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MULTIPLIER REPRESENTATIONS AND AN APPLICATION TO THE PROBLEM WHETHER $A \otimes_\varepsilon X$ DETERMINES A AND/OR X

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Abstract.

In this paper we define and investigate a new class of Banach space invariants. These invariants are constructed by means of the multiplier algebra Mult (X) of a Banach space X. Mult (X) consists of all linear and continuous operators T: $X \rightarrow X$ such that every extreme functional is an eigenvector of the transposed operator T'.

With each k in the Choquėt boundary of Mult (X) we associate a Banach space X_k. If Mult (X) does not behave too pathologically X can be regarded as a space of sections in the product space of the X_k such that the $T \in$ Mult (X) correspond to multiplication operators associated with certain continuous functions. Such a representation will be called a *multiplier representation.*

We investigate these multiplier representations in the slightly more general setting of Banach spaces of vector-valued functions which are *A-modules* for an arbitrary function algebra A. Our results generalize theorems for function modules which correspond to the special case when A is a CK-space.

In the last chapter the theory is applied to the study of $A \otimes_\varepsilon X$, where A is a function algebra and X is a Banach space such that Mult (X) is finite-dimensional. We obtain necessary and sufficient conditions concerning X such that A and/or X can be reconstructed from $A \otimes_\varepsilon X$. In addition it is possible to describe the structure of all isometries of $A \otimes_\varepsilon X$.

1. Introduction.

Multiplication operators on algebras have been generalized in a number of ways. We will consider a definition which concerns operators in arbitrary Banach spaces.

1.1. Definition. ([1], [4]). Let X be a Banach space, E_X the set of extreme functionals, i.e. the set of extreme points of the unit ball of X'. A linear

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continuous operator $T\colon X \to X$ is called a *multiplier*, if there is a function a_T from E_X into the scalar field $\boldsymbol{K}$ ($=\mathbb{R}$ or $\mathbb{C}$) such that $p \circ T = a_T(p)p$ for every $p \in E_X$.

The collection Mult (X) of all multipliers on X is obviously a commutative Banach algebra in $[X]$ which contains the identity operator. A certain subalgebra of Mult (X), the *centralizer* $Z(X)$ of X, plays an important role in M-structure theory; we refer the reader to [2] for a survey of this theory.

Whereas $Z(X)$ is a CK-space, Mult (X) can only be considered as a *function algebra*, i.e. a closed subalgebra of a CK-space which separates the points and contains the constant functions; this will be proved in proposition 2.1 below.

For each k in the Choquet boundary ch (Mult (X)) of Mult (X) we define a Banach space X_k. There is a natural map ω from X into the Banach space product $\prod^\infty \{X_k \mid k \in \text{ch}\,(\text{Mult}\,(X))\}$ which is an isometrical isomorphism if, in a sense, ch (Mult (X)) is not too small. If this is the case, X can be identified with $\omega(X)$, and the $T \in$ Mult (X) can be regarded as multiplication operators associated with continuous functions. We will then say that X admits a *multiplier representation.*

Several properties of $\omega(X)$ (and therefore of X if X has a multiplier representation) are related in a simple way with the properties of the X_k. Since these results only depend on the fact that $\omega(X)$ as a subspace of the product of the X_k satisfies some semicontinuity and module conditions we discusss a slightly more general situation: we consider arbitrary function algebras A, families of Banach spaces $(X_k)_{k \in \text{ch}\, A}$, and closed subspaces X of $\prod^\infty \{X_k \mid k \in \text{ch}\, A\}$ which are A-modules in a natural way and for which the mappings $k \mapsto \|x(k)\|$ are upper semi-continuous for every $x \in X$. Such spaces X, which will be called *A-modules*, share some properties with function modules. For example, we obtain a theorem by which the extreme functionals of X can be described in a simple way by means of the extreme functionals of the X_k, a result which strictly generalizes the theorem of Cunningham, Effros, and Roy [6].

Of particular interest will be the multiplier representation of $A \otimes_\varepsilon X$, where A is a function algebra. This multiplier representation can easily be determined if the multiplier of X is finite-dimensional. Using this we will be able to derive necessary and sufficient conditions concerning X such that A and/or X can be reconstructed from $A \otimes_\varepsilon X$. For example, we will show that $A \otimes_\varepsilon X$ and $B \otimes_\varepsilon Y$ are isometrically isomorphic iff $A \cong B$ and $X \cong Y$ (A, B function algebras, X, Y Banach spaces with one-dimensional multiplier algebra). Also we obtain a result of Cambern by which it is possible to describe all isometries of $A \otimes_\varepsilon X$, when A is a function algebra and X is a reflexive Banach space which contains no nontrivial M-summands (I am grateful to Professor Cambern for making available to me a preprint containing this theorem; this paper has been the starting point of the present investigations).

Finally we note that our theory gives new results only in the case of complex scalars. In the real case the multiplier algebra coincides with the centralizer, and the corresponding theorems concerning the centralizer have already been published (see chapter 4 in [2]).

2. Mult (X) and the associated family $\{X_k \mid k \in \operatorname{ch}(\operatorname{Mult}(X))\}$.

X will always denote a nonzero Banach space. At first we collect together some simple properties of Mult (X).

2.1. PROPOSITION.

(i) *If* $T: X \to X$ *is an operator such that there is a subset* $E \subset E_X$ *which is weak*-dense in* E_X *and a function* $a_T: E \to \mathbf{K}$ *such that* $p \circ T = a_T(p)p$ *for* $p \in E$, *then* T *is a multiplier. Further,* a_T *can be extended as a weak*-continuous function to all of* $\bar{E_X} \setminus \{0\}$ ($\bar{E_X}$ *= the weak*-closure of* E_X).

(ii) *There are a compact Hausdorff space* K_0 *and an isometric algebra homomorphism* ϱ_0: Mult $(X) \to CK_0$ *such that* $\varrho_0(\operatorname{Mult}(X))$ *is a function algebra over* K_0. *Also there is a map* $v: E_X \to K_0$ *such that* $a_T(p) = (\varrho_0(T))(v(p))$ *for* $p \in E_X$, $T \in$ Mult (X).

(iii) *Let* $\varepsilon > 0$ *and* $S, T \in$ Mult (X) *such that*

$$|\varrho_0(T)(k)| \leqq |\varrho_0(S)(k)| + \varepsilon \quad \text{for every } k \in K_0 .$$

Then $\|Tx\| \leqq \|Sx\| + \varepsilon\|x\|$ *for every* $x \in X$.

PROOF. (i) We first show that every $p_0 \in \bar{E_X} \setminus \{0\}$ is an eigenvector of T'. We choose a net $(p_i)_{i \in I}$ in E such that $w^* - \lim p_i = p_0$ and an element $x_0 \in X$ such that $p_0(x_0) = 1$. $p_i \to p_0$ implies $p_i(x_0) \to 1$ and

$$p_i(Tx_0) = a_T(p_i)p_i(x_0) \to p_0(Tx_0)$$

so that $(a_T(p_i))_i$ is convergent; let

$$a_T(p_0) := \lim a_T(p_i) .$$

We then have

$$p_0(Tx) = \lim p_i(Tx) = \lim a_T(p_i)p_i(x) = a_T(p_0)p_0(x)$$

for every $x \in X$ so that $p_0 \circ T = a_T(p_0)p_0$.

In this way a_T is extended to all of $\bar{E_X} \setminus \{0\}$, and this extension is obviously unique. It will also be denoted by a_T in the sequel. a_T is weak*-continuous since the mappings $p \mapsto a_T(p)p(x)$ are continuous (they coincide with $p \mapsto p(Tx)$), and for every p_0 there is a vector x_0 such that $p(x_0) \neq 0$ for p in an neighbourhood of p_0.

(ii) Let L be the locally compact space $E_X^{-} \setminus \{0\}$. By $T \mapsto a_T$ we may embed Mult (X) as a subspace of $C^b L$, the space of bounded continuous functions on L. Note that this mapping is an isometric algebra homomorphism. We identify $C^b L$ with $C(\beta L)$, where βL denotes the Stone–Čech compactification of L. Finally, setting

$$\text{“}k \sim l \text{ iff } a_T(k) = a_T(l) \text{ for every } T \in \text{Mult}(X)\text{” (for } k, l \in \beta L)$$

we define $K_0 := \beta L / \sim$.

With

$\nu :=$ the canonical map $p \mapsto [p]$ from E_X to K_0,

$\varrho_0 :=$ the composition of $T \mapsto a_T$ and the canonical map which assigns to a_T the induced map from K_0 to $\boldsymbol{K}$, ν and ϱ_0 have the properties claimed.

(iii) By (ii) we have

$$|a_T(p)| \leqq |a_S(p)| + \varepsilon \qquad \text{for every } p \in E_X .$$

Now, for $x \in X$, choose $p \in E_X$ such that $p(Tx) = \|Tx\|$. We then have

$$\begin{aligned} \|Tx\| &= |p(Tx)| = |a_T(p)p(x)| \leqq |a_S(p)p(x)| + \varepsilon |p(x)| \\ &= |p(Sx)| + \varepsilon |p(x)| \leqq \|Sx\| + \varepsilon \|x\| . \end{aligned}$$

Now let ϱ: Mult $(X) \to CK$ be any representation of Mult (X) as a function algebra, i.e. ϱ is an isometric algebra homomorphism and $\varrho(\text{Mult}(X))$ is a function algebra over K.

