equipped with the pointwise topology. Of course Z is a compact metrizable space. The mapping F goes from $\mathbf{Q}\times[-1,1]$ into Z and is defined by putting the value of $F(q,r)$ at the point $(x,y)\in\mathbf{Q}^2$ equal to

$$F(q,r)(x,y) = 2(x-q)(y-r)/((x-q)^2+(y-r^2))$$

whenever this quotient is defined, and equal to zero, if the quotient is undefined. Now it is easy to show that F is separately continuous on $\mathbf{Q}\times[-1,1]$ and the joint continuity points $(q,r)\in\mathbf{Q}\times[-1,1]$ are exactly those points where r is irrational! Hence Q is not a Namioka space. This finishes the proof of Theorem 1.

We shall now use the class of Namioka spaces to extend substantially the class of spaces for which Johnson's theorem (see [3]) holds. Many results in automatic continuity theory follows easily from Namioka's theorem and this result.

A Hausdorff space X has countable chain condition on the open sets if any family O_i $(i\in I)$ of pairwise disjoint non empty open sets is at most countable.

THEOREM 2. *Let X be a Namioka space with countable chain condition on the open sets, and let $f\colon X\times Y\to Z$ be a separately continuous mapping, where Y is compact and Z is metrizable. Then the set of continuous mappings from Y into Z given by $\{f(x,.)\mid x\in X\}$ is separable in the metric*

$$D(\varphi,\psi) = \sup\{d(\varphi(y),\psi(y)) \mid y\in Y\}$$

where d is any metric on Z.

PROOF. Let us first reduce to the case, where Z equals $[-1,1]$ with its usual metric. To do this we consider

$$K = \{k\colon Z\to[-1,1] \mid \forall_{z_1,z_2\in Z}\colon |k(z_1)-k(z_2)| \leqq d(z_1,z_2)\}\ .$$

Obviously K is a compact Hausdorff space with the topology of pointwise convergence. We put $\tilde{Y}=Y\times K$ and $f\colon X\times\tilde{Y}\to[-1,1]$ is defined by $f((x,(y,k))=k(f(x,y))$. Now we easily see that

$$\sup\{|\tilde{f}(x_1,\tilde{y})-\tilde{f}(x_2,\tilde{y})| \mid \tilde{y}\in\tilde{Y}\} \leqq \alpha < 1$$

imply

$$\sup\{d(f(x_1,y),f(x_2,y)) \mid y\in Y\} \leqq \alpha .$$

Hence if we can prove the theorem for $Z=[-1,1]$ and arbitrary compact Y, we have really done. Accordingly we now assume $Z=[-1,1]$.

It is easy to show that any point of the section $\{x\}\times Y$ is a joint continuity point for $f\colon X\times Y\to[-1,1]$ if and only if x is a continuity point for the associated mapping

$$F\colon X \to C(Y)$$

defined by $F(x)(y)=f(x,y)$ (with respect to norm topology on $C(Y)$, see [2]). Let us choose a *maximal* family O_i^n ($i\in I_n$) of non empty, pairwise disjoint open sets in X with diam $(F(O_i^n))\leqq 1/n$. Clearly I_n is non empty and countable. From the assumption that X is a Namioka space, we conclude that $G_n=\bigcup_{i\in I_n}O_i^n$ is *dense* in X. Let $f_n\in C(Y)$ be a sequence, such that each of the sets $F(O_i^n)$ contains at least one f_n. On Y we define the equivalence relation

$$y_1 \sim y_2 \Leftrightarrow \forall_n : f_n(y_1) = f_n(y_2) .$$

The set of equivalence classes $\tilde{Y}$ is a compact metric space in an obvious way and the continuous functions on $\tilde{Y}$ will be identified with those continuous functions on Y which respects $\sim$ (see [2] for a similar argument). Let $\mathscr{F}_\sim(Y)$ be the space of *all* real functions on Y respecting $\sim$, and let us consider $\mathscr{F}_\sim(Y)$ with the topology of pointwise convergence. Of course $\mathscr{F}_\sim(Y)$ is a closed subspace of $\mathscr{F}(Y)$ all real functions on Y with pointwise convergence). Let

$$C_n = \{f\in\mathscr{F}(Y) \mid \|f\|_\infty \leqq 1/n\} ,$$

then C_n is compact in $\mathscr{F}(Y)$ and therefore

$$\mathscr{F}_n(Y) = \mathscr{F}_\sim(Y)+C_n$$

is closed in $\mathscr{F}(Y)$. We conclude, that the set $F^{-1}(\mathscr{F}_n(Y))$ is closed in X. Since this set contains G_n, we have that it equals X. Consequently for

$$\mathscr{F}_\infty(Y) = \bigcap_{n=1}^{\infty} \mathscr{F}_n(Y)$$

we have $F^{-1}(\mathscr{F}_\infty(Y))=X$. It is fairly obvious that $\mathscr{F}_\infty(Y)\subseteqq\mathscr{F}_\sim(Y)$, and we conclude that for each $x\in X$, the function $F(x)$ belongs to the Banach space $C(\tilde{Y})\subseteqq C(Y)$. But $C(\tilde{Y})$ is separable, since $\tilde{Y}$ is metrizable (and compact) and this conclude the proof of Theorem 2.

Theorem 2 was proved originally for compact spaces X supporting a positive Radon measure (see [3]). The method of proof was very different from the present paper. Encouraged by discussions with F. Topsøe and J. Hoffmann-Jørgensen we realized an unexpected connection between Namioka's theorem (see [4]) and R. E. Johnson's theorem. At the same time the proofs are simplified and the theorems are substantially generalized.

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ADDED IN PROOF SEPTEMBER 1982. The attention of the author has been drawn to a paper by Jean Saint Raymond (Jeux topologiques et espaces de Namioka, to appear in Proc. Amer. Math. Soc.) where it is shown that the metrizable Namioka spaces are precisely the (metrizable) Baire spaces! He uses the methods of the present author (slightly improved) and some new tricks.

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