2.2. LEMMA. *Suppose that k is a point in the Choquet boundary of* $\varrho(\text{Mult}(X))$. *Further, let S, T be multipliers on X such that $|\varrho(S)(k)|$, $|\varrho(T)(k)| > 1$. Then, for every $\varepsilon > 0$, there is an R in* Mult (X) *such that*

$$|\varrho(R)(k)| > 1$$

and

$$\|Sx\| \leqq \|Rx\| + \varepsilon \|x\|, \qquad \|Tx\| \leqq \|Rx\| + \varepsilon \|x\| \qquad \textit{for every } x \in X .$$

PROOF. Consider the mapping $\varrho \circ \varrho_0^{-1}$: range $\varrho_0 \to$ range ϱ (ϱ_0 as in 2.1(ii)). $\delta_k: f \mapsto f(k)$ is a multiplicative functional on range ϱ, which is extreme in the dual unit ball. Since $\varrho \circ \varrho_0^{-1}$ is an isometric algebra isomorphism, $\delta_k \circ \varrho \circ \varrho_0^{-1}$ is also extreme. Thus, there is a $k_0 \in K_0$ in the Choquet boundary of range ϱ_0 such that $\delta_{k_0} = \delta_k \circ \varrho \circ \varrho_0^{-1}$. We have

$$(*) \qquad \varrho_0(T)(k_0) = \varrho(T)(k) \quad \text{for every } T \in \text{Mult}(X)$$

so that in particular

$$1+\eta := \min\{|\varrho_0(T)(k_0)|, |\varrho_0(S)(k_0)|\} > 1 .$$

Choose a neighbourhood U of k_0 such that

$$\min\{|\varrho_0(T)(\tilde{k}_0), |\varrho_0(S)(\tilde{k}_0)|\} > 1+\eta/2 \quad \text{for } \tilde{k}_0 \in U .$$

Since k_0 lies in the Choquet boundary of range ϱ_0, there is a multiplier R_0 on X such that

$$\varrho_0(R_0)(k_0) = 1 = \|\varrho_0(R_0)\| ,$$

$$|\varrho_0(R_0)(\tilde{k}_0)| \leqq \varepsilon(1+\eta/2)^{-1} \quad \text{for } \tilde{k}_0 \notin U;$$

cf. Theorem 2.3 below.

With $R := (1+\eta/2)R_0$, we then have, for every $\tilde{k}_0 \in K_0$,

$$|\varrho_0(R_0)(\tilde{k}_0)| \leqq \min\{|\varrho_0(T)(\tilde{k}_0)|, |\varrho_0(S)(\tilde{k}_0)|\} + \varepsilon ,$$

and the assertion follows immediately from 2.1(iii) and (*).

In the preceding proof we have used a theorem from the theory of function algebras which will be of importance frequently in this paper. We cite this theorem for the sake of easy reference:

2.3. THEOREM. *Let A be a function algebra over a compact Hausdorff space K. Then, for every k_0 in the Choquet boundary of A, every neighbourhood U of k_0 and every $\varepsilon > 0$, there is an f in A such that*

$$f(k_0) = \|f\| = 1, \ |f(k)| \leqq \varepsilon \quad \textit{for every } k \notin U .$$

PROOF. [7, Theorem 22].

The following definition is fundamental for our investigations:

2.4. DEFINITION. For $x \in X$ and k in the Choquet boundary of $\varrho(\text{Mult}(X))$ let

$$|x|_\varrho(k) := \inf\{\|Tx\| \mid T \in \text{Mult}(X), |\varrho(T)(k)| > 1\} .$$

2.5. PROPOSITION.

(i) $x \mapsto |x|_\varrho(k)$ *is a semi-norm for every* $k \in \text{ch}(\text{range}\,\varrho)$.
(ii) $k \mapsto |x|_\varrho(k)$ *is upper semicontinuous for every* $x \in X$.
(iii) $|Tx|_\varrho(k) = |\varrho(T)(k)|\,|x|_\varrho(k)$ *for* $x \in X$, $T \in \text{Mult}(X)$, *and* $k \in \text{ch}(\text{range}\,\varrho)$.

PROOF. (i) For $S, T \in \operatorname{Mult}(X)$ with $|\varrho(S)(k)|, |\varrho(T)(k)| > 1$ and $\varepsilon > 0$ we choose R as in 2.2. It follows that, for $x \in X$,

$$\begin{aligned} |x+y|_\varrho(k) &\leqq \|R(x+y)\| \\ &\leqq \|Rx\| + \|Ry\| \\ &\leqq \|Sx\| + \|Ty\| + \varepsilon(\|x\| + \|y\|) . \end{aligned}$$

Hence

$$|x+y|_\varrho(k) \leqq |x|_\varrho(k) + |y|_\varrho(k) + \varepsilon(\|x\| + \|y\|)$$

and thus, since $\varepsilon > 0$ was arbitrary,

$$|x+y|_\varrho(k) \leqq |x|_\varrho(k) + |y|_\varrho(k) .$$

The proof of the remaining assertions is routine.

(ii) This follows immediately from the definition and the fact that the mappings $k \mapsto \varrho(T)(k)$ are continuous.

(iii) First, let T_0 be a multiplier such that $\varrho(T_0)(k) = 0$ and $\varepsilon > 0$. Using Theorem 2.3 we obtain an $S \in \operatorname{Mult}(X)$ such that $|\varrho(S)(k)| > 1$ and $\|\varrho(T_0)\varrho(S)\| \leqq \varepsilon$. It follows that

$$|T_0 x|_\varrho(k) \leqq \|ST_0 x\| \leqq \varepsilon\|x\| \qquad \text{for every } \varepsilon > 0$$

so that $|T_0 x|_\varrho(k) = 0$.

For arbitrary $T \in \operatorname{Mult}(X)$ define $T_0 := T - [\varrho(T)(k)]\,\mathrm{Id}$. By the first part of the proof we have $|Tx - \varrho(T)(k)x|_\varrho(k) = 0$ so that, since $|\cdot|_\varrho(k)$ is a semi-norm,

$$|Tx|_\varrho(k) = |\varrho(T)(k)||x|_\varrho(k) .$$

2.6. DEFINITION. Let K and ϱ be as above and $k \in \operatorname{ch}(\operatorname{range} \varrho)$. We define $X_{k,\varrho}$ to be the Banach space associated with the semi-normed space $(X, |\cdot|_\varrho(k))$, i.e. the completion of

$$X / \{x \mid |x|_\varrho(k) = 0\} ,$$

the quotient provided with the norm $\|[x]\| := |x|_\varrho(k)$.

2.7. UNIQUENESS THEOREM. *Let* $\varrho_1 \colon \operatorname{Mult}(X) \to CK_1$ *and* $\varrho_2 \colon \operatorname{Mult}(X) \to CK_2$ *be isometric algebra isomorphisms from* Mult (X) *onto functions algebras over* K_1 *and* K_2, *respectively.*

Then there are a homeomorphism $t \colon \operatorname{ch}(\operatorname{range} \varrho_1) \to \operatorname{ch}(\operatorname{range} \varrho_2)$ *and a family of isometric isomorphisms*

$$u_k \colon X_{k,\varrho_1} \to X_{t(k),\varrho_2} .$$

Thus the family $(X_{k,\varrho})_{k\in \mathrm{ch}(\mathrm{range}\,\varrho)}$ *is essentially the same for every representation* ϱ: Mult $(X) \to CK$.

PROOF. $\varrho_2 \circ \varrho_1^{-1}$ is an algebra isomorphism from range ϱ_1 onto range ϱ_2. Thus there is a homeomorphism t: ch(range ϱ_1) $\to$ ch(range ϱ_2) such that

$$[\varrho_2 \circ \varrho_1^{-1}(f)](t(k)) = f(k)$$

for every $f \in$ range ϱ_1 and every $k \in$ ch(range ϱ_1). It follows that, for $T \in$ Mult (X),

$$|(\varrho_1(T))(k)| > 1 \quad \text{iff} \quad |(\varrho_2(T))(t(k))| > 1 .$$

Hence

$$|x|_{\varrho_1}(k) = |x|_{\varrho_2}(t(k))$$

for every x so that $X_{k,\varrho_1} \cong X_{t(k),\varrho_2}$.

As an illustration we consider a simple

EXAMPLE. Let $X = A$ be any function algebra over a compact Hausdorff space K. Since A is isometrically isomorphic with the restriction of A to its Shilov boundary we may assume that ch A is dense in K. It is then easy to see that

$$\mathrm{Mult}\,(A) = \{M_h \mid h \in A\} \cong A ,$$

where M_h denotes the multiplication operator $f \mapsto hf$:

Every M_h is in Mult (A), since E_A is just the set

$$\{\lambda\delta_k \mid k \in \mathrm{ch}\,A, |\lambda| = 1\} .$$

Conversely, for T in Mult (A) let $h := T\mathbf{1}$. Then, for $f \in A$, Tf and hf coincide on ch A so that by continuity $T = M_h$.

Now let ϱ: $A \to CK$ be the identical representation. Theorem 2.3 gives $|f|_\varrho(k) = |f(k)|$ for $f \in A$ and $k \in$ ch A; it follows that the $X_{k,\varrho}$ are just the scalar field.

Further examples will be discussed later.

Let ϱ: Mult $(X) \to CK$ be a fixed representation of Mult (X) as above. For simplicity we will write X_k instead of $X_{k,\varrho}$ for $k \in$ ch(range ϱ).

We will try to identify X with a space of sections in the Banach space product $\prod^\infty \{X_k \mid k \in \mathrm{ch}(\mathrm{range}\,\varrho)\}$ (the subspace of the product which consists of those tupels for which the supremum of the norms of the elements is finite).

The construction is modeled after the similar construction in the case of function modules.

2.8. THEOREM. $\omega: X \to \prod^{\infty} X_k$, *defined by* $[\omega(x)](k) :=$ *the residue class of* x *in* X_k, *has the following properties*:

(i) a) ω *is a linear continuous map with* $\|\omega\| \leqq 1$.
b) $k \mapsto \|[\omega(x)](k)\|$ *is upper semicontinuous for every* $x \in X$.
c) $\omega(X)$ *is a* (range ϱ)*-module. More precisely*: *for* $x \in X$ *and* $T \in \mathrm{Mult}\,(X)$, *we have* $\omega(Tx) = \varrho(T)\,\omega(x)$ (*pointwise multiplication*).
d) $\{[\omega(x)](k) \mid x \in X\}$ *is dense in* X_k *for every* $k \in \mathrm{ch}(\mathrm{range}\,\varrho)$.

(ii) ω *is unique in the following sense*: *If* ϱ_i: $\mathrm{Mult}\,(X) \to CK_i$ $(i = 1, 2)$ *are representations of* Mult (X), *and the corresponding* ω's *and* X_k's *are denoted by* ω_1, ω_2 *and* X_k^1, X_k^2, *then there are a homeomorphism* t: $\mathrm{ch}(\mathrm{range}\,\varrho_1) \to \mathrm{ch}(\mathrm{range}\,\varrho_2)$ *and a family of isometric isomorphisms* u_k: $X_k^1 \to X_k^2$ *such that*

$$u_k([\omega_1(x)](k)) = [\omega_2(x)](t(k))$$

for $x \in X$ *and* $k \in \mathrm{ch}(\mathrm{range}\,\varrho_1)$.

PROOF. (i) a), b), and d) follow immediately from properties of $|\cdot|_\varrho(k)$, which have been proved in Proposition 2.5. For the proof of c), let $T \in \mathrm{Mult}\,(X)$, $x \in X$, and $k \in \mathrm{ch}(\mathrm{range}\,\varrho)$ be given. $\varrho(T) - \varrho(T)(k)\mathbf{1}$ vanishes at k, so that

$$|Tx - \varrho(T)(k)x|_\varrho(k) = 0$$

by 2.5(iii). By the definition of ω this means

$$[\omega(Tx - \varrho(T)(k)x)](k) = 0\,,$$

that is $(\omega(Tx))(k) = (\varrho(T)(k))(\omega(x))(k)$.

(ii) This is just a restatement of theorem 2.7.

Unfortunately $\omega(X)$ contains less information than X in general. A counterexample will be given in section 4. However, the complexity of the known counterexamples indicates that they represent an exceptional situation.

2.9. DEFINITION. X is said to have a *multiplier representation*, if ω is an isometrical isomorphism. Since $\|\omega\| \leqq 1$ this means that $\|\omega(x)\| \geqq \|x\|$ for every x or, equivalently, that

$$\sup_p |x|_\varrho(k) = \|x\|\,.$$

The notation "multiplier representation" is justified by the fact that on this case we may identify X with $\omega(X)$, so that the $T \in \text{Mult}(X)$ are precisely the multiplication operators M_h, h in the function algebra $\text{Mult}(X)$.

We have already noted that our construction coincides with the construction of the function module representation if $Z(X)$ is all of $\text{Mult}(X)$. Since X is always a subspace of the function module representation (cf. Chapter 4 in [2]), this means that X has a multiplier representation in this case. The $T \in Z(X)$ are characterized by the fact that there is an $S \in \text{Mult}(X)$ such that $a_S = \overline{a_T}$ (complex conjugate). Hence, trivially, we have $Z(X) = \text{Mult}(X)$ if the scalars are real. However, there are a number of complex spaces for which this is also true:

2.10. Proposition. *Both of the following conditions imply that* $Z(X) = \text{Mult}(X)$, *so that* X *has a multiplier representation*:

(i) X *can be embedded as a selfadjoint subspace of a* CK*-space*

(ii) $\text{Mult}(X)$ *is finite-dimensional.*

Proof. (i) Suppose that $X \subset CK$ and that $\bar{x} \in X$ for every $x \in X$. For $T \in \text{Mult}(X)$ we define $S: X \to X$ by $Sx := \overline{(T\bar{x})}$. It is easy to see that $S \in \text{Mult}(X)$ and that $a_S = \overline{a_T}$ (note that every $p \in E_X$ has the form $\lambda\delta_k$ for suitable $k \in K$, $|\lambda| = 1$).

(ii) We have shown in 2.1 (ii) how $\text{Mult}(X)$ can be regarded in a natural way as a function algebra over a quotient of $\beta(E_{\bar{X}} \setminus \{0\})$. Since every finite-dimensional function algebra is selfadjoint (see e.g. Corollary 1 in section 3.4 of [7]) it follows that for every $T \in \text{Mult}(X)$, there is an S such that $a_S = \overline{a_T}$.

Further examples of Banach spaces which have a multiplier representation, will be given in the next sections.

3. *A*-modules.

If X has a multiplier representation, X can be identified with $\omega(X)$. The properties of $\omega(X)$ as a subspace of $\prod^\infty X_k$ (Theorem 2.8) lead to the following definition:

3.1. Definition. Let A be a function algebra over a compact Hausdorff space, $(X_k)_{k \in \text{ch}\, A}$ a family of Banach spaces, and X a closed subspace of $\prod^\infty \{X_k \mid k \in \text{ch}\, A\}$; we will regard X as a space of vector-valued functions on $\text{ch}\, A$.

X is called an *A-module*, if

(i) the (pointwise defined) product fx is in X for $f \in A$ and $x \in X$,

(ii) $k \mapsto \|x(k)\|$ is upper semicontinuous for every $x \in X$,

(iii) $\{x(k) \mid x \in X\}^- = X_k$ for every $k \in \text{ch}\, A$,

(iv) $\{k \mid k \in \text{ch}\, A,\ X_k \neq \{0\}\}$ is dense in $\text{ch}\, A$.

EXAMPLES/REMARKS.

1. If A is a self-adjoint function algebra, i.e. a CK-space, then the definition coincides with the definition of function modules. Function modules have been investigated in detail in Chapter 4 of [2].

2. Let X_0 be an arbitrary Banach space, and A a function algebra over K as above. For $k \in \text{ch}\, A$, define $X_k := X_0$. Then $A \otimes X_0$, the injective tensor product of A and X_0, can be embedded in a natural way as a subspace of $\prod^\infty X_k$; we define

$$(f \otimes x_0)(k) := f(k)x_0$$

and extend this definition in the obvious way to all of $A \otimes_\varepsilon X_0$. It is easy to see that $A \otimes_\varepsilon X_0$ satisfies 3.1 (i)–(iv).

3. If X is any Banach spaces which has a multiplier representation, then $X \cong \omega(X)$ is a Mult (X)-module by Theorem 2.8.

4. The most important properties are (i) and (ii). (iii) guarantees that the fiber spaces X_k are not greater than necessary, and by (iv) the $f \in A$ can be identified with the multiplication operators $x \mapsto fx$ on X.

3.2. LEMMA. *Let X be an A-module.*

(i) *For $k_0 \in \text{ch}\, A$ and $x_0 \in X_{k_0}$, there is an $x \in X$ such that $x_0 = x(k_0)$. Thus* 3.1 (iii) *can be replaced by*

$$\text{“}\{x(k) \mid x \in X\} = X_k \text{ for every } k \in \text{ch}\, A\text{.”}$$

(ii) *For $k_0 \in \text{ch}\, A$, $x \in X$, U a neighbourhood of k_0, and $\varepsilon > 0$, there is an $f \in A$ such that $\|f\| = f(k_0) = 1$,*

$$\|fx\| \leqq \|x(k_0)\| + \varepsilon,\ \|fx|_{(\text{ch}\, A)\setminus U}\| \leqq \varepsilon\ .$$

PROOF. (i) By 3.1 (iii), we may write $x_0 = \sum x_n(k_0)$, where $x_n \in X$ and $\sum \|x_n(k_0)\| < \infty$. For $n \in \mathbf{N}$, choose neighbourhoods U_n of k_0 such that

$$\|x_n(k)\| \leqq \|x_n(k_0)\| + 2^{-n}$$

for $k \in U_n \cap (\text{ch}\, A)$ and $f_n \in A$ such that

$$f_n(k_0) = \|f_0\| = 1, \quad \|f_n|_{K \setminus U_n}\| \leqq \|x_n(k_0)\|(1+\|x_n\|)^{-1} \text{ ,.}$$

We have

$$(f_n x_n)(k_0) = x_n(k_0) \quad \text{and} \quad \|f_n x_n\| \leqq \|x_n(k_0)\| + 2^{-n}$$

so that $x := \sum f_n x_n$ (which obviously has the claimed properties) exists in X.

(ii) Choose a neighbourhood V of k_0 contained in U such that

$$\|x(k)\| \leqq \|x(k_0)\| + \varepsilon$$

for $k \in V \cap (\mathrm{ch}\, A)$ and an $f \in A$ such that

$$f(k_0) = \|f\| = 1, \ \|f|_{K \setminus V}\| \leqq \varepsilon(1+\|x\|)^{-1} .$$

Such an f obviously satisfies $\|fx\| \leqq \|x(k_0)\| + \varepsilon$.

The following theorem is a generalization of the Cunningham–Effros–Roy theorem ([6]; for another proof see Theorem 4.5 in [2]).

3.3. THEOREM. *Let X be an A-module as in 3.1. By $\tilde{E}_X$ we denote the set of functionals*

$$\delta_k \otimes p_0 : \ x \mapsto p_0(x(k)) ,$$

where $k \in \mathrm{ch}\, A$ such that $X_k \neq \{0\}$ and $p_0 \in E_{X_k}$.

(i) *$\tilde{E}_X$ is contained in E_X, the set of extreme functionals on X.*
(ii) *$E_X \subset \bar{\tilde{E}}_X$ (weak*-closure).*
(iii) *$E_X \subset \tilde{E}_X$ is not true in general.*
(iv) *If $\mathrm{ch}\, A$ is closed, then $E_X \subset \tilde{E}_X$.*
(v) *If X is the A-modul $A \otimes_\varepsilon X_0$ (see Example 2 above), then $E_X \subset \tilde{E}_X$.*

PROOF. (i) The proof is similar to the proof of Theorem 4.5 in [2].

Let $k \in \mathrm{ch}\, A$ with $X_k \neq \{0\}$ and $p_0 \in E_{X_k}$ be given. We assume that $\delta_k \otimes p_0 = 1/2(r_1 + r_2)$ (with r_1, r_2 in the unit ball of X') and we have to show that $r_1 = r_2 = \delta_k \otimes p_0$.

Suppose that we have shown that

(*) $\quad r_1(x) = r_2(x) = 0$ for every $x \in X$ such that $x(k) = 0$.

We then define p_i: $X_k \to \boldsymbol{K}$ by $x(k) \mapsto r_i(x)$ $(i = 1, 2)$. Because of (*) and 3.2(i), the p_i are well-defined linear functionals on X_k, and 3.2(ii) implies that $\|p_1\|, \|p_2\| \leqq 1$. We have $p_0 = 1/2(p_1 + p_2)$ so that $p_0 = p_1 = p_2$. This proves that $\delta_k \otimes p_0 = r_1 = r_2$.

It thus remains to prove (*). Let $x \in X$ with $x(k)=0$ and $\|x\|=1$ be given, and $\varepsilon>0$. We choose a neighbourhood U of k such that $\|x(k')\| \leqq \varepsilon$ for $k' \in U \cap (\operatorname{ch} A)$ and an $\tilde{x} \in X$ such that $\|\tilde{x}(k)\|=1$, $\|\tilde{x}\| \leqq 1+\varepsilon$, $\|\tilde{x}|_{(\operatorname{ch} A)\setminus U}\| \leqq \varepsilon$, $|p_0(\tilde{x}(k))| \geqq 1-\varepsilon$. Such an x can be constructed by using Lemma 3.2(ii) and the fact that $X_k \neq \{0\}$.

It follows that $\|bx+\tilde{x}\| \leqq 1+2\varepsilon$ for every $b \in \boldsymbol{K}$ with $|b|=1$. Consequently we have $|r_i(bx+\tilde{x})| \leqq 1+2\varepsilon$ $(i=1,2)$ for these b. On the other hand,

$$\begin{aligned} 1-\varepsilon &\leqq |p_0[(bx+\tilde{x})(k)]| \\ &= |(\delta_k \otimes p_0)(bx+\tilde{x})| \\ &\leqq (1/2)(|r_1(bx+\tilde{x})|+|r_2(bx+\tilde{x})|) \end{aligned}$$

is valid so that necessarily $|r_i(bx+\tilde{x})| \geqq 1-4\varepsilon$ $(i=1,2$, all $b \in \boldsymbol{K}$ with $|b|=1)$. Because of

$$|r_i(\tilde{x})| \leqq 1+\varepsilon \quad \text{and} \quad |r_i(\tilde{x})| \geqq 1-3\varepsilon$$

(this follows from $\delta_k \otimes p_0 = 1/2(r_1+r_2)$ and $|p_0(\tilde{x}(k))| \geqq 1-\varepsilon$) this is only possible if $|r_i(x)| \leqq 5\varepsilon$ $(i=1,2)$, and since ε was arbitrary this proves that $r_1(x) = r_2(x) = 0$.

(ii) It suffices to show that the closed convex hull (closure with respect to the weak*-topology) of $\tilde{E}_X$ is the unit ball of X'. Suppose that there is a q with $\|q\| \leqq 1$ which is not in this closure. The Hahn–Banach theorem provides us with an $x \in X$ such that $|p(x)| \leqq 1$ for $p \in \tilde{E}_X$ and $|q(x)|>1$. It follows that

$$\|x\| = \sup \|x(k)\| = \sup \{|p_0(x(k))| \mid k \in \operatorname{ch} A,\ p_0 \in E_{X_k}\} \leqq 1,$$

but this is not possible since $|q(x)|>1$ and $\|q\| \leqq 1$.

(iii) Let A be a function algebra over a compact Hausdorff space K such that there are:

a k_0 in $(\operatorname{ch} A)^- \setminus \operatorname{ch} A$,

a continuous function $h_0\colon K \to [1/2, 1]$ such that $h_0(k_0)=1$, $h_0(k)<1$ for every $k \neq k_0$.

We define $X_k = \mathbf{C}$ for every $k \in \operatorname{ch} A$ and regard $X := h_0 A$ in a natural way as an A-module in $\prod^\infty \{X_k \mid k \in \operatorname{ch} A\}$. Then $h_0\mathbf{1}$ does not attain the norm on $\tilde{E}_X$ so that $\tilde{E}_X \subsetneqq E_X$ in this case.

(iv) This assertion will be proved by using a similar property of function modules.

We define $Y \subset \prod^\infty X_k$ by

$$Y := \{x_1 + \bar{f} x_2 \mid x_1, x_2 \in X,\ f \in A\}^-$$

($\bar{f}$= the complex conjugate of f). We have $X \subset Y$, and Y is a function module over $\mathrm{ch}\, A$ in the sense of [2, Definition 4.1]:

Y is an $(A+\bar{A})$-module and thus a $C(\mathrm{ch}\, A)$-module,

$k \mapsto \|y(k)\|$ is upper semi-continuous for every $y \in Y$ (it clearly suffices to consider elements of the form $x_1 + \bar{f}x_2$; if $\|x_1(k_0)+\bar{f}(k_0)x_2(k_0)\| < a$, the element $x_1 + \bar{f}(k_0)x_2$ is in X so that 3.1(ii) and the continuity of $\bar{f}$ provide us with a neighbourhood U of k_0 such that $\|(x_1+\bar{f}x_2)(k)\| < a$ for $k \in U$).

Now let $p \in E_X$ be given. We choose a $q \in E_Y$ such that $q|_X = p$. By the Cunningham–Effros–Roy theorem q (and thus p) has the form $\delta_k \otimes p_0$ with $k \in \mathrm{ch}\, A$ such that $X_k \neq \{0\}$ and $p_0 \in E_{X_k}$.

(v) $A \otimes_\varepsilon X_0$ is a subspace pf $C(K, X_0)$ (the space of continuous functions from K to X_0) in a natural way. Thus, every $p \in E_X$ has by Singer's theorem the form $\delta_k \otimes p_0$ with $k \in K$, $p_0 \in E_{X_0}$ (cf. [2, Corollary 4.6]). k is necessarily contained in $\mathrm{ch}\, A$ since otherwise δ_k and thus p would admit a proper decomposition by norm one functionals.

3.4. COROLLARY . *If X is an A-module, then $h \mapsto M_h$ is an isometric algebra homomorphism from A into* Mult(X); *M_h denotes the multiplication operator $x \mapsto hx$.*

PROOF. M_h is well-defined by 3.1(i), and the multiplier property follows immediately from 2.1(i) and 3.3(iii). Finally, we have $\|M_h\| = \|h\|$ by 3.1(iv) and 3.2(ii).

3.5. COROLLARY. *Let X be an A-module and T: $X \to X$ a linear continuous operator.*

(i) *If T is a multiplier, then there are multipliers T_k in* Mult(X_k) *(all $k \in$* ch *A) such that $T = \prod T_k$, that is $(Tx)(k) = T_k(x(k))$ for every k and x.*

(ii) *Conversely, if T has the form $T = \prod T_k$, where every T_k is a multiplier, then T is also a multiplier.*

PROOF. (i) We define T_k: $X_k \to X_k$ by $T_k(x(k)) := (Tx)(k)$ for $x \in X$. T_k is defined on all of X_k by 3.2(i). T_k is also well-defined: if $x(k) = 0$, then

$$p_0[(Tx)(k)] \;=\; (\delta_k \otimes p_0(Tx) \;=\; a_T(\delta_k \otimes p_0)(\delta_k \otimes p_0)(x) \;=\; 0$$

for every $p_0 \in E_{X_k}$, so that $(Tx)(k) = 0$.

The T_k are in Mult(X_k), since $p_0 \circ T_k = a_T(\delta_k \otimes p_0)p_0$ for every $p_0 \in E_{X_k}$. Finally, we have $T = \prod T_k$ by definition.

(ii) The assumption implies that every $p \in \tilde{E}_X$ is an eigenvector of T'. Therefore T is a multiplier by 2.1(i) and 3.3(ii).

3.6. COROLLARY. *Let X be an A-module such that* Mult (X_k) *is one-dimensional for every $k \in \operatorname{ch} A$. Then*

$$\operatorname{Mult}(X) = \{M_h \mid h\colon \operatorname{ch} A \to \boldsymbol{K}$$

a bounded function such that $M_h X \subset X\}$.

PROOF. "$\supset$" follows immediately from 2.1(i) and 3.3(ii), and "$\subset$" is a consequence of 3.5(i).

NOTE. If M_h: ch $A \to \boldsymbol{K}$ is any function such that $M_h X \subset X$, then h is bounded on $\{k \mid X_k \neq \{0\}\}$. To show this note that $M_h\colon X \to X$ is a closed operator (the proof is routine) so that $\|M_h\|$ is finite. With the help of 3.2, one can then easily show that $|h(k)| \leqq \|M_h\|$, whenever $X_k \neq \{0\}$. It follows that $M_h \in \operatorname{Mult}(X)$ also in this case.

We now return to the problem of finding Banach spaces which admit a multiplier representation.

Let X be an A-module such that for every $T \in \operatorname{Mult}(X)$ there is an $h \in A$ with $T = M_h$. Choosing ϱ: Mult $(X) \to A$ as the mapping $M_h \mapsto h$, it is easy to see (by combining 2.3 with 3.1(ii)) that $|x|_\varrho(k) = \|x(k)\|$ for $x \in X$ and $k \in \operatorname{ch} A$, so that the $X_{k,\varrho}$ as constructed in 2.6 are precisely the fiber spaces X_k. In addition, the map $\omega\colon X \to \prod^\infty X_k$ is just the natural embedding in this case so that X has a multiplier representation.

This simple observation enables us to investigate Banach spaces as follows:

Given X, determine a "natural" function algebra associated with X such that X can be identified with an A-module. If it is possible to show that Mult (X) $\subset \{M_h \mid h \in A\}$, then the representation under consideration is a multiplier representation.

EXAMPLE. Let X_0 be a Banach space such that $\operatorname{Mult}(X) = \boldsymbol{K}\,\mathrm{Id}$ and A a function algebra. We regard $A \otimes_\varepsilon X_0$ as an A-module over $(\operatorname{ch} A)^-$ as in Example 2 above and *we claim that this representation is a multiplier representation.* To this end, let $T \in \operatorname{Mult}(A \otimes_\varepsilon X_0)$ be given. By Corollary 3.6, there is a bounded function $h\colon \operatorname{ch} A \to \boldsymbol{K}$ such that $Tx = hx$ for every $x \in X$, and it remains to show that there is an $f_h \in A$ such that $f_h|_{\operatorname{ch} A} = h$. Choose any $x_0 \in X_0 \setminus \{0\}$ and a $p_0 \in X_0'$ with $p_0(x_0) = 1$. Since $A \otimes_\varepsilon X_0$ is a subspace of $C(K, X_0)$ (where $K = (\operatorname{ch} A)^-$), p_0 gives rise to an operator

$$P\colon\ A \otimes_\varepsilon X \to CK\colon\ (Px)(k) := p_0(x(k))\,.$$

The range of P is contained in A, since P maps the $f \otimes x_0$ into A, and these elements generate $A \otimes_\varepsilon X_0$. In particular,

$$f_h := P(T(\mathbf{1} \otimes x_0)) \in A ,$$

and evaluation at the $k \in \operatorname{ch} A$ gives $f_h|_{\operatorname{ch} A} = h$.

4. Two counterexamples.

In this section we will show that

I. there is a Banach space which has no multiplier representation,

II. it is not reasonable to consider the Banach spaces X_k in 2.6 or 3.1 for every k in the Shilov boundary $(\operatorname{ch} A)^-$ of A (recall that we have always restricted ourselves to the k in the Choquet boundary).

I. Let A be a necessarily complex function algebra over a compact Hausdorff space K such that $\operatorname{ch} A$ is not closed. We fix a k_0 in $(\operatorname{ch} A)^- \setminus \operatorname{ch} A$ and suppose that $\operatorname{ch} A$ is dense in K.

Further, let $B \in C_{\mathbf{C}}([0,1])$ be a closed subspace such that

B1: $\mathbf{1} \in B$,

B2: for any $t \in [0,1]$, there is a $g \in B$ which attains its norm precisely at t,

B3: there is a $g_0 \in B$ such that $h \in B$, $hg_0 \in B$ always imply that $h = \text{constant}$;

(for example, we could define B as the complex linear span of the functions $\mathbf{1}$, x, e^x, with $g_0(x) := e^x$).

Let L be the disjoint union of $[0,1]$ with K, where $0 \in [0,1]$ and k_0 have been identified; K and $[0,1]$ can be thought of as subsets of K, and $k_0 = 0$.

We define X to be the complex Banach space of all continuous functions f from L to $\mathbf{C}$ for which $f|_K \in A$ and $f|_{[0,1]} \in B$. At first we determine $\operatorname{Mult}(X)$. Since $\mathbf{1} \in B$, we have $A \subset X$ in a natural way so that the mappings δ_k: $f \mapsto f(k)$ are in E_X for every $k \in \operatorname{ch} A$. The δ_k with $k \in [0,1]$ are also in E_X by B2. Now let $T \in \operatorname{Mult}(X)$ be given and $h := T\mathbf{1}$. We have $(Tf)(k) = (hf)(k)$ for $f \in X$ and $k \in (\operatorname{ch} A) \cup [0,1]$ since the associated δ_k are in E_X, and this yields $T = M_h$ by continuity. By B3 $h|_{[0,1]}$ is constant; this proves that

$$\operatorname{Mult}(X) \subset \{M_h \mid h \in X,\ h|_{[0,1]} = \text{const.}\} \cong A ,$$

and the reverse inclusion is trivially valid. With $\varrho =$ the isomorphism $M_h \mapsto h|_K$ (from $\operatorname{Mult}(X)$ onto A), we get $\omega =$ the restriction mapping from X to $X|_{\operatorname{ch} A}$ for our example, and since $\|\omega(f)\| = \|f|_{\operatorname{ch} A}\|$ (which is strictly less than $\|f\|$ in general) for $f \in X$, X cannot have a multiplier representation.

II. The reader will have observed that we used frequently Theorem 2.3 which guarantees that points in the Choquet boundary behave much nicer than arbitrary points of the base space under consideration. However, it is not only this property why we restricted ourselves to the k in the Choquet boundary. To make this statement clear, we give two further definitions of "A-modules" which differ slightly from Definition 3.1:

DEFINITION. Let A be a function algebra over a compact Hausdorff space K, $(X_k)_{k \in K}$ a family of Banach spaces, $X \subset \prod^\infty X_k$ a closed subspace. X is called an A-module*, if X satisfies the conditions 3.1(i)–(iv) (with K instead of ch A).

DEFINITION. Similarly, we define an A-module**: ch A has to be replaced by the Shilov boundary in 3.1.

Every Banach space X is isometrically isomorphic with a (Mult(X))-module*.

(Sketch of proof: Consider ϱ_0: Mult$(X) \to CK_0$ as in 2.1(ii), define

$$X_{[p]} := \text{completion of } X|_{[p]}$$

for $p \in \beta L$ and embed X by $x \mapsto (x|_{[p]})_{[p]}$ into $\prod^\infty X_{[p]}$.)

Thus in a sense the $T \in$ Mult(X) can always be considered as multiplication operators associated with continuous functions. However, neither the definition "A-module*" nor "A-module**" is reasonable, if one tries to associate the X_k with X in an invariant way, i.e. if one wants to have an analogue of Theorem 2.8. This is easy to see in the case of the definition "A-module*": If A is any function algebra over a compact Hausdorff space K, then K is isometrically isomorphic with the restriction of A to $(\text{ch } A)^-$, so that the k in $K \setminus (\text{ch } A)^-$ have no meaning if A is considered as a Banach space. As to the second case we have the following

COUNTEREXAMPLE. There are a function algebra A and two A-modules** X, Y (component spaces: X_k and Y_k for $k \in (\text{ch } A)^-$, respectively) which are isometrically isomorphic such that

$$\text{Mult}(X) = \{M_h \mid h \in A\} \quad \text{and} \quad \text{Mult}(Y) = \{M_h \mid h \in A\},$$

but there is a k_0 such that X_{k_0} is not isometrically isomorphic with any Y_k.

To construct this counterexample. let ε, δ be positive numbers such that there is a $c \in \mathbf{R}$ with

$$(1) \qquad (1/2)(1-\varepsilon)^{-1} \leqq c \leqq ((1+\varepsilon)(1+2\delta))^{-1}.$$

Further, choose a fumction algebra A such that there are a $k_0 \in (\text{ch } A)^- \setminus \text{ch } A$, an open neighbourhood U of k_0, a probability measure μ on $(\text{ch } A)^- \setminus \{k_0\}$ which represents k_0 such that

$$\mu(U) \leqq \delta \,. \tag{2}$$

(Take e.g.

$A = \{f \mid f \text{ in the disk algebra}, f(0)=f(1)\}$,
$K = \{z \mid |z|=1\}, \quad k_0=\{0,1\}$,
μ = the normalized Lebesgue measure on K,
$U = \{e^{2\pi i t} \mid |t| \leqq \delta/2\}$.)

Now let $\|\cdot\|_1$, $\|\cdot\|_2$ be norms on $\mathbf{C}^2$ such that

(3) $(1-\varepsilon)\|(a,b)\|_i \leqq |a|+|b| \leqq (1+\varepsilon)\|(a,b)\|_i \quad (i=1,2,\ a,b \in \mathbf{C})$ and

(4) $(\mathbf{C}^2, \|\cdot\|_1)$, $(\mathbf{C}^2, \|\cdot\|_2)$, and $(\mathbf{C}^2, L^1\text{-norm})$ are pairwise not isometrically isomorphic.

We define the fiber spaces by

$$X_k := Y_k := (\mathbf{C}^2, L^1\text{-norm}) \quad \text{if } k \in (\operatorname{ch} A)^- \setminus \{k_0\}$$

$$X_{k_0} := (\mathbf{C}^2, \|\cdot\|_1), \quad Y_{k_0} := (\mathbf{C}^2, \|\cdot\|_2)\,,$$

and the spaces X and Y as $\{(f,g)h_0 \mid f,g \in A\}$ in $\prod^\infty X_k$ and $\prod^\infty Y_k$, respectively; in this definition h_0 denotes the function

$$h_0(k) := \begin{cases} 1 & \text{if } k \in (\operatorname{ch} A)^- \setminus U \\ 1/2 & \text{if } k \in U \setminus \{k_0\} \\ c & \text{if } k = k_0 \end{cases}$$

We claim that

a) X and Y are A-modules**,
b) $X \cong Y$,
c) $\operatorname{Mult}(X)=\{M_h \mid h \in A\}$, $\quad \operatorname{Mult}(Y)=\{M_h \mid h \in A\}$,

and X_{k_0} is isometrically isomorphic with no Y_k by construction.

a) The proof us routine; the only less obvious fact is the upper semicontinuity of the $k \mapsto \|((f,g)h_0)(k)\|$, and this follows from the first inequalities in (1) and (3).

b) It suffices to show that

$$\|(f,g)h_0\| = \|(f,g)h_0|_{(\operatorname{ch} A)^- \setminus \{k_0\}}\|$$

in X and Y. This implies that $X \cong Y$ since both modules coincide on $(\operatorname{ch} A)^- \setminus \{k_0\}$.

Let $f,g \in A$ be given with $\|(f,g)h_0|_{(\operatorname{ch} A)^- \setminus \{k_0\}}\| = 1$. This means in particular that

$$|f(k)|+|g(k)| \leqq \begin{cases} 1 & \text{if } k \in (\operatorname{ch} A)^- \setminus U \\ 2 & \text{if } k \in U \setminus \{k_0\}, \end{cases}$$

so that

$$\begin{aligned} |f(k_0)|+|g(k_0)| &= \left|\int f\,d\mu\right| + \left|\int g\,d\mu\right| \\ &\leqq \int |f(k)|\,d\mu(k) + \int |g(k)|\,d\mu(k) \\ &\leqq \int_K d\mu + 2\int_U d\mu \\ &\leqq 1+2\delta\,. \end{aligned}$$

The second inequalities in (1) and (3) now imply that $\|((f,g)h_0)(k_0)\|_i \leqq 1$ which proves b).

c) By a) and b), X and Y are even A-modules in the sense of 3.1 when restricted to $\operatorname{ch} A$. Thus every multiplication operator M_h is a multiplier. Conversely, if $T \in \operatorname{Mult}(X)$ (respectively $\operatorname{Mult}(Y)$), there is an $h: \operatorname{ch} A \to \mathbf{C}$ with $T = M_h$ by 3.6. Since $(1,1)h_0 \in X$ (respectively Y) we have $(h,h)h_0 \in X$ (respectively Y), so that there is an $f \in A$ with $(f,f)h_0 = (h,h)h_0$. Hence $h \in A$, and the proof of c) is complete.

5. Spaces for which Mult (X) is finite-dimensional.

In view of the applications of our theory in section 6 we investigate now the class of Banach spaces X for which $\operatorname{Mult}(X)$ is finite-dimensional.

5.1. Proposition. Mult (X) *is finite-dimensional iff X can be written as a finite product* $X = \prod_{i=1}^{n\,\infty} X_i$ *where every* Mult (X_i) *is one-dimensional.*

Proof. Suppose that $\operatorname{Mult}(X)$ is n-dimensional. We have already shown in Proposition 2.10 that $\operatorname{Mult}(X)$ coincides with the centralizer of X in this case. Therefore X can be written as $X = \prod_{i=1}^{n\,\infty} X_i$, where every X_i has one-dimensional centralizer ([2, 5.3b]). The $\operatorname{Mult}(X_i)$ are necessarily one-dimensional, since

$$\operatorname{Mult}(X) \cong \prod_{i=1}^{n} \operatorname{Mult}(X_i)$$

in a natural way so that $\dim(\mathrm{Mult}(X_{i_0}))>1$ would imply $\mathrm{Mult}(X)>n$, a contradiction.

The reverse implication follows immediately from

$$\mathrm{Mult}\,(\textstyle\prod^{\infty} X_i) \cong \prod \mathrm{Mult}\,(X_i)\,.$$

Therefore the spaces we have in mind are built up from spaces for which Mult (X) is one-dimensional. There are a number of well-known spaces with this property.

5.2. PROPOSITION. *Each of the following conditions implies that* Mult (X) *is one-dimensional*:

(i) *X is smooth.*

(ii) *X contains a nontrivial L^p-summand for any p in* $]1,\infty[$ *(recall that an L^p-summand, $1 \leqq p < \infty$, is a closed subspace J of X such that there exists an L^p-complement of J,* i.e. *a closed subspace $J^\perp$ of X such that X is the algebraic direct sum of J and $J^\perp$ and $\|x+x^\perp\|^p = \|x\|^p + \|x^\perp\|^p$ for $x \in J$ and $x^\perp \in J^\perp$; J is called nontrivial if $J \neq \{0\}, X$).*

(iii) $\dim X \geqq 3$, and X contains a nontrivial L^1-summand.

PROOF. (i) Let X be a smooth Banach space and $T \in \mathrm{Mult}\,(X)$. Since $E_{\bar{X}}$ contains the boundary of the dual unit ball every $p \in X'$ is an eigenvector of T'. It follows that T' (and thus T) is a multiple of the identity operator.

(ii) We will use the following simple fact from linear algebra: Let V be a vector space, $S\colon V \to V$ a linear map, $x_0, x_1, \ldots, x_n \in V$ such that $x_0 = a_1 x_1 + \ldots + a_n x_n$ with $a_i \neq 0$ for $i = 1, \ldots, n$.

(*) If the $x_1, \ldots, x_n$ are linearly independent and $x_0, \ldots, x_n$ are eigenvectors of S, then $x_0, \ldots, x_n$ lie in the same eigenspace of S.

Now let X be a Banach space containing a nontrivial L^p-summand J. It is well-known (see e.g. Lemma 1.4 in [3]) that the annihilators J^π and $(J^\perp)^\pi$ of J and $J^\perp$ in X' are L^q-summands, where $1/p + 1/q = 1$. An elementary calculation shows that

$$E_X = \{t^{1/q} p_1 + (1-t)^{1/q} p_2 \mid p_1, p_2 \in E_X,\ p_1 \in J^\pi,\ p_2 \in (J^\perp)^\pi,\ 0 \leqq t \leqq 1\}\,,$$

so that a $T \in \mathrm{Mult}\,(X)$ lies in $\boldsymbol{K}\,\mathrm{Id}$, iff a_T is constant on

$$E := \{p \mid p \in E_X,\ p \in J^\pi \text{ or } p \in (J^\perp)^\pi\}\,.$$

Let $T \in \mathrm{Mult}\,(X)$ be given and

$$p_1 \in J^\pi \cap E_X, \quad p_2 \in (J^\perp)^\pi \cap E_X .$$

$p_0 := (1/2)^{1/q}(p_1+p_2)$ is also in E_X, and an application of (*) with $x_i := p_i$ $(i=0,1,2)$, $V := X'$, $S := T'$ gives $a_T(p_0) = a_T(p_1) = a_T(p_2)$. It follows that a_T is constant on E so that $T \in \boldsymbol{K}\,\mathrm{Id}$ as claimed.

(iii) In this case the elements of E_X are of the form $p+q$, where p and q are extremal in the unit ball of J^π and $(J^\perp)^\pi$, respectively; this follows at once from $X' = J^\pi \oplus_\infty (J^\perp)^\pi$ ([2, 1.5]). Suppose that $\dim J^\pi \geqq 2$ (if $\dim J^\pi = 1$, we consider $(J^\perp)^\pi$ instead of J^π). We choose any $p_1, p_2 \in J^\pi$ and $q \in (J^\perp)^\pi$ which are extremal in the respective unit balls such that p_1, p_2 are linearly independent. With $x_0 := -p_2+q$, $x_1 := p_1+q$, $x_2 := -p_1+q$, $x_3 := p_2+q$ an application of (*) (from the proof of (ii)) yields $a_T(p_1+q) = a_T(p_2+q) = a_T(-p_1+q) = a_T(-p_2+q)$, whenever $T \in \mathrm{Mult}(X)$. If follows that a_T is constant on E_X so that $\mathrm{Mult}(X) = \boldsymbol{K}\,\mathrm{Id}$.

NOTES. 1. In (iii) the condition "$\dim X \geqq 3$" can be dropped in the case of complex scalars. This follows at once from [2, Theorem 1.13] and the fact that Mult (X) coincides with the centralizer for finite-dimensional spaces by 2.10.

However, the real l_2^∞ contains a nontrial L^1-summand whereas Mult (l_2^∞) is two-dimensional.

2. In particular, every L-space which is not two-dimensional has only trivial multipliers.

By the next theorem, Mult (X) can only be great if X contains subspaces which are arbitrarily close to c_0.

5.3. THEOREM. *Let X be a Banach space such that* Mult (X) *is infinite-dimensional. Then, for every $\varepsilon > 0$, there are a subspace Y of X and an isomorphism $I\colon c_0 \to Y$ such that $\|I\|, \|I^{-1}\| \leqq 1+\varepsilon$.*

PROOF. We represent Mult (X) as a function algebra over K_0 as described in the proof of 2.1 (ii).

Let $\varepsilon > 0$ be given. For $\varepsilon_0 > 0$ (which will be fixed later), we choose positive numbers $\varepsilon_1, \varepsilon_2, \ldots$ such that $\varepsilon_1 + \varepsilon_2 + \ldots \leqq \varepsilon_0$. Since Mult (X) is infinite-dimensional, we may select a sequence (U_n) of disjoint open subsets of K_0 and points $k_n \in U_n \cap \mathrm{ch}(\mathrm{range}\,\varrho_0)$. For $n \in \mathbf{N}$ we choose $T_n \in \mathrm{Mult}(X)$ such that

$$\|T_n\| = (\varrho_0(T_n))(k_n) = 1, \quad \|(\varrho_0(T_n))|_{K_0 \setminus U_n}\| \leqq \varepsilon_n$$

(cf. Theorem 2.3). By the definition of the operator norm there are $y_n \in X$, $\|y_n\| = 1$, such that $\|T_n y_n\| > 1 - \varepsilon_0$. We define $x_n := T_n y_n$ for $n \in \mathbf{N}$ and claim that

$$Y := \overline{\text{lin}}\,\{x_n \mid n \in \mathbf{N}\}$$

is isomorphic with c_0.

To this end, we define I from the space of finite sequences to Y by

$$\sum_{i=1}^{n} a_i e_i \mapsto \sum_{i=1}^{n} a_i x_i$$

(e_i = the usual ith unit vector). We will prove that

$$(*) \qquad (1-2\varepsilon_0)\max_i |a_i| \leqq \left\|\sum_{i=1}^{n} a_i x_i\right\| \leqq (1+\varepsilon_0)\max_i |a_i|$$

for every finite sequence $\sum a_i e_i$ in c_0. A routine computation then shows that I can be extended as an isomorphism from c_0 to Y with

$$\|I\|, \|I^{-1}\| \leqq \max\{1+\varepsilon_0, (1-2\varepsilon_0)^{-1}\}$$

($\leqq 1+\varepsilon$ for suitable ε_0).

Thus it remains to prove (*). Let $p \in E_X$ be a arbitrarily given. With ν as in 2.1(ii), $\nu(p)$ lies in at most one U_n, say U_{n_0}. By the choice of the T_n, we have

$$|a_{T_n}(p)| = |(\varrho_0(T))(\nu(p))| \leqq \varepsilon_n$$

for $n \neq n_0$ and $|a_{T_{n_0}}(p)| \leqq 1$ so that

$$\left|p\left(\sum_{i=1}^{n} a_i x_i\right)\right| = \left|\sum_{i=1}^{n} a_i a_{T_i}(p) p(y_i)\right|$$
$$\leqq (\max|a_i|)(1+\varepsilon_1+\varepsilon_2+\ldots)$$
$$\leqq (\max|a_i|)(1+\varepsilon_0)$$

and this inequality is a fortiori valid if $\nu(p)$ is contained in no U_n). It follows that $\|\sum a_i x_i\| \leqq (\max|a_i|)(1+\varepsilon_0)$.

On the other hand, if $i_0 \in \{1,\ldots,n\}$ is arbitrarily given, choose $p \in E_X$ such that

$$|p(x_{i_0})| = |a_{T_{i_0}}(p)|\,|p(y_{i_0})| \geqq 1-\varepsilon_0 .$$

We have $|a_{T_{i_0}}(p)| \geqq 1-\varepsilon_0$ so that $\nu(p)$ lies in U_{i_0} and therefore in no U_i with $i \neq i_0$. Hence

$$|p(a_i x_i)| = |a_{T_i}(p)|\,|a_i p(y_i)| \leqq \varepsilon_i |a_i|$$

for $i \neq i_0$ and thus

$$\|\sum a_i x_i\| \geqq |p(\sum a_i x_i)|$$
$$\geqq |p(a_{i_0} x_{i_0})| - \sum_{i \neq i_0} |p(a_i x_i)|$$

$$\geqq |a_{i_0}|(1-\varepsilon_0)-(\max|a_i|)(\varepsilon_1+\varepsilon_2+\dots)$$

$$\geqq |a_{i_0}|(1-\varepsilon_0)-\varepsilon_0(\max|a_i|)\,.$$

This proves that $\|\sum a_i x_i\| \geqq (1-2\varepsilon_0)\max|a_i|$.

5.4. COROLLARY. Mult(X) *is finite-dimensional for every reflexive Banach space.*

5.5. COROLLARY. *A reflexive function algebra is finite-dimensional.*

PROOF. For function algebras we have $A \cong$ Mult(X) in a natural way (cf. the example in section 2).

5.6. COROLLARY. *If X is a reflexive space which contains no nontrivial M-summand, then* Mult$(X)=\boldsymbol{K}$ Id *(M-summands are defined similarly to L^p-summands; the relevant norm condition here is $\|x+x^\perp\|=\max\{\|x\|,\|x^\perp\|\}$).*

PROOF. By 5.4 and 2.10, $Z(X)=$ Mult(X) is n-dimensional with $n<\infty$. It follows that X can be written as a sum of n nonzero M-summands (Proposition 5.1). We have $n=1$, since X is the only nonzero M-summand.

5.7. COROLLARY. *An infinite-dimensional function algebra contains subspaces which are arbitrarily close to c_0.*

6. On the problem whether $A\otimes_\varepsilon X$ determines A and/or X.

In this section we want to apply our theory to tensor products. In a sense the results may be thought of as variants of theorems of the Banach–Stone type where K and/or X are reconstructed from $CK\otimes_\varepsilon X=C(K,X)$ (see [2]). We resist the temptation to derive theorems for very general situations, only the case of finite-dimensional Mult (X) is considered. Generalizations can be obtained similarly to the Banach–Stone case; cf. Chapters 9–11 in [2].

To prepare our investigations we restate the uniqueness theorem 2.7 in the A-module setting:

6.1. THEOREM. *Let X and Y be an A-module and a B-module, respectively, such that* Mult $(X)=\{M_h \mid h\in A\}$ *and* Mult $(Y)=\{M_h \mid h\in B\}$ *(A and B functions algebras).*

Then, for every isometric isomorphism $I\colon X\to Y$, there are a homeomorphism $t\colon \mathrm{ch}\,B\to \mathrm{ch}\,A$ and a family of isometric isomorphisms $u_l\colon X_{t(l)}\to Y_l$ such that $(Ix)(l)=u_l(x(t(l)))$ for $x\in X$ and $l\in \mathrm{ch}\,B$.

In the special case of tensor products we can prove much more:

6.2. THEOREM: *Let A and B be function algebras over K and L, respectively; as before we may assume that $(\operatorname{ch} A)^- = K$ and $(\operatorname{ch} B)^- = L$. If X_0 and Y_0 are Banach spaces with one-dimensional multiplier algebra, then for every isometric isomorphism $I\colon A \otimes_\varepsilon X_0 \to B \otimes_\varepsilon Y_0$ there are a homeomorphism $t\colon L \to K$ and a family of isometric isomorphisms $u_l\colon X_0 \to Y_0$ (all $l \in L$) such that*

$$(Ix)(l) = u_l(x(t(l)))$$

for $l \in L$ and $x \in A \otimes_\varepsilon X_0$; in these expressions we regard $A \otimes_\varepsilon X_0$ and $B \otimes_\varepsilon Y_0$ as subspaces of $C(K, X_0)$ and $C(L, Y_0)$.

Further, $f \mapsto f \circ t$ is an algebra isomorphism from A onto B, and for $x_0 \in X_0$ and $y_0 \in Y_0$ the mappings $l \mapsto u_l(x_0)$ and $k \mapsto u_{t^{-1}(k)}^{-1}(y_0)$ are contained in $B \otimes_\varepsilon Y_0$ and $A \otimes_\varepsilon X_0$, respectively (so that in particular $l \mapsto u_l$ and $l \mapsto u_l^{-1}$ are strongly continuous).

PROOF. We have already shown at the end of section 3 how multiplier representations of $A \otimes_\varepsilon X_0$ and $B \otimes_\varepsilon Y_0$ can be obtained. I induces an isometric algebra isomorphism

$$i\colon \operatorname{Mult}(B \otimes_\varepsilon Y_0) \cong B \to \operatorname{Mult}(A \otimes_\varepsilon X_0) \cong A$$

by $i(T) = I^{-1} \circ T \circ I$. Hence there is a homeomorphism $t\colon L \to K$ such that $f \mapsto f \circ t$ is an algebra isomorphism from A onto B. Theorem 6.1 provides us with a family of isometric isomorphisms $u_l\colon X_0 \to Y_0$ ($l \in \operatorname{ch} B$) such that

$$(Ix)(l) = u_l(x(t(l))) \quad \text{for } x \in A \otimes_\varepsilon X_0 \text{ and } l \in \operatorname{ch} B\,.$$

We have only to show that the u_l can be defined for all l in L in such a way that

$$(Ix)(l) = u_l(x(t(l))), \quad (l \mapsto u_l(x_0)) \in B \otimes_\varepsilon Y_0, \text{ and}$$

$$(k \mapsto u_{t^{-1}(k)}^{-1}(y_0)) \in A \otimes_\varepsilon X_0\,.$$

The construction is straightforward: For $l \in L \setminus \operatorname{ch} B$, $x_0 \in X_0$, define $u_l(x_0)\colon = (Ix)(l)$, where x is any element of $A \otimes_\varepsilon X_0$ such that $x(t(l)) = x_0$.

u_l is welldefined: If $x(t(l)) = 0$, choose a net l_i in $\operatorname{ch} B$ with $t(l_i) \to t(l)$; $x(t(l)) = 0$ implies that $x(t(l_i)) \to 0$ so that

$$(Ix)(l) = \lim (Ix)(l_i) = \lim u_{l_i}(x(t(l_i))) = 0\,.$$

$\|u_l\| \leqq 1$:

$$\|u_l(x_0)\| = \|(I(1 \otimes x_0))(l)\| \leqq \|1 \otimes x_0\| = \|x_0\|\,.$$

u_l is an isometrical isomorphism: A direct proof of this fact can easily be

obtained. However, it is sufficient to consider I^{-1} instead of I, and a similar construction as above provides us with an inverse u_l^{-1} of u_l.

It is clear from the construction that $(Ix)(l) = u_l(x(t(l)))$ for every l and x, and this identity implies that

$$(l \mapsto u_l(x_0)) = I(1 \otimes x_0) \in B \otimes_\varepsilon Y_0 \text{ and}$$

$$(k \mapsto u_{t^{-1}(k)}^{-1}(y_0)) = I^{-1}(1 \otimes y_0) \in A \otimes_\varepsilon X_0 .$$

6.3. Corollary. *(A, B, X_0, Y_0 as in 6.2) $A \otimes_\varepsilon X_0$ and $B \otimes_\varepsilon Y_0$ are isometrically isomorphic iff $A \cong B$ (as algebras) and $X_0 \cong Y_0$.*

In the special case, when A and B are the same algebra, Theorem 6.2 can be improved.

6.4. Lemma. *Let A be a function algebra over a compact Hausdorff space and X_0 a Banach space. Further let $\tilde{I}\colon A \otimes_\varepsilon X_0 \to A \otimes_\varepsilon X_0$ be an isometrical isomorphism such that there are isometric isomorpisms $u_k\colon X_0 \to X_0$ ($k \in (\operatorname{ch} A)^-$) with $(\tilde{I}x)(k) = u_k(x(k))$ for every x and k.*

Then there are isometrical isomorphisms $u_k\colon X_0 \to X_0$ for every $k \in K$ with $(\tilde{I}x)(k) = u_k(x(k))$ ($x \in A \otimes_\varepsilon X_0$, $k \in K$).

Proof. Let $k_0 \in K$ be given. Suppose that we have shown that

$$(*) \qquad (\tilde{I}x)(k_0) = 0 \quad \text{whenever } x(k_0) = 0 .$$

Then it is routine to prove that $u_{k_0}(x(k_0)) := (\tilde{I}x)(k_0)$ ($x \in A \otimes_\varepsilon X_0$) defines a welldefined isometrical isomorphism on X_0 (for the construction of $u_{k_0}^{-1}$ consider $\tilde{I}^{-1}$ instead of $\tilde{I}$). Since $(\operatorname{ch} A)^-$ is a boundary for $A \otimes_\varepsilon X_0$ and not only for A (this follows from $A \cong A|_{(\operatorname{ch} A)^-}$) two elements of $A \otimes_\varepsilon X_0$ are identical, iff their restrictions to $\operatorname{ch} A$ coincide. In particular, $\tilde{I}(f \otimes x_0)$ and $fI(1 \otimes x_0)$ have this property so that

$$(**) \qquad (\tilde{I}(f \otimes x_0))(k_0) = f(k_0)(\tilde{I}(1 \otimes x_0))(k_0) \qquad (f \in A,\ x_0 \in X_0) .$$

We will now prove (*). Let $x \in A \otimes_\varepsilon X_0$ with $x(k_0) = 0$ be given and $\varepsilon > 0$. We have

$$\left\| x - \sum_{i=1}^{n} f_i \otimes x_0^i \right\| \leqq \varepsilon$$

for suitable $f_i \in A$, $x_0^i \in X_0$. It follows from (**) that

$$\|(Ix)(k_0)\| \leqq \|\tilde{I}(\sum f_i \otimes x_0^i)(k_0)\| + \varepsilon$$

$$\begin{aligned} &= \|\sum f_i(k_0)(\tilde{I}(1\otimes x_0^i))(k_0)\| + \varepsilon \\ &= \|\big(\tilde{I}(1\otimes(\sum f_i(k_0)x_0^i))\big)(k_0)\| + \varepsilon \\ &\leqq \|\tilde{I}(1\otimes x(k_0))\| + 2\varepsilon \\ &\leqq 2\varepsilon \end{aligned}$$

so that $(\tilde{I}x)(k_0) = 0$.

6.5. THEOREM. *Let X_0 be a Banach space with* $\mathrm{Mult}(X_0) = \boldsymbol{K}\,\mathrm{Id}$ *and A a function algebra over a compact Hausdorff space K. Then for every isometric isomorphism $I\colon A\otimes_\varepsilon X_0 \to A\otimes_\varepsilon X_0$, there are an algebra isomorphism φ on A, and a family of isometric isomorphisms $u_k\colon X_0 \to X_0$ $(k\in K)$ such that $(Ix)(k) = u_k((\varphi^\sim x)(k))$; here $\varphi^\sim\colon A\otimes_\varepsilon X_0 \to A\otimes_\varepsilon X_0$ denotes the isomorphism* $\varphi\otimes\mathrm{Id}$.
Further, $k\mapsto u_k(x_0)$ and $k\mapsto u_k^{-1}(x_0)$ are in $A\otimes_\varepsilon X_0$ for every $x_0\in X_0$.

PROOF. We construct t and the u_k $(k\in(\mathrm{ch}\,A)^-)$ as in Theorem 6.2 and define $\varphi\colon A\to A$ by $\varphi(f) :=$ the canonical extension of $f\circ t$. We then have

$$(Ix)(k) = u_k((\varphi^\sim x)(k)) \quad \text{for } x\in A\otimes_\varepsilon X_0 \text{ and } k\in(\mathrm{ch}\,A)^- .$$

This is obvious if $x = f\otimes x_0$, and the general case follows by an approximation argument.

With $\tilde{I} := I\circ(\varphi^\sim)^{-1}$ we may apply the preceding lemma by which we get isometric isomorphisms u_k on X_0 for every $k\in K$ such that $(Ix)(k) = u_k((\varphi^\sim x)(k))$ for every k and x. Finally, this expression implies that

$$(k\mapsto u_k(x_0)) = I(1\otimes x_0)$$

and

$$(k\mapsto u_k^{-1}(x_0)) = (\varphi^\sim\circ I^{-1})(1\otimes x_0) \quad \text{for } x_0\in X_0 ,$$

and both functions are in $A\otimes_\varepsilon X_0$ as claimed.

NOTE. Cambern [4] has obtained this theorem by using T-set methods in the case of reflexive Banach spaces which contain no nontrivial M-summand. This case is contained in the theorem by Corollary 5.6.

Theorem 6.2 will now be extended to situations, where the spaces under consideration are composed from simple parts.

6.6. LEMMA. *Let $X_1,\ldots,X_n$ be Banach spaces which admit a multiplier representation. Then $\prod_{i=1}^{n}{}^{\infty} X_i$ has also a multiplier representation.*

PROOF. This follows immeduately from Mult $(\prod^\infty X_i) \cong \prod \text{Mult}(X_i)$.

NOTE. In the A-module setting, Lemma 6.6 can be stated as follows: If the X_i are A_i-modules such that

$$\text{Mult}(X_i) = \{M_h \mid h \in A_i\} \quad \text{for } i = 1, \ldots, n,$$

then $\prod^\infty X_i$ is an $\prod^\infty A_i$-module, and

$$\text{Mult}(\prod^\infty X_i) = \{M_h \mid h \in \prod^\infty A_i\}.$$

6.7. THEOREM. *Let $A_1, \ldots, A_n, B_1, \ldots, B_m$ be function algebras and $X_0^1, \ldots, X_0^n, Y_0^1, \ldots, Y_0^m$ Banach spaces with one-dimensional multiplier algebra such that the X_0^i (respectively the Y_0^j) are pairwise not isometrically isomorphic. Further, let*

$$I: \prod^\infty A_i \otimes_\varepsilon X_0^i \to \prod^\infty B_j \otimes_\varepsilon Y_0^j$$

be an isometric isomorphism. Then $n = m$, and there is a permutation

$$\omega: \{1, \ldots, n\} \to \{1, \ldots, n\}$$

such that the restriction of I to $A_i \otimes X_0^i$ is an isometric isomorphism onto $B_{\omega(i)} \otimes_\varepsilon Y_0^{\omega(i)}$. Thus I is the product of a family of isometric isomorphisms

$$I_i: A_i \otimes X_0^i \to B_{\omega(i)} \otimes_\varepsilon Y_0^{\omega(i)}$$

(which can be described by using 6.2).

PROOF. A multiplier representation of $X := \prod^\infty A_i \otimes_\varepsilon X_0^i$ can be derived from (the note after) Lemma 6.6: With $A := \prod^\infty A_i$, we represent A as a function algebra over the disjoint union of the $(\text{ch}\, A_i)^-$. We have $\text{ch}\, A = \dot{\bigcup} \text{ch}\, A_i$, and with $X_k := X_0^i$ for $k \in \text{ch}\, A_i$, we can embed X into $\prod^\infty \{X_k \mid k \in \text{ch}\, A\}$ in a natural way such that

$$\text{Mult}(X) = \{M_h \mid h \in A\}.$$

$Y := \prod^\infty B_j \otimes_\varepsilon Y_0^j$ is represented similarly as a B-module over $\dot{\bigcup} (\text{ch}\, B_j)^-$ $(B := \prod^\infty B_j)$.

By 6.1 there is a homeomorphism $t: \text{ch}\, B \to \text{ch}\, A$, and the fiber space $X_{t(l)}$ is always isometrically isomorphic with Y_l. Since these fiber spaces are the X_0^i and the Y_0^j respectively, we have $n = m$ and there is a permutation $\omega: \{1, \ldots, n\} \to \{1, \ldots, n\}$ such that $X_0^i \cong Y_0^{\omega(i)}$; also t induces a homeomorphism from $\text{ch}\, B_{\omega(i)}$ onto $\text{ch}\, A_i$, and I maps $A_i \otimes_\varepsilon X_0^i$ onto all of $B_{\omega(i)} \otimes_\varepsilon Y_0^{\omega(i)}$. The remaining assertions now follow from Theorem 6.2.

6.8. THEOREM. *Let* X_0 *and* Y_0 *be Banach spaces such that the multiplier algebras of* X_0 *and* Y_0 *are finite-dimensional.*

By 5.1 *we may write* X_0 *(and* Y_0*) as*

$$X_0 = \prod_{\sigma=1}^{s}{}^{\infty} X_\sigma^{n_\sigma} \quad \left(Y_0 = \prod_{\varrho=1}^{r}{}^{\infty} Y_\varrho^{m_\varrho} \right),$$

where the X_σ *(the* Y_ϱ*) have one-dimensional multiplier algebra and are pairwise not isometrically isomorphic.*

Then, for function algebras A *and* B, $A \otimes_\varepsilon X_0$ *and* $B \otimes_\varepsilon Y_0$ *are isometrically isomorphic iff,*

$r = s$, *there is a permutation* ω *on* $\{1, \ldots, s\}$ *such that* $X_\sigma \cong Y_{\omega(\sigma)}$ $(\sigma = 1, \ldots, s)$, $A^{n_\sigma} = B^{m_{\omega(\sigma)}}$ *as algebras for all* σ.

PROOF. We have natural isomorphisms

$$A \otimes_\varepsilon X^n \cong A^n \otimes_\varepsilon X \quad \text{and} \quad A \otimes_\varepsilon \prod{}^{\infty} X_i \cong \prod{}^{\infty} A \otimes_\varepsilon X_i ,$$

so that we may regard $A \otimes_\varepsilon X_0$ and $B \otimes_\varepsilon Y_0$ as $\prod^\infty (A^{n_\sigma} \otimes_\varepsilon X_\sigma)$ and $\prod^\infty (B^{m_\varrho} \otimes_\varepsilon Y_\varrho)$, respectively. The assertion is thus a corollary to the preceding theorem.

6.9. COROLLARY. *Let* X_0 *be as in Theorem* 6.8. *Then* $A \otimes_\varepsilon X_0 \cong B \otimes_\varepsilon X_0$ *always implies that* $A \cong B$ *(for function algebras* A, B*) iff* $\min n_\sigma = 1$.

PROOF. If $\min n_\sigma = 1$, then $A \otimes_\varepsilon X_0 \cong B \otimes_\varepsilon X_0$ yields $A \cong B$ by Theorem 6.8. For the proof of the converse we refer the reader to Theorem 11.16 in [2], where it is shown that there are non-isometric CK-spaces A and B with $A \otimes_\varepsilon X_0 \cong B \otimes_\varepsilon X_0$, if $\min n_\sigma > 1$.

6.10. COROLLARY. *Let* X_0 *and* Y_0 *be as in Theorem* 6.8 *and suppose that*

$$n_1 = \ldots = n_s = m_1 = \ldots = m_r = 1 .$$

Then, for function algebras A *and* B, $A \otimes_\varepsilon X_0 \cong B \otimes_\varepsilon Y_0$, *iff* $A \cong B$ *and* $X_0 \cong Y_0$.

